

# Statistical Mechanics of Recurrent Neural Networks II – Dynamics

A.C.C. COOLEN

*Department of Mathematics, King's College London Strand, London WC2R 2LS, UK*

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## 1. Introduction

This paper, on solving the dynamics of recurrent neural networks using nonequilibrium statistical mechanical techniques, is the sequel of [1], which was devoted to solving the statics using equilibrium techniques. I refer to [1] for a general introduction to recurrent neural networks and their properties.

*Equilibrium* statistical mechanical techniques can provide much detailed quantitative information on the behavior of recurrent neural networks, but they obviously have serious restrictions. The first one is that, by definition, they will only provide information on network properties in the stationary state. For associative memories, for instance, it is not clear how one can calculate quantities like sizes of domains of attraction without solving the dynamics. The second, and more serious, restriction is that for equilibrium statistical mechanics to apply the dynamics of the network under study must obey detailed balance, i.e. absence of microscopic probability currents in the stationary state. As we have seen in [1], for recurrent networks in which the dynamics take the form of a stochastic alignment of neuronal firing rates to postsynaptic potentials which, in turn, depend linearly on the firing rates, this requirement of detailed balance usually implies symmetry of the synaptic matrix. From a physiological point of view this requirement is clearly unacceptable, since it is violated in any network that obeys Dale's law as soon as an excitatory neuron is connected to an inhibitory one. Worse still, we saw in [1] that in any network of graded-response neurons detailed balance will always be violated, even when the synapses are symmetric. The situation will become even worse when we turn to networks of yet more realistic (spike-based) neurons, such as integrate-and-fire ones. In contrast to this, *nonequilibrium* statistical mechanical techniques, it will turn out, do not impose such biologically nonrealistic restrictions on neuron types and synaptic symmetry, and they are consequently the more appropriate avenue for future theoretical research aimed at solving biologically more realistic models.

The common strategy of all nonequilibrium statistical mechanical studies is to derive and solve dynamical laws for a suitable small set of relevant macroscopic quantities from the dynamical laws of the underlying microscopic neuronal system. In order to make progress, as in equilibrium studies, one is initially forced to pay the price of having relatively simple model neurons, and of not having a very complicated spatial wiring structure in the network under study; the networks described and analyzed in this paper will consequently be either fully connected, or randomly diluted. When attempting to obtain exact dynamical solutions within this class, one then soon finds a clear separation of network models into two distinct complexity classes, reflecting in the dynamics a separation which we also found in the statics. In statics one could get away with relatively simple mathematical techniques as long as the number of attractors of the dynamics was small compared to the number  $N$  of

neurons. As soon as the number of attractors became of the order of  $N$ , on the other hand, one entered the complex regime, requiring the more complicated formalism of replica theory. In dynamics we will again find that we can get away with relatively simple mathematical techniques as long as the number of attractors remains small, and find closed deterministic differential equations for macroscopic quantities with just a single time argument. As soon as we enter the complex regime, however, we will no longer find closed equations for one-time macroscopic objects: we will now have to work with correlation and response functions, which have two time arguments, and turn to the less trivial generating functional techniques.<sup>1</sup>

In contrast to the situation in statics [1], I cannot in this paper give many references to textbooks on the dynamics, since these are more or less nonexistent. There would appear to be two reasons for this. Firstly, in most physics departments nonequilibrium statistical mechanics (as a subject) is generally taught and applied far less intensively than equilibrium statistical mechanics, and thus the nonequilibrium studies of recurrent neural networks have been considerably less in number and later in appearance in literature than their equilibrium counterparts. Secondly, many of the popular textbooks on the statistical mechanics of neural networks were written around 1989, roughly at the point in time where nonequilibrium statistical mechanical studies just started being taken up. When reading such textbooks one could be forgiven for thinking that solving the dynamics of recurrent neural networks is generally ruled out, whereas, in fact, nothing could be further from the truth. Thus the references in this paper will, out of necessity, be mainly to research papers. I regret that, given constraints on page numbers and given my aim to explain ideas and techniques in a lecture notes style (rather than display encyclopedic skills), I will inevitably have left out relevant references. Another consequence of the scarce and scattered nature of the literature on the nonequilibrium statistical mechanics of recurrent neural networks is that a situation has developed where many mathematical procedures, properties and solutions are more or less known by the research community, but without there being a clear reference in literature where these were first formally derived (if at all). Examples of this are the fluctuation–dissipation theorems (FDTs) for parallel dynamics and the nonequilibrium analysis of networks with graded response neurons; often the separating boundary between accepted general knowledge and *published* accepted general knowledge is somewhat fuzzy.

The structure of this paper mirrors more or less the structure of [1]. Again I will start with relatively simple networks, with a small number of attractors (such as systems with uniform synapses, or with a small number of patterns stored with Hebbian-type rules), which can be solved with relatively simple mathematical techniques. These will now also include networks that do not evolve to a stationary

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1 A brief note about terminology: strictly speaking, in this paper we will apply these techniques only to models in which time is measured in discrete units, so that we should speak about generating functions rather than generating functionals. However, since these techniques can and have also been applied intensively to models with continuous time, they are in literature often referred to as generating functional techniques, for both discrete and continuous time.

state, and networks of graded response neurons, which could not be studied within equilibrium statistical mechanics at all. Next follows a detour on correlation and response functions and their relations (i.e. FDTs), which serves as a prerequisite for the last section on generating functional methods, which are indeed formulated in the language of correlation and response functions. In this last, more mathematically involved, section I study symmetric and nonsymmetric attractor neural networks close to saturation, i.e. in the complex regime. I will show how to solve the dynamics of fully connected as well as extremely diluted networks, emphasizing the (again) crucial issue of presence (or absence) of synaptic symmetry, and compare the predictions of the (exact) generating functional formalism to both numerical simulations and simple approximate theories.

## 2. Attractor neural networks with binary neurons

The simplest nontrivial recurrent neural networks consist of  $N$  binary neurons  $\sigma_i \in \{-1, 1\}$  (see [1]) which respond stochastically to postsynaptic potentials (or local fields)  $h_i(\boldsymbol{\sigma})$ , with  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ . The fields depend linearly on the instantaneous neuron states,  $h_i(\boldsymbol{\sigma}) = \sum_j J_{ij}\sigma_j + \theta_i$ , with the  $J_{ij}$  representing synaptic efficacies, and the  $\theta_i$  representing external stimuli and/or neural thresholds.

### 2.1. Closed macroscopic laws for sequential dynamics

First I show how for sequential dynamics (where neurons are updated one after the other) one can calculate, from the microscopic stochastic laws, differential equations for the probability distribution of suitably defined macroscopic observables. For mathematical convenience our starting point will be the continuous-time master equation for the microscopic probability distribution  $p_t(\boldsymbol{\sigma})$

$$\frac{d}{dt}p_t(\boldsymbol{\sigma}) = \sum_i \{w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma}) - w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\}, \quad w_i(\boldsymbol{\sigma}) = \frac{1}{2}[1 - \sigma_i \tanh[\beta h_i(\boldsymbol{\sigma})]] \quad (1)$$

with  $F_i\Phi(\boldsymbol{\sigma}) = \Phi(\sigma_1, \dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots, \sigma_N)$  (see [1]). I will discuss the conditions for the evolution of these macroscopic state variables to become *deterministic* in the limit of infinitely large networks and, in addition, be governed by a *closed* set of equations. I then turn to specific models, with and without detailed balance, and show how the macroscopic equations can be used to illuminate and understand the dynamics of attractor neural networks away from saturation.

