LETTER TO THE EDITOR

The XY spin glass with slow dynamic couplings

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Abstract. We investigate an XY spin-glass model in which both spins and couplings evolve in time: the spins change rapidly according to Glauber-type rules, whereas the couplings evolve slowly with a dynamics involving spin correlations and Gaussian disorder. For large times the model can be solved using replica theory. In contrast to the XY-model with static disordered couplings, solving the present model requires two levels of replicas, one for the spins and one for the couplings. Relevant order parameters are defined and a phase diagram is obtained upon making the replica-symmetric ansatz. The system exhibits two different spin-glass phases, with distinct de Almeida–Thouless lines, marking continuous replica-symmetry breaking: one describing freezing of the spins only, and one describing freezing of both spins and couplings.

Recently various models with a coupled dynamics of fast Ising spins and slow couplings have been studied (see e.g. [1–5] and references therein). In addition to physical motivations, such as understanding the simultaneous learning and retrieval in recurrent neural networks or the influence of slow atomic diffusion processes in disordered magnetic systems, there is a greater theoretical interest in such models in that they generate the replica formalism for a *finite* number of replicas n. Moreover, the replica number is found to have a physical meaning as the ratio of two temperatures (characterizing the stochasticity in the spin dynamics and the coupling dynamics, respectively). In this letter we extend the methods and results obtained for Ising spin models to a classical XY spin glass with dynamic couplings, whose spin variables are physically more realistic than Ising ones. In addition, the XY model is closely related to neural network models of coupled oscillators, which provide a phenomenological description of neuronal firing synchronization in brain tissue. We solve our model upon making the replica-symmetric ansatz, and calculate the de Almeida–Thouless (AT) lines [6] (of which here there are two types), where continuous transitions occur to phases with broken replica symmetry. In doing so we also improve the calculations of [3]. As in the Ising case we find two qualitatively different types of spin-glass phases. In one spin-glass phase the spins do freeze in random directions, but on the timescales of the coupling dynamics these 'frozen directions' change. In the second spin-glass phase the spins as well as the couplings freeze, such that even on the large timescales the 'frozen directions' of the spins remain stationary.

We choose a system of N classical two-component spin variables $\boldsymbol{S}_i = (\cos\theta_i, \sin\theta_i)$, $i = 1 \ldots N$, and symmetric exchange interactions (or couplings) J_{ij}, with a Glauber-type

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dynamics such that for *stationary* choices of the couplings the microscopic spin probability density would evolve towards a Boltzmann distribution, with the standard Hamiltonian $H(\{S_i\}, \{J_{ij}\}) = -\sum_{k<\ell} J_{k\ell} S_k \cdot S_\ell$ and with inverse temperature $\beta = T^{-1}$. The couplings J_{ij} are taken to be of infinite range. They will now themselves be allowed to evolve in time in a stochastic manner, partially in response to the states of the spins and to externally imposed biases. However, we assume that the spin dynamics is very fast compared with that of the couplings, such that on the timescales of the couplings, the spins are effectively in equilibrium (i.e. we take the adiabatic limit). For the dynamics of the couplings the following Langevin form is proposed (which is the natural adaptation to XY spins of the choices originally made in [1, 3] for Ising spins):

$$\frac{\mathrm{d}}{\mathrm{d}t} J_{ij} = \frac{\langle S_i \cdot S_j \rangle + K_{ij}}{N} - \mu J_{ij} + \frac{\eta_{ij}(t)}{N^{1/2}} \qquad i < j = 1 \ldots N. \tag{1}$$

The term $\langle S_i \cdot S_j \rangle$, representing local spin correlations associated with the coupling J_{ij}, is a thermodynamic average over the Boltzmann distribution of the spins, given the instantaneous couplings $\{J_{k\ell}\}$. External biases $K_{ij} = \mu N B_{ij}$ serve to steer the weights to some preferred values†. The B_{ij} are choosen to be quenched random variables, drawn independently from a Gaussian probability distribution with mean B_0/N and variance $\tilde{B}/N$. The decay term μJ_{ij} in (1) is added to limit the magnitude of the couplings. Finally, the terms $\eta_{ij}(t)$ represent Gaussian white noise contributions, of zero mean and covariance $\langle \eta_{ij}(t)\eta_{kl}(t') \rangle = 2\tilde{T} \delta_{ik}\delta_{jl}\delta(t - t')$, with associated temperature $\tilde{T} = \tilde{\beta}^{-1}$. Factors of N are introduced in order to ensure non-trivial behaviour in the thermodynamic limit $N \to \infty$.