#### 2.1.1. A toy model

Let me illustrate the basic ideas with the help of a simple (infinite range) toy model:  $J_{ij} = (J/N)\eta_i\xi_j$  and  $\theta_i = 0$  (the variables  $\eta_i$  and  $\xi_i$  are arbitrary, but may not depend on  $N$ ). For  $\eta_i = \xi_i = 1$  we get a network with uniform synapses. For  $\eta_i = \xi_i \in \{-1, 1\}$  and  $J > 0$  we recover the Hopfield [2] model with one stored pattern. Note: the synaptic matrix is nonsymmetric as soon as a pair  $(ij)$  exists such

that  $\eta_i \xi_j \neq \eta_j \xi_i$ , so in general equilibrium statistical mechanics will not apply. The local fields become  $h_i(\boldsymbol{\sigma}) = J \eta_i m(\boldsymbol{\sigma})$  with  $m(\boldsymbol{\sigma}) = \frac{1}{N} \sum_k \xi_k \sigma_k$ . Since they depend on the microscopic state  $\boldsymbol{\sigma}$  only through the value of  $m$ , the latter quantity appears to constitute a natural macroscopic level of description. The probability density of finding the macroscopic state  $m(\boldsymbol{\sigma}) = m$  is given by  $\mathcal{P}_t[m] = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[m - m(\boldsymbol{\sigma})]$ . Its time derivative follows upon inserting (1):

$$\begin{aligned} \frac{d}{dt} \mathcal{P}_t[m] &= \sum_{\boldsymbol{\sigma}} \sum_{k=1}^N p_t(\boldsymbol{\sigma}) w_k(\boldsymbol{\sigma}) \left\{ \delta \left[ m - m(\boldsymbol{\sigma}) + \frac{2}{N} \xi_k \sigma_k \right] - \delta[m - m(\boldsymbol{\sigma})] \right\} \\ &= \frac{d}{dm} \left\{ \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[m - m(\boldsymbol{\sigma})] \frac{2}{N} \sum_{k=1}^N \xi_k \sigma_k w_k(\boldsymbol{\sigma}) \right\} + \mathcal{O}\left(\frac{1}{N}\right). \end{aligned}$$

Inserting our expressions for the transition rates  $w_i(\boldsymbol{\sigma})$  and the local fields  $h_i(\boldsymbol{\sigma})$  gives:

$$\frac{d}{dt} \mathcal{P}_t[m] = \frac{d}{dm} \left\{ \mathcal{P}_t[m] \left[ m - \frac{1}{N} \sum_{k=1}^N \xi_k \tanh[\eta_k \beta J m] \right] \right\} + \mathcal{O}(N^{-1}).$$

In the limit  $N \rightarrow \infty$  only the first term survives. The general solution of the resulting Liouville equation is  $\mathcal{P}_t[m] = \int dm_0 \mathcal{P}_0[m_0] \delta[m - m(t|m_0)]$ , where  $m(t|m_0)$  is the solution of

$$\frac{d}{dt} m = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \xi_k \tanh[\eta_k \beta J m] - m, \quad m(0) = m_0. \quad (2)$$

This describes deterministic evolution; the only uncertainty in the value of  $m$  is due to uncertainty in initial conditions. If at  $t = 0$  the quantity  $m$  is known exactly, this will remain the case for finite time-scales;  $m$  turns out to evolve in time according to (2).

### 2.1.2. Arbitrary synapses

Let us now allow for less trivial choices of the synaptic matrix  $\{J_{ij}\}$  and try to calculate the evolution in time of a given set of macroscopic observables  $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = (\Omega_1(\boldsymbol{\sigma}), \dots, \Omega_n(\boldsymbol{\sigma}))$  in the limit  $N \rightarrow \infty$ . There are no restrictions yet on the form or the number  $n$  of these state variables; these will, however, arise naturally if we require the observables  $\boldsymbol{\Omega}$  to obey a closed set of deterministic laws, as we will see. The probability density of finding the system in macroscopic state  $\boldsymbol{\Omega}$  is given by:

$$\mathcal{P}_t[\boldsymbol{\Omega}] = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]. \quad (3)$$

Its time derivative is obtained by inserting (1). If in those parts of the resulting expression which contain the operators  $F_i$  we perform the transformations  $\boldsymbol{\sigma} \rightarrow F_i \boldsymbol{\sigma}$ , we arrive at

$$\frac{d}{dt} \mathcal{P}_t[\mathbf{\Omega}] = \sum_i \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) w_i(\boldsymbol{\sigma}) \{ \delta[\mathbf{\Omega} - \mathbf{\Omega}(F_i \boldsymbol{\sigma})] - \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] \}.$$

Upon writing  $\Omega_\mu(F_i \boldsymbol{\sigma}) = \Omega_\mu(\boldsymbol{\sigma}) + \Delta_{j\mu}(\boldsymbol{\sigma})$  and making a Taylor expansion in powers of  $\{\Delta_{j\mu}(\boldsymbol{\sigma})\}$ , we finally obtain the so-called Kramers–Moyal expansion:

$$\frac{d}{dt} \mathcal{P}_t[\mathbf{\Omega}] = \sum_{\ell \geq 1} \frac{(-1)^\ell}{\ell!} \sum_{\mu_1=1}^n \cdots \sum_{\mu_\ell=1}^n \frac{\partial^\ell}{\partial \Omega_{\mu_1} \cdots \partial \Omega_{\mu_\ell}} \left\{ \mathcal{P}_t[\mathbf{\Omega}] F_{\mu_1 \cdots \mu_\ell}^{(\ell)}[\mathbf{\Omega}; t] \right\}. \quad (4)$$

It involves conditional averages  $\langle f(\boldsymbol{\sigma}) \rangle_{\mathbf{\Omega}; t}$  and the ‘discrete derivatives’  $\Delta_{j\mu}(\boldsymbol{\sigma}) = \Omega_\mu(F_j \boldsymbol{\sigma}) - \Omega_\mu(\boldsymbol{\sigma})$ :<sup>2</sup>

$$F_{\mu_1 \cdots \mu_\ell}^{(\ell)}[\mathbf{\Omega}; t] = \left\langle \sum_{j=1}^N w_j(\boldsymbol{\sigma}) \Delta_{j\mu_1}(\boldsymbol{\sigma}) \cdots \Delta_{j\mu_\ell}(\boldsymbol{\sigma}) \right\rangle_{\mathbf{\Omega}; t},$$

$$\langle f(\boldsymbol{\sigma}) \rangle_{\mathbf{\Omega}; t} = \frac{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] f(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})]}. \quad (5)$$

Retaining only the  $\ell = 1$  term in (4) would lead us to a Liouville equation, which describes deterministic flow in  $\mathbf{\Omega}$  space. Including also the  $\ell = 2$  term leads us to a Fokker–Planck equation which, in addition to flow, describes diffusion of the macroscopic probability density. Thus a sufficient condition for the observables  $\mathbf{\Omega}(\boldsymbol{\sigma})$  to evolve in time deterministically in the limit  $N \rightarrow \infty$  is:

$$\lim_{N \rightarrow \infty} \sum_{\ell \geq 2} \frac{1}{\ell!} \sum_{\mu_1=1}^n \cdots \sum_{\mu_\ell=1}^n \sum_{j=1}^N \langle |\Delta_{j\mu_1}(\boldsymbol{\sigma}) \cdots \Delta_{j\mu_\ell}(\boldsymbol{\sigma})| \rangle_{\mathbf{\Omega}; t} = 0. \quad (6)$$

In the simple case where all observables  $\Omega_\mu$  scale similarly in the sense that all ‘derivatives’  $\Delta_{j\mu} = \Omega_\mu(F_j \boldsymbol{\sigma}) - \Omega_\mu(\boldsymbol{\sigma})$  are of the same order in  $N$  (i.e. there is a monotonic function  $\tilde{\Delta}_N$  such that  $\Delta_{j\mu} = \mathcal{O}(\tilde{\Delta}_N)$  for all  $j\mu$ ), for instance, criterion (6) becomes:

$$\lim_{N \rightarrow \infty} n \tilde{\Delta}_N \sqrt{N} = 0. \quad (7)$$

If for a given set of observables condition (6) is satisfied we can for large  $N$  describe the evolution of the macroscopic probability density by a Liouville equation:

$$\frac{d}{dt} \mathcal{P}_t[\mathbf{\Omega}] = - \sum_{\mu=1}^n \frac{\partial}{\partial \Omega_\mu} \left\{ \mathcal{P}_t[\mathbf{\Omega}] F_\mu^{(1)}[\mathbf{\Omega}; t] \right\}$$

2 Expansion (4) is to be interpreted in a distributional sense, i.e. only to be used in expressions of the form  $\int d\mathbf{\Omega} \mathcal{P}_t(\mathbf{\Omega}) G(\mathbf{\Omega})$  with smooth functions  $G(\mathbf{\Omega})$ , so that all derivatives are well-defined and finite. Furthermore, (4) will only be useful if the  $\Delta_{j\mu}$ , which measure the sensitivity of the macroscopic quantities to single neuron state changes, are sufficiently small. This is to be expected: for finite  $N$  any observable can only assume a finite number of possible values; only for  $N \rightarrow \infty$  may we expect smooth probability distributions for our macroscopic quantities.

whose solution describes deterministic flow:  $\mathcal{P}_t[\mathbf{\Omega}] = \int d\mathbf{\Omega}_0 \mathcal{P}_0[\mathbf{\Omega}_0] \delta[\mathbf{\Omega} - \mathbf{\Omega}(t|\mathbf{\Omega}_0)]$  with  $\mathbf{\Omega}(t|\mathbf{\Omega}_0)$  given, in turn, as the solution of

$$\frac{d}{dt}\mathbf{\Omega}(t) = \mathbf{F}^{(1)}[\mathbf{\Omega}(t); t], \quad \mathbf{\Omega}(0) = \mathbf{\Omega}_0. \quad (8)$$

In taking the limit  $N \rightarrow \infty$ , however, we have to keep in mind that the resulting deterministic theory is obtained by taking this limit for *finite*  $t$ . According to (4) the  $\ell > 1$  terms do come into play for sufficiently large times  $t$ ; for  $N \rightarrow \infty$ , however, these times diverge by virtue of (6).

### 2.1.3. The issue of closure

Eq. (8) will in general not be autonomous; tracing back the origin of the explicit time dependence in the right-hand side of (8) one finds that to calculate  $\mathbf{F}^{(1)}$  one needs to know the microscopic probability density  $p_t(\boldsymbol{\sigma})$ . This, in turn, requires solving Eq. (1) (which is exactly what one tries to avoid). We will now discuss a mechanism via which to eliminate the offending explicit time dependence, and to turn the observables  $\mathbf{\Omega}(\boldsymbol{\sigma})$  into an autonomous level of description, governed by *closed* dynamic laws. The idea is to choose the observables  $\mathbf{\Omega}(\boldsymbol{\sigma})$  in such a way that there is no explicit time dependence in the flow field  $\mathbf{F}^{(1)}[\mathbf{\Omega}; t]$  (if possible). According to (5) this implies making sure that there exist functions  $\Phi_\mu[\mathbf{\Omega}]$  such that

$$\lim_{N \rightarrow \infty} \sum_{j=1}^N w_j(\boldsymbol{\sigma}) \Delta_{j\mu}(\boldsymbol{\sigma}) = \Phi_\mu[\mathbf{\Omega}(\boldsymbol{\sigma})] \quad (9)$$

in which case the time dependence of  $\mathbf{F}^{(1)}$  indeed drops out and the macroscopic state vector simply evolves in time according to:

$$\frac{d}{dt}\mathbf{\Omega} = \mathbf{\Phi}[\mathbf{\Omega}], \quad \mathbf{\Phi}[\mathbf{\Omega}] = (\Phi_1[\mathbf{\Omega}], \dots, \Phi_n[\mathbf{\Omega}]).$$

Clearly, for this closure method to apply, a suitable separable structure of the synaptic matrix is required. If, for instance, the macroscopic observables  $\Omega_\mu$  depend linearly on the microscopic state variables  $\boldsymbol{\sigma}$  (i.e.  $\Omega_\mu(\boldsymbol{\sigma}) = \frac{1}{N} \sum_{j=1}^N \omega_{\mu j} \sigma_j$ ), we obtain with the transition rates defined in (1):

$$\frac{d}{dt}\Omega_\mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=1}^N \omega_{\mu j} \tanh(\beta h_j(\boldsymbol{\sigma})) - \Omega_\mu \quad (10)$$

in which case the only further condition for (9) to hold is that all local fields  $h_k(\boldsymbol{\sigma})$  must (in leading order in  $N$ ) depend on the microscopic state  $\boldsymbol{\sigma}$  only through the values of the observables  $\mathbf{\Omega}$ ; since the local fields depend linearly on  $\boldsymbol{\sigma}$  this, in turn, implies that the synaptic matrix must be separable: if  $J_{ij} = \sum_\mu K_{i\mu} \omega_{\mu j}$  then indeed  $h_i(\boldsymbol{\sigma}) = \sum_\mu K_{i\mu} \Omega_\mu(\boldsymbol{\sigma}) + \theta_i$ . Next I will show how this approach can be applied to



networks for which the matrix of synapses has a separable form (which includes most symmetric and nonsymmetric Hebbian type attractor models). I will restrict myself to models with  $\theta_i = 0$ ; introducing nonzero thresholds is straightforward and does not pose new problems.

2.2. Application to separable attractor networks

2.2.1. Separable models: description at the level of sublattice activities

We consider the following class of models, in which the interaction matrix has the form

$$J_{ij} = \frac{1}{N} Q(\xi_i; \xi_j), \quad \xi_i = (\xi_i^1, \dots, \xi_i^p). \tag{11}$$

The components  $\xi_i^\mu$ , representing the information (‘patterns’) to be stored or processed, are assumed to be drawn from a finite discrete set  $\Lambda$ , containing  $n_\Lambda$  elements (they are not allowed to depend on  $N$ ). The Hopfield model [2] corresponds to choosing  $Q(\mathbf{x}; \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$  and  $\Lambda \equiv \{-1, 1\}$ . One now introduces a partition of the system  $\{1, \dots, N\}$  into  $n_\Lambda^p$  so-called sublattices  $I_\mathbf{\eta}$ :

$$I_\mathbf{\eta} = \{i | \xi_i = \mathbf{\eta}\}, \quad \{1, \dots, N\} = \bigcup_{\mathbf{\eta}} I_\mathbf{\eta}, \quad \mathbf{\eta} \in \Lambda^p. \tag{12}$$