The three independent global symmetries of our model, which can be expressed efficiently in terms of the Pauli spin matrices σ_x and σ_z, are the following:

$$\begin{array}{llll}
\text{inversion of both spin axes:} & S_i \to -S_i & \text{for all } i & \\
\text{inversion of one spin axis:} & S_i \to \sigma_z S_i & \text{for all } i & (2)\\
\text{permutation of spin axes:} & S_i \to \sigma_x S_i & \text{for all } i. &
\end{array}$$

Upon using algebraic relations such as $\sigma_x \sigma_z \sigma_x = -\sigma_z$ and $\sigma_z \sigma_x \sigma_z = -\sigma_x$ we see that in the high T (ergodic) regime these three global symmetries generate the following local identities, respectively:

$$\langle S_i \rangle = \mathbf{0} \qquad \langle S_i \cdot \sigma_x S_j \rangle = 0 \qquad \langle S_i \cdot \sigma_z S_j \rangle = 0. \tag{3}$$

The equilibrium solution of the probability density associated with the stochastic equation (1) for the couplings follows from the fact that (1) is conservative, i.e. that it can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} J_{ij} = -\frac{1}{N} \frac{\partial}{\partial J_{ij}} \tilde{H}(\{J_{ij}\}) + \frac{\eta_{ij}(t)}{N^{1/2}} \tag{4}$$

with the following effective Hamiltionian for the couplings:

$$\tilde{H}(\{J_{ij}\}) = -\frac{1}{\beta} \log \mathcal{Z}_\beta(\{J_{ij}\}) + \frac{1}{2}\mu N \sum_{k<\ell} J_{k\ell}^2 - \mu N \sum_{k<\ell} B_{k\ell} J_{k\ell}. \tag{5}$$

In this expression $\mathcal{Z}_\beta(\{J_{ij}\}) = \mathrm{Tr}_{\{S_i\}} \exp[\beta \sum_{k<\ell} J_{k\ell} S_k \cdot S_\ell]$ is the partition function of the XY spins with instantaneous couplings $\{J_{ij}\}$. Thus, the stationary probability density for the couplings is also of a Boltzmann form, with the Hamiltonian (5), and the

† This notation follows that of [3].

thermodynamics of the slow system (the couplings) are generated by the partition function $\tilde{\mathcal{Z}}_{\tilde{\beta}} = \int \prod_{k<\ell} dJ_{k\ell} \exp[-\tilde{\beta}\tilde{H}(\{J_{ij}\})]$, leading to (modulo irrelevant prefactors)

$$\tilde{\mathcal{Z}}_{\tilde{\beta}} = \int \prod_{k<\ell} dJ_{k\ell} \, [\mathcal{Z}_\beta(\{J_{ij}\})]^n \exp\left[\mu\tilde{\beta}N \sum_{k<\ell} B_{k\ell} J_{k\ell} - \tfrac{1}{2}\mu\tilde{\beta}N \sum_{k<\ell} J_{k\ell}^2 \right]. \quad (6)$$

Finally, we define the disorder-averaged free energy per site $\tilde{f} = -(\tilde{\beta}N)^{-1}\langle \log \tilde{\mathcal{Z}}_{\tilde{\beta}} \rangle_B$, in which $\langle \cdot \rangle_B$ is an average over the $\{B_{ij}\}$. In contrast to standard systems with frozen disorder, the replica number n is here given by the ratio $n = \tilde{\beta}/\beta$, and can take any real non-negative value. The limit $n \to 0$ corresponds to a situation in which the coupling dynamics is driven by the Gaussian white noise, rather than by spin correlations; in the limit $n \to \infty$ the influence of spin correlations dominates.