The number of neurons in sublattice  $I_\mathbf{\eta}$  is denoted by  $|I_\mathbf{\eta}|$  (this number will have to be large). If we choose as our macroscopic observables the average activities (‘magnetisations’) within these sublattices, we are able to express the local fields  $h_k$  solely in terms of macroscopic quantities:

$$m_\mathbf{\eta}(\boldsymbol{\sigma}) = \frac{1}{|I_\mathbf{\eta}|} \sum_{i \in I_\mathbf{\eta}} \sigma_i, \quad h_k(\boldsymbol{\sigma}) = \sum_{\mathbf{\eta}} p_\mathbf{\eta} Q(\xi_k; \mathbf{\eta}) m_\mathbf{\eta} \tag{13}$$

with the relative sublattice sizes  $p_\mathbf{\eta} = |I_\mathbf{\eta}|/N$ . If all  $p_\mathbf{\eta}$  are of the same order in  $N$  (which, for example, is the case if the vectors  $\xi_i$  have been drawn at random from the set  $\Lambda^p$ ) we may write  $\Delta_{j\mathbf{\eta}} = \mathcal{O}(n_\Lambda^p N^{-1})$  and use (7). The evolution in time of the sublattice activities is then found to be deterministic in the  $N \rightarrow \infty$  limit if  $\lim_{N \rightarrow \infty} p/\log N = 0$ . Furthermore, condition (9) holds, since

$$\sum_{j=1}^N w_j(\boldsymbol{\sigma}) \Delta_{j\mathbf{\eta}}(\boldsymbol{\sigma}) = \tanh \left[ \beta \sum_{\mathbf{\eta}'} p_{\mathbf{\eta}'} Q(\mathbf{\eta}; \mathbf{\eta}') m_{\mathbf{\eta}'} \right] - m_\mathbf{\eta}.$$

We may conclude that the situation is that described by (10), and that the evolution in time of the sublattice activities is governed by the following autonomous set of differential Eqs. [3]:

$$\frac{d}{dt} m_\mathbf{\eta} = \tanh \left[ \beta \sum_{\mathbf{\eta}'} p_{\mathbf{\eta}'} Q(\mathbf{\eta}; \mathbf{\eta}') m_{\mathbf{\eta}'} \right] - m_\mathbf{\eta} \tag{14}$$

We see that, in contrast to the equilibrium techniques as described in [1], here there is no need at all to require symmetry of the interaction matrix or absence of self-interactions. In the symmetric case  $Q(\mathbf{x}; \mathbf{y}) = Q(\mathbf{y}; \mathbf{x})$  the system will approach equilibrium; if the kernel  $Q$  is positive definite this can be shown, for instance, by inspection of the Lyapunov function<sup>3</sup>  $\mathcal{L}\{m_\eta\}$ :

$$\mathcal{L}\{m_\eta\} = \frac{1}{2} \sum_{\eta\eta'} p_\eta m_\eta Q(\eta; \eta') m_{\eta'} p_{\eta'} - \frac{1}{\beta} \sum_{\eta} p_\eta \log \cosh \left[ \beta \sum_{\eta'} Q(\eta; \eta') m_{\eta'} p_{\eta'} \right]$$

which is bounded from below and obeys:

$$\frac{d}{dt} \mathcal{L} = - \sum_{\eta\eta'} \left[ p_\eta \frac{d}{dt} m_\eta \right] Q(\eta; \eta') \left[ p_{\eta'} \frac{d}{dt} m_{\eta'} \right] \leq 0. \tag{15}$$

Note that from the sublattice activities, in turn, follow the ‘overlaps’  $m_\mu(\sigma)$  (see [1]):

$$m_\mu(\sigma) = \frac{1}{N} \sum_{i=1}^N \xi_i^\mu \sigma_i = \sum_{\eta} p_\eta \eta_\mu m_\eta. \tag{16}$$

Simple examples of relevant models of the type (11), the dynamics of which are for large  $N$  described by Eq. (14), are for instance the ones where one applies a non-linear operation  $\Phi$  to the standard Hopfield-type [2] (or Hebbian-type) interactions. This nonlinearity could result from e.g. a clipping procedure or from retaining only the *sign* of the Hebbian values:

$$J_{ij} = \frac{1}{N} \Phi \left( \sum_{\mu \leq p} \xi_i^\mu \xi_j^\mu \right):$$

$$\text{e.g. } \Phi(x) = \begin{cases} -K & \text{for } x \leq -K \\ x & \text{for } -K < x < K \\ K & \text{for } x \geq K \end{cases} \quad \text{or} \quad \Phi(x) = \text{sgn}(x).$$

The effect of introducing such nonlinearities is found to be of a quantitative nature, giving rise to little more than a re-scaling of critical noise levels and storage capacities. I will not go into full details, these can be found in e.g. [4], but illustrate this statement by working out the  $p = 2$  equations for randomly drawn pattern bits  $\xi_i^\mu \in \{-1, 1\}$ , where there are only four sublattices, and where  $p_\eta = \frac{1}{4}$  for all  $\eta$ . Using  $\Phi(0) = 0$  and  $\Phi(-x) = -\Phi(x)$  (as with the above examples) we obtain from (14):

$$\frac{d}{dt} m_\eta = \tanh \left[ \frac{1}{4} \beta \Phi(2)(m_\eta - m_{-\eta}) \right] - m_\eta. \tag{17}$$

Here the choice made for  $\Phi(x)$  shows up only as a rescaling of the temperature. From (17) we further obtain  $\frac{d}{dt} (m_\eta + m_{-\eta}) = -(m_\eta + m_{-\eta})$ . The system decays ex-

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3 A function of the state variables which is bounded from below and whose value decreases monotonically during the dynamics, see e.g. [5]. Its existence guarantees evolution towards a stationary state (under some weak conditions).

ponentially towards a state where, according to (16),  $m_{\mathbf{\eta}} = -m_{-\mathbf{\eta}}$  for all  $\mathbf{\eta}$ . If at  $t = 0$  this is already the case, we find (at least for  $p = 2$ ) decoupled equations for the sublattice activities.

### 2.2.2. Separable models: description at the level of overlaps

Equations (14) and (16) suggest that at the level of overlaps there will be, in turn, closed laws if the kernel  $Q$  is bilinear:<sup>4</sup>  $Q(\mathbf{x}; \mathbf{y}) = \sum_{\mu\nu} x_{\mu} A_{\mu\nu} y_{\nu}$ , or:

$$J_{ij} = \frac{1}{N} \sum_{\mu\nu=1}^p \xi_i^{\mu} A_{\mu\nu} \xi_j^{\nu}, \quad \xi_i = (\xi_i^1, \dots, \xi_i^p). \quad (18)$$

We will see that now the  $\xi_i^{\mu}$  need not be drawn from a finite discrete set (as long as they do not depend on  $N$ ). The Hopfield model corresponds to  $A_{\mu\nu} = \delta_{\mu\nu}$  and  $\xi_i^{\mu} \in \{-1, 1\}$ . The fields  $h_k$  can now be written in terms of the overlaps  $m_{\mu}$ :

$$h_k(\boldsymbol{\sigma}) = \xi_k \cdot \mathbf{A}\mathbf{m}(\boldsymbol{\sigma}), \quad \mathbf{m} = (m_1, \dots, m_p), \quad m_{\mu}(\boldsymbol{\sigma}) = \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu} \sigma_i. \quad (19)$$