We carry out the disorder average using the identity $\log \tilde{\mathcal{Z}}_{\tilde{\beta}} = \lim_{r\to 0} r^{-1}[\tilde{\mathcal{Z}}_{\tilde{\beta}}^r - 1]$, evaluating the latter by analytic continuation from integer r. Our system with partition function $\tilde{\mathcal{Z}}_{\tilde{\beta}}$ is thus replicated r times; we label each replica by a Roman index. Each of the r functions $\tilde{\mathcal{Z}}_{\tilde{\beta}}$, in turn, is given by (6), and involves $\mathcal{Z}_\beta(\{J_{ij}\})^n$ which is replaced by the product of n further replicas, labelled by Greek indices. For non-integer n, analytic continuation is again made from integer n. Performing the disorder average in $\tilde{f}$ results in an expression involving nr coupled replicas of the original system: $\{S_i\} \to \{S_{ia}^\alpha\}$, with $\alpha = 1 \ldots n$ and $a = 1 \ldots r$. For $N \to \infty$ this expression can be evaluated in the familiar fashion of replica mean-field theory [7], by saddle-point integration. This procedure induces the following order parameters:

$$m_a^\alpha = \frac{1}{N} \sum_i \left\langle \overline{\langle S_{ia}^\alpha \rangle} \right\rangle_B \qquad\qquad q_{ab}^{\alpha\beta} = \frac{1}{N} \sum_i \left\langle \overline{\langle S_{ia}^\alpha \cdot S_{ib}^\beta \rangle} \right\rangle_B$$

$$u_{ab}^{\alpha\beta} = \frac{1}{N} \sum_i \left\langle \overline{\langle S_{ia}^\alpha \cdot \sigma_x S_{ib}^\beta \rangle} \right\rangle_B \qquad v_{ab}^{\alpha\beta} = \frac{1}{N} \sum_i \left\langle \overline{\langle S_{ia}^\alpha \cdot \sigma_z S_{ib}^\beta \rangle} \right\rangle_B.$$

The horizontal bar denotes thermal averaging over the coupling dynamics with fixed biases $\{B_{ij}\}$. Comparison with (3) shows that the order parameters m_a^α, $u_{ab}^{\alpha\beta}$ and $v_{ab}^{\alpha\beta}$ measure the breaking of the global symmetries (2). For simplicity we choose $B_0 = 0$. We make the usual assumption that, in the absence of global symmetry-breaking forces, phase transitions can lead to at most *local* violation of the identities (3). Thus the latter will remain valid if averaged over all sites, at any temperature, which implies that $m_a^\alpha = 0$ and $u_{ab}^{\alpha\beta} = v_{ab}^{\alpha\beta} = 0$. The spin-glass order parameters $q_{ab}^{\alpha\beta}$, on the other hand, are not related to simple global symmetries, and serve to characterize the various phases.

The final stage of the calculation is to make the replica symmetry (RS) ansatz. Since observables with identical Roman indices refer to system copies with identical couplings, whereas observables with identical Roman indices *and* identical Greek indices refer to system copies with identical couplings *and* identical spins, in the present model the RS ansatz takes the form $q_{ab}^{\alpha\beta} = \delta_{ab}\{\delta_{\alpha\beta} + q_1[1-\delta_{\alpha\beta}]\} + q_0[1-\delta_{ab}]$†. The remaining two order parameters are determined as the solutions of the following coupled saddle-point equations:

$$q_0 = \int dx \, P(x) \left\{ \frac{\int dz \, P(z)[I_0(z\Xi)]^{n-1} I_1(z\Xi) I_1(zx\beta\Xi^{-1}\sqrt{\tfrac{1}{2}\tilde{B}q_0})}{\int dz \, P(z)[I_0(z\Xi)]^n I_0(zx\beta\Xi^{-1}\sqrt{\tfrac{1}{2}\tilde{B}q_0})} \right\}^2 \quad (7)$$
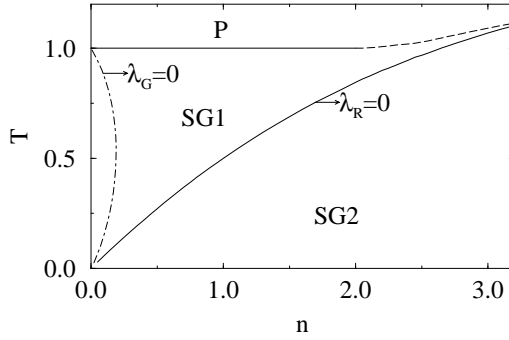
† $S_a^\alpha \cdot S_a^\alpha = 1$.

Figure 1. Phase diagram of the *XY* spin glass with slow dynamic couplings, drawn in the *n–T* plane; for $B_0 = 0$, $\tilde{B} = 1$ and $\tilde{J} = 3$. P: paramagnetic phase, $q_1 = q_0 = 0$. SG1: first spin-glass phase, $q_1 > 0$ and $q_0 = 0$ (freezing on spin timescales only). SG2: second spin-glass phase, $q_1 > 0$ and $q_0 > 0$ (freezing on all timescales). AT lines: $\lambda_R = 0$ (Roman replicon), $\lambda_G = 0$ (Greek replicon).

$$q_1 = \int dx \, P(x) \left\{ \frac{\int dz \, P(z)[I_0(z\Xi)]^{n-2}[I_1(z\Xi)]^2 I_0(zx\beta\Xi^{-1}\sqrt{\frac{1}{2}\tilde{B}q_0})}{\int dz \, P(z)[I_0(z\Xi)]^n I_0(zx\beta\Xi^{-1}\sqrt{\frac{1}{2}\tilde{B}q_0})} \right\} \quad (8)$$

with the two shorthands $\tilde{J} = 1/\mu\tilde{\beta}$, $\Xi = \beta\sqrt{\frac{1}{2}(\tilde{J} + \tilde{B})q_1 - \frac{1}{2}\tilde{B}q_0}$, with $P(x) = xe^{-\frac{1}{2}x^2}\theta[x]$, and where the functions $I_n(x)$ are the modified Bessel functions [8]. Their physical meaning is given by

$$q_0 = \frac{1}{N}\sum_i \langle\overline{\langle S_i\rangle}^2\rangle_B \qquad q_1 = \frac{1}{N}\sum_i \langle\overline{\langle S_i\rangle^2}\rangle_B. \quad (9)$$

It is clear that $0 \leqslant q_0 \leqslant q_1 \leqslant 1$.

We have studied the fixed-point equations (7), (8) after having first eliminated the parameter redundancy by putting $\tilde{B} = 1$ and $\tilde{J} = 3$, which resulted in the phase diagram in the *n-T* plane as shown in figure 1. In addition to a paramagnetic phase (P), where $q_0 = q_1 = 0$, one finds two distinct spin-glass phases: SG1, where $q_1 > 0$ but $q_0 = 0$, and SG2, where both $q_1 > 0$ and $q_0 > 0$. The SG1 phase describes freezing of the spins on the fast timescales only (where spin equilibration occurs); on the large timescales, where coupling equilibration occurs, one finds that, due to the slow motion of the couplings, the frozen spin directions continually change. In the SG2 phase, on the other hand, both spins and couplings freeze, with the net result that even on the large timescales the frozen spin directions are 'pinned'. The SG1–SG2 transition is always second order. The transition SG1–P is second order for $n < 2$ (in which case its location is given by $\tilde{B} + \tilde{J} = 4T^2$), but first order for $n > 2$. When *n* further increases to $n > 3.5$, the SG1 phase disappears, and the system exhibits a first-order transition from P to SG2.