For this choice of macroscopic variables we find  $\Delta_{j\mu} = \mathcal{O}(N^{-1})$ , so the evolution of the vector  $\mathbf{m}$  becomes deterministic for  $N \rightarrow \infty$  if, according to (7),  $\lim_{N \rightarrow \infty} p/\sqrt{N} = 0$ . Again (9) holds, since

$$\sum_{j=1}^N w_j(\boldsymbol{\sigma}) \Delta_{j\mu}(\boldsymbol{\sigma}) = \frac{1}{N} \sum_{k=1}^N \xi_k \tanh[\beta \xi_k \cdot \mathbf{A}\mathbf{m}] - \mathbf{m}.$$

Thus the evolution in time of the overlap vector  $\mathbf{m}$  is governed by a closed set of differential equations:

$$\frac{d}{dt} \mathbf{m} = \langle \xi \tanh[\beta \xi \cdot \mathbf{A}\mathbf{m}] \rangle_{\xi} - \mathbf{m}, \quad \langle \Phi(\xi) \rangle_{\xi} = \int d\xi \rho(\xi) \Phi(\xi) \quad (20)$$

with  $\rho(\xi) = \lim_{N \rightarrow \infty} N^{-1} \sum_i \delta[\xi - \xi_i]$ . Symmetry of the synapses is not required. For certain nonsymmetric matrices  $A$  one finds stable limit-cycle solutions of (20). In the symmetric case  $A_{\mu\nu} = A_{\nu\mu}$  the system will approach equilibrium; the Lyapunov function (15) for positive definite matrices  $A$  now becomes:

$$\mathcal{L}\{\mathbf{m}\} = \frac{1}{2} \mathbf{m} \cdot \mathbf{A}\mathbf{m} - \frac{1}{\beta} \langle \log \cosh[\beta \xi \cdot \mathbf{A}\mathbf{m}] \rangle_{\xi}.$$

Fig. 1 shows in the  $m_1, m_2$ -plane the result of solving the macroscopic laws (20) numerically for  $p = 2$ , randomly drawn pattern bits  $\xi_i^{\mu} \in \{-1, 1\}$ , and two choices of the matrix  $A$ . The first choice (upper row) corresponds to the Hopfield model; as the noise level  $T = \beta^{-1}$  increases the amplitudes of the four attractors (corresponding to the two patterns  $\xi^{\mu}$  and their mirror images  $-\xi^{\mu}$ ) continuously decrease, until at the

<sup>4</sup> Strictly speaking, it is already sufficient to have a kernel which is linear in  $\mathbf{y}$  only, i.e.  $Q(\mathbf{x}; \mathbf{y}) = \sum_{\nu} f_{\nu}(\mathbf{x}) y_{\nu}$ .

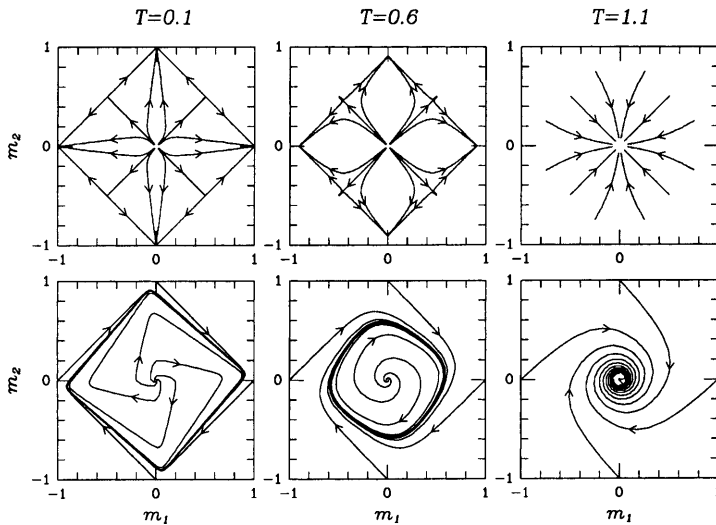


Fig. 1. Flow diagrams obtained by numerically solving Eq. (20) for  $p = 2$ . Upper row:  $A_{\mu\nu} = \delta_{\mu\nu}$  (the Hopfield model); lower row:  $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$  (here the critical noise level is  $T_c = 1$ ).

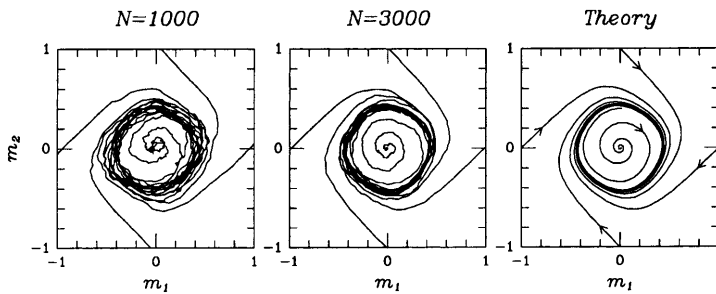


Fig. 2. Comparison between simulation results for finite systems ( $N = 1000$  and  $N = 3000$ ) and the  $N = \infty$  analytical prediction (20), for  $p = 2$ ,  $T = 0.8$  and  $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ .

critical noise level  $T_c = 1$  (see also [1]) they merge into the trivial attractor  $\mathbf{m} = (0, 0)$ . The second choice corresponds to a nonsymmetric model (i.e. without detailed balance); at the macroscopic level of description (at finite time scales) the system clearly does not approach equilibrium; macroscopic order now manifests itself in the form of a limit-cycle (provided the noise level  $T$  is below the critical value  $T_c = 1$  where this limit-cycle is destroyed). To what extent the laws (20) are in agreement with the result of performing the actual simulations in finite systems is illustrated in Fig. 2. Other examples can be found in [6,7].

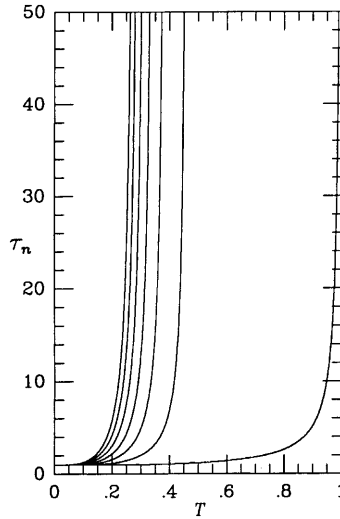


Fig. 3. Asymptotic relaxation times  $\tau_n$  of the mixture states of the Hopfield model as a function of the noise level  $T = \beta^{-1}$ . From bottom to top:  $n = 1, 3, 5, 7, 9, 11, 13$ .

As a second simple application of the flow Eq. (20) we turn to the relaxation times corresponding to the attractors of the Hopfield model (where  $A_{\mu\nu} = \delta_{\mu\nu}$ ). Expanding (20) near a stable fixed-point  $\mathbf{m}^*$ , i.e.  $\mathbf{m}(t) = \mathbf{m}^* + \mathbf{x}(t)$  with  $|\mathbf{x}(t)| \ll 1$ , gives the linearized equation

$$\frac{d}{dt}x_\mu = \sum_\nu \left[ \beta \langle \xi_\mu \xi_\nu \tanh[\beta \xi \cdot \mathbf{m}^*] \rangle_\xi - \delta_{\mu\nu} \right] x_\nu + \mathcal{O}(\mathbf{x}^2). \quad (21)$$

The Jacobian of (20), which determines the linearized Eq. (21), turns out to be *minus* the curvature matrix of the free energy surface at the fixed-point (c.f. the derivations in [1]). The asymptotic relaxation towards any stable attractor is generally exponential, with a characteristic time  $\tau$  given by the inverse of the smallest eigenvalue of the curvature matrix. If, in particular, for the fixed point  $\mathbf{m}^*$  we substitute an  $n$ -mixture state, i.e.  $m_\mu = m_n$  ( $\mu \leq n$ ) and  $m_\mu = 0$  ( $\mu > n$ ), and transform (21) to the basis where the corresponding curvature matrix  $\mathbf{D}^{(n)}$  (with eigenvalues  $D_\lambda^n$ ) is diagonal,  $\mathbf{x} \rightarrow \tilde{\mathbf{x}}$ , we obtain

$$\tilde{x}_\lambda(t) = \tilde{x}_\lambda(0)e^{-tD_\lambda^n} + \dots$$

so  $\tau^{-1} = \min_\lambda D_\lambda^n$ , which we have already calculated (see [1]) in determining the character of the saddle-points of the free-energy surface. The result is shown in Fig. 3. The relaxation time for the  $n$ -mixture attractors decreases monotonically with the degree of mixing  $n$ , for any noise level. At the transition where a macroscopic state  $\mathbf{m}^*$  ceases to correspond to a local minimum of the free energy surface, it also destabilizes in terms of the linearized dynamic Eq. (21) (as it should). The Jacobian develops a zero eigenvalue, the relaxation time diverges, and the long-time behavior is no longer

obtained from the linearized equation. This gives rise to critical slowing down (power law relaxation as opposed to exponential relaxation). For instance, at the transition temperature  $T_c = 1$  for the  $n = 1$  (pure) state, we find by expanding (20):

$$\frac{d}{dt} m_\mu = m_\mu \left[ \frac{2}{3} m_\mu^2 - \mathbf{m}^2 \right] + \mathcal{O}(\mathbf{m}^5)$$

which gives rise to a relaxation towards the trivial fixed-point of the form  $\mathbf{m} \sim t^{-\frac{1}{2}}$ .

If one is willing to restrict oneself to the limited class of models (18) (as opposed to the more general class (11)) and to the more global level of description in terms of  $p$  overlap parameters  $m_\mu$  instead of  $n_\Lambda^p$  sublattice activities  $m_\eta$ , then there are two rewards. Firstly there will be no restrictions on the stored pattern components  $\xi_i^\mu$  (for instance, they are allowed to be real-valued); secondly the number  $p$  of patterns stored can be much larger for the deterministic autonomous dynamical laws to hold ( $p \ll \sqrt{N}$  instead of  $p \ll \log N$ , which from a biological point of view is not impressive).

### 2.3. Closed macroscopic laws for parallel dynamics

We now turn to the parallel dynamics counterpart of (1), i.e. the Markov chain

$$p_{\ell+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_\ell(\boldsymbol{\sigma}') \quad W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] = \prod_{i=1}^N \frac{1}{2} [1 + \sigma_i \tanh[\beta h_i(\boldsymbol{\sigma}')] ] \quad (22)$$

(with  $\sigma_i \in \{-1, 1\}$ , and with local fields  $h_i(\boldsymbol{\sigma})$  defined in the usual way). The evolution of macroscopic probability densities will here be described by discrete mappings, instead of differential equations.

#### 2.3.1 The toy model

Let us first see what happens to our previous toy model:  $J_{ij} = (J/N) \eta_i \xi_j$  and  $\theta_i = 0$ . As before we try to describe the dynamics at the (macroscopic) level of the quantity  $m(\boldsymbol{\sigma}) = \frac{1}{N} \sum_k \xi_k \sigma_k$ . The evolution of the macroscopic probability density  $\mathcal{P}_t[m]$  is obtained by inserting (22):

$$\mathcal{P}_{t+1}[m] = \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \delta[m - m(\boldsymbol{\sigma})] W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_t(\boldsymbol{\sigma}') = \int dm' \tilde{W}_t[m, m'] \mathcal{P}_t[m'] \quad (23)$$

with

$$\tilde{W}_t[m, m'] = \frac{\sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \delta[m - m(\boldsymbol{\sigma})] \delta[m' - m(\boldsymbol{\sigma}')] W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_t(\boldsymbol{\sigma}')}{\sum_{\boldsymbol{\sigma}'} \delta[m' - m(\boldsymbol{\sigma}')] p_t(\boldsymbol{\sigma}')}$$

We now insert our expression for the transition probabilities  $W[\boldsymbol{\sigma}; \boldsymbol{\sigma}']$  and for the local fields. Since the fields depend on the microscopic state  $\boldsymbol{\sigma}$  only through  $m(\boldsymbol{\sigma})$ , the distribution  $p_t(\boldsymbol{\sigma}')$  drops out of the above expression for  $\tilde{W}_t$  which thereby loses its explicit time dependence,  $\tilde{W}_t[m, m'] \rightarrow \tilde{W}[m, m']$ :

$$\tilde{W}[m, m'] = e^{-\sum_i \log \cosh(\beta J m' \eta_i)} \left\langle \delta[m - m(\boldsymbol{\sigma})] e^{\beta J m' \sum_i \eta_i \sigma_i} \right\rangle_{\boldsymbol{\sigma}}$$

with  $\langle \dots \rangle_{\boldsymbol{\sigma}} = 2^{-N} \sum_{\boldsymbol{\sigma}} \dots$

Inserting the integral representation for the  $\delta$ -function allows us to perform the average:

$$\tilde{W}[m, m'] = \left[ \frac{\beta N}{2\pi} \right] \int dk e^{N\Psi(m, m', k)},$$

$$\Psi = i\beta km + \langle \log \cosh \beta[J\eta m' - ik\xi] \rangle_{\eta, \xi} - \langle \log \cosh \beta[J\eta m'] \rangle_{\eta}.$$

Since  $\tilde{W}[m, m']$  is (by construction) normalized,  $\int dm \tilde{W}[m, m'] = 1$ , we find that for  $N \rightarrow \infty$  the expectation value with respect to  $\tilde{W}[m, m']$  of any sufficiently smooth function  $f(m)$  will be determined only by the value  $m^*(m')$  of  $m$  in the relevant saddle-point of  $\Psi$ :

$$\int dm f(m) \tilde{W}[m, m'] = \frac{\int dm dk f(m) e^{N\Psi(m, m', k)}}{\int dm dk e^{N\Psi(m, m', k)}} \rightarrow f(m^*(m')) \quad (N \rightarrow \infty).$$

Variation of  $\Psi$  with respect to  $k$  and  $m$  gives the two saddle-point equations:

$$m = \langle \xi \tanh \beta[J\eta m' - \xi k] \rangle_{\eta, \xi}, \quad k = 0.$$

We may now conclude that  $\lim_{N \rightarrow \infty} \tilde{W}[m, m'] = \delta[m - m^*(m')]$  with  $m^*(m') = \langle \xi \tanh(\beta J \eta m') \rangle_{\eta, \xi}$ , and that the macroscopic Eq. (23) becomes:

$$\mathcal{P}_{t+1}[m] = \int dm' \delta[m - \langle \xi \tanh(\beta J \eta m') \rangle_{\eta, \xi}] \mathcal{P}_t[m'] \quad (N \rightarrow \infty).$$