Additional transitions occur, corresponding to a continuous breaking of replica symmetry. The stability of the RS solutions is, as always, expressed in terms of the eigenvalues of the matrix of second derivatives of quadratic fluctuations at the saddle point [6]. We have calculated all eigenvalues and their multiplicity following [3, 6] (full details will be published elsewhere [9]). We find two replicon eigenvalues: λ_G, associated with the Greek replicas and λ_R, associated with the Roman replicas. We have drawn the corresponding AT-lines, where λ_G and λ_R are zero, respectively, in the phase diagram as chain curves. At low spin temperature *T* replica symmetry is broken with respect to the
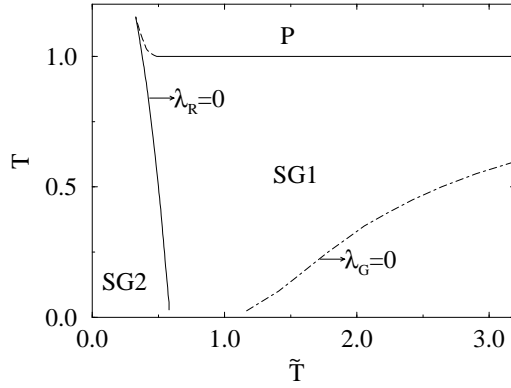
Figure 2. Phase diagram of the XY spin glass with slow dynamic couplings, drawn in the $\tilde{T}$–T plane; for $B_0 = 0$, $\tilde{B} = 1$ and $\tilde{J} = 3$. Further notation as in figure 1.
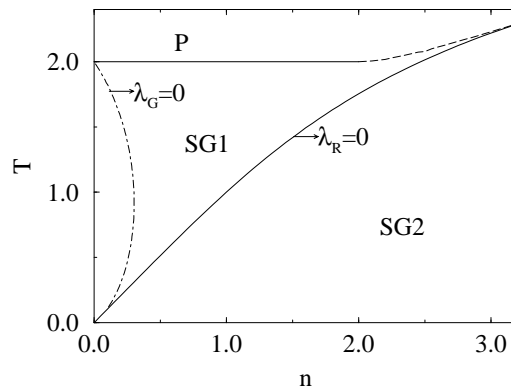

Figure 3. Phase diagram of the Ising spin glass with slow dynamic couplings [1], drawn in the n–T plane; for $B_0 = 0$, $\tilde{B} = 1$ and $\tilde{J} = 3$. Further notation as in figure 1.

Greek replicas ($\lambda_G = 0$), whereas at low coupling temperature $\tilde{T}$ replica symmetry is broken with respect to the Roman replicas ($\lambda_R = 0$). This is illustrated more clearly by drawing the phase diagram in the $\tilde{T}$–T plane, as in figure 2. This second figure also shows that there is no re-entrance from SG1 to RSB, when T is varied for fixed $\tilde{T}$.

Qualitatively the phase diagram of the present system is very similar to that of the Ising spin glass with dynamic couplings as studied in [3], see figure 3, including the behaviour of both the Greek AT-line $\lambda_G = 0$ and the Roman AT-line $\lambda_R = 0$. Here our results differ from, and improve upon, those of [3] (which is why we present figure 3, rather than just refer to [3]). The set of eigenvalues given in [3] turn out to satisfy only part of the relevant orthogonality conditions used in their calculation. The replica symmetric solution in SG1 is always stable with respect to the Roman replicas. In fact, we can show analytically that the Roman AT-line coincides with the SG1–SG2 transition line. Our model, with dynamics on two different timescales, is reminiscent of a simple XY model with one step replica-symmetry breaking (1RSB), and our eigenvalues formally resemble e.g. those describing the stability of the 1RSB solution in the perceptron model of [10].

In conclusion, we have solved a classical XY spin-glass model in which both the spins and their couplings evolve stochastically, according to coupled equations, but on different

timescales. The solution of our model in RS ansatz is mathematically similar to that of the XY model with static couplings, but with 1RSB. Qualitatively, the phase diagram, which exhibits two different spin-glass types and both first- and second-order transitions, resembles closely that of the Ising spin glass with dynamic couplings, provided appropriate adjustment of the calculation of the AT-line in [3] is carried out. Our calculation shows how the methods used for solving the Ising case can be easily adapted to more complicated spin types, and illustrates the robustness of the structure and peculiarities of phase diagrams describing the behaviour of large spin systems with dynamic couplings.

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