This describes deterministic evolution. If at  $t = 0$  we know  $m$  exactly, this will remain the case for finite time scales, and  $m$  will evolve according to a discrete version of the sequential dynamics law (2):

$$m_{t+1} = \langle \xi \tanh[\beta J \eta m_t] \rangle_{\eta, \xi} \quad (24)$$

### 2.3.2. Arbitrary synapses

We now try to generalize the above approach to less trivial classes of models. As for the sequential case we will find in the limit  $N \rightarrow \infty$  closed deterministic evolution equations for a more general set of intensive macroscopic state variables  $\mathbf{\Omega}(\boldsymbol{\sigma}) = \Omega_1(\boldsymbol{\sigma}), \dots, \Omega_n(\boldsymbol{\sigma})$  if the local fields  $h_i(\boldsymbol{\sigma})$  depend on the microscopic state  $\boldsymbol{\sigma}$  only through the values of  $\mathbf{\Omega}(\boldsymbol{\sigma})$ , and if the number  $n$  of these state variables necessary to do so is not too large. The evolution of the ensemble probability density (3) is now obtained by inserting the Markov Eq. (22):

$$\mathcal{P}_{t+1}[\mathbf{\Omega}] = \int d\mathbf{\Omega}' \tilde{W}_t[\mathbf{\Omega}, \mathbf{\Omega}'] \mathcal{P}_t[\mathbf{\Omega}'] \quad (25)$$

$$\begin{aligned} \tilde{W}_t[\mathbf{\Omega}, \mathbf{\Omega}'] &= \frac{\sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] \delta[\mathbf{\Omega}' - \mathbf{\Omega}(\boldsymbol{\sigma}')] W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_t(\boldsymbol{\sigma}')}{\sum_{\boldsymbol{\sigma}'} \delta[\mathbf{\Omega}' - \mathbf{\Omega}(\boldsymbol{\sigma}')] p_t(\boldsymbol{\sigma}')} \\ &= \langle \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] \langle e^{\sum_i [\beta \sigma_i h_i(\boldsymbol{\sigma}') - \log \cosh(\beta h_i(\boldsymbol{\sigma}'))]} \rangle_{\mathbf{\Omega}'; t} \rangle_{\boldsymbol{\sigma}} \end{aligned} \quad (26)$$

with  $\langle \dots \rangle_{\sigma} = 2^{-N} \sum_{\sigma} \dots$ , and with the conditional (or sub-shell) average defined as in (5). It is clear from (26) that in order to find autonomous macroscopic laws, i.e. for the distribution  $p_t(\sigma)$  to drop out, the local fields must depend on the microscopic state  $\sigma$  only through the macroscopic quantities  $\mathbf{\Omega}(\sigma)$ :  $h_i(\sigma) = h_i[\mathbf{\Omega}(\sigma)]$ . In this case  $\tilde{W}_t$  loses its explicit time dependence,  $\tilde{W}_t[\mathbf{\Omega}, \mathbf{\Omega}'] \rightarrow \tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}']$ . Inserting integral representations for the  $\delta$ -functions leads to:

$$\tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}'] = \left[ \frac{\beta N}{2\pi} \right]^n \int d\mathbf{K} e^{N\Psi(\mathbf{\Omega}, \mathbf{\Omega}', \mathbf{K})},$$

$$\Psi = i\beta \mathbf{K} \cdot \mathbf{\Omega} + \frac{1}{N} \log \left\langle e^{\beta [\sum_i \sigma_i h_i[\mathbf{\Omega}'] - iN\mathbf{K} \cdot \mathbf{\Omega}(\sigma)]} \right\rangle_{\sigma} - \frac{1}{N} \sum_i \log \cosh[\beta h_i[\mathbf{\Omega}']].$$

Using the normalization  $\int d\mathbf{\Omega} \tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}'] = 1$ , we can write expectation values with respect to  $\tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}']$  of macroscopic quantities  $f[\mathbf{\Omega}]$  as

$$\int d\mathbf{\Omega} f[\mathbf{\Omega}] \tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}'] = \frac{\int d\mathbf{\Omega} d\mathbf{K} f[\mathbf{\Omega}] e^{N\Psi(\mathbf{\Omega}, \mathbf{\Omega}', \mathbf{K})}}{\int d\mathbf{\Omega} d\mathbf{K} e^{N\Psi(\mathbf{\Omega}, \mathbf{\Omega}', \mathbf{K})}}. \quad (27)$$

For saddle-point arguments to apply in determining the leading order in  $N$  of (27), we encounter restrictions on the number  $n$  of our macroscopic quantities (as expected), since  $n$  determines the dimension of the integrations in (27). The restrictions can be found by expanding  $\Psi$  around its maximum  $\Psi^*$ . After defining  $\mathbf{x} = (\mathbf{\Omega}, \mathbf{K})$ , of dimension  $2n$ , and after translating the location of the maximum to the origin, one has

$$\Psi(\mathbf{x}) = \Psi^* - \frac{1}{2} \sum_{\mu\nu} x_{\mu} x_{\nu} H_{\mu\nu} + \sum_{\mu\nu\rho} x_{\mu} x_{\nu} x_{\rho} L_{\mu\nu\rho} + \mathcal{O}(\mathbf{x}^4)$$

giving

$$\begin{aligned} & \frac{\int d\mathbf{x} g(\mathbf{x}) e^{N\Psi(\mathbf{x})}}{\int d\mathbf{x} e^{N\Psi(\mathbf{x})}} - g(\mathbf{0}) \\ &= \frac{\int d\mathbf{x} [g(\mathbf{x}) - g(\mathbf{0})] \exp(-\frac{1}{2}N\mathbf{x} \cdot \mathbf{H}\mathbf{x} + N \sum_{\mu\nu\rho} x_{\mu} x_{\nu} x_{\rho} L_{\mu\nu\rho} + \mathcal{O}(N\mathbf{x}^4))}{\int d\mathbf{x} \exp(-\frac{1}{2}N\mathbf{x} \cdot \mathbf{H}\mathbf{x} + N \sum_{\mu\nu\rho} x_{\mu} x_{\nu} x_{\rho} L_{\mu\nu\rho} + \mathcal{O}(N\mathbf{x}^4))} \\ &= \frac{\int d\mathbf{y} [g(\mathbf{y}/\sqrt{N}) - g(\mathbf{0})] \exp(-\frac{1}{2}\mathbf{y} \cdot \mathbf{H}\mathbf{y} + \sum_{\mu\nu\rho} y_{\mu} y_{\nu} y_{\rho} L_{\mu\nu\rho} / \sqrt{N} + \mathcal{O}(\mathbf{y}^4/N))}{\int d\mathbf{y} \exp(-\frac{1}{2}\mathbf{y} \cdot \mathbf{H}\mathbf{y} + \sum_{\mu\nu\rho} y_{\mu} y_{\nu} y_{\rho} L_{\mu\nu\rho} / \sqrt{N} + \mathcal{O}(\mathbf{y}^4/N))} \\ &= \frac{\int d\mathbf{y} [N^{-\frac{1}{2}}\mathbf{y} \cdot \nabla g(\mathbf{0}) + \mathcal{O}(\mathbf{y}^2/N)] \exp(-\frac{1}{2}\mathbf{y} \cdot \mathbf{H}\mathbf{y}) \left[ 1 + \sum_{\mu\nu\rho} y_{\mu} y_{\nu} y_{\rho} L_{\mu\nu\rho} / \sqrt{N} + \mathcal{O}(\mathbf{y}^6/N) \right]}{\int d\mathbf{y} \exp(-\frac{1}{2}\mathbf{y} \cdot \mathbf{H}\mathbf{y}) \left[ 1 + \sum_{\mu\nu\rho} y_{\mu} y_{\nu} y_{\rho} L_{\mu\nu\rho} / \sqrt{N} + \mathcal{O}(\mathbf{y}^6/N) \right]} \\ &= \mathcal{O}(n^2/N) + \mathcal{O}(n^4/N^2) + \text{nondominant terms, } (N, n \rightarrow \infty) \end{aligned}$$

with  $\mathbf{H}$  denoting the Hessian (curvature) matrix of the surface  $\Psi$  at the minimum  $\Psi^*$ . We thus find



$$\lim_{N \rightarrow \infty} n/\sqrt{N} = 0 : \quad \lim_{N \rightarrow \infty} \int d\mathbf{\Omega} f[\mathbf{\Omega}] \tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}'] = f[\mathbf{\Omega}^*(\mathbf{\Omega}')],$$

where  $\mathbf{\Omega}^*(\mathbf{\Omega}')$  denotes the value of  $\mathbf{\Omega}$  in the saddle-point where  $\Psi$  is minimized. Variation of  $\Psi$  with respect to  $\mathbf{\Omega}$  and  $\mathbf{K}$  gives the saddle-point equations:

$$\mathbf{\Omega} = \frac{\langle \mathbf{\Omega}(\boldsymbol{\sigma}) e^{\beta [\sum_i \sigma_i h_i[\mathbf{\Omega}'] - iN\mathbf{K} \cdot \mathbf{\Omega}(\boldsymbol{\sigma})]} \rangle_{\boldsymbol{\sigma}}}{\langle e^{\beta [\sum_i \sigma_i h_i[\mathbf{\Omega}'] - iN\mathbf{K} \cdot \mathbf{\Omega}(\boldsymbol{\sigma})]} \rangle_{\boldsymbol{\sigma}}}, \quad \mathbf{K} = 0.$$

We may now conclude that  $\lim_{N \rightarrow \infty} \tilde{W}[\mathbf{\Omega}, \mathbf{\Omega}'] = \delta[\mathbf{\Omega} - \mathbf{\Omega}^*(\mathbf{\Omega}')]$ , with

$$\mathbf{\Omega}^*(\mathbf{\Omega}') = \frac{\langle \mathbf{\Omega}(\boldsymbol{\sigma}) e^{\beta \sum_i \sigma_i h_i[\mathbf{\Omega}']} \rangle_{\boldsymbol{\sigma}}}{\langle e^{\beta \sum_i \sigma_i h_i[\mathbf{\Omega}']} \rangle_{\boldsymbol{\sigma}}}$$

and that for  $N \rightarrow \infty$  the macroscopic Eq. (25) becomes  $\mathcal{P}_{t+1}[\mathbf{\Omega}] = \int d\mathbf{\Omega}' \delta[\mathbf{\Omega} - \mathbf{\Omega}^*(\mathbf{\Omega}')] \mathcal{P}_t[\mathbf{\Omega}']$ . This relation again describes deterministic evolution. If at  $t = 0$  we know  $\mathbf{\Omega}$  exactly, this will remain the case for finite time scales and  $\mathbf{\Omega}$  will evolve according to

$$\mathbf{\Omega}(t+1) = \frac{\langle \mathbf{\Omega}(\boldsymbol{\sigma}) e^{\beta \sum_i \sigma_i h_i[\mathbf{\Omega}(t)]} \rangle_{\boldsymbol{\sigma}}}{\langle e^{\beta \sum_i \sigma_i h_i[\mathbf{\Omega}(t)]} \rangle_{\boldsymbol{\sigma}}}. \quad (28)$$

As with the sequential case, in taking the limit  $N \rightarrow \infty$  we have to keep in mind that the resulting laws apply to finite  $t$ , and that for sufficiently large times terms of higher order in  $N$  do come into play. As for the sequential case, a more rigorous and tedious analysis shows that the restriction  $n/\sqrt{N} \rightarrow 0$  can in fact be weakened to  $n/N \rightarrow 0$ . Finally, for macroscopic quantities  $\mathbf{\Omega}(\boldsymbol{\sigma})$  which are linear in  $\boldsymbol{\sigma}$ , the remaining  $\boldsymbol{\sigma}$ -averages become trivial, so that [8]:

$$\Omega_{\mu}(\boldsymbol{\sigma}) = \frac{1}{N} \sum_i \omega_{\mu i} \sigma_i : \quad \Omega_{\mu}(t+1) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \omega_{\mu i} \tanh[\beta h_i[\mathbf{\Omega}(t)]] \quad (29)$$

(to be compared with (10), as derived for sequential dynamics).

## 2.4. Application to separable attractor networks

### 2.4.1. Separable models: sublattice activities and overlaps

The separable attractor models (11), described at the level of sublattice activities (13), indeed have the property that all local fields can be written in terms of the macroscopic observables. What remains to ensure deterministic evolution is meeting the condition on the number of sublattices. If all relative sublattice sizes  $p_{\mathfrak{n}}$  are of the same order in  $N$  (as for randomly drawn patterns) this condition again translates into  $\lim_{N \rightarrow \infty} p/\log N = 0$  (as for sequential dynamics). Since the sublattice activities are linear functions of the  $\sigma_i$ , their evolution in time is governed by Eq. (29), which acquires the form:

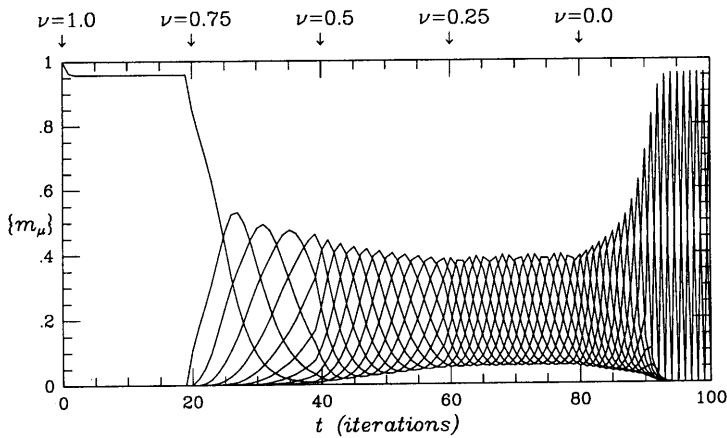


Fig. 4. Evolution of overlaps  $m_\mu(\boldsymbol{\sigma})$ , obtained by numerical iteration of the macroscopic parallel dynamics laws (31), for the synapses  $J_{ij} = \frac{\nu}{N} \sum_\mu \xi_i^\mu \xi_j^\mu + \frac{1-\nu}{N} \sum_\mu \xi_i^{\mu+1} \xi_j^\mu$ , with  $p = 10$  and  $T = 0.5$ .

$$m_{\boldsymbol{\eta}}(t+1) = \tanh \left[ \beta \sum_{\boldsymbol{\eta}'} p_{\boldsymbol{\eta}'} Q(\boldsymbol{\eta}; \boldsymbol{\eta}') m_{\boldsymbol{\eta}'}(t) \right]. \quad (30)$$

As for sequential dynamics, symmetry of the interaction matrix does not play a role.

At the more global level of overlaps  $m_\mu(\boldsymbol{\sigma}) = N^{-1} \sum_i \xi_i^\mu \sigma_i$  we, in turn, obtain autonomous deterministic laws if the local fields  $h_i(\boldsymbol{\sigma})$  can be expressed in terms of  $\mathbf{m}(\boldsymbol{\sigma})$  only, as for the models (18) (or, more generally, for all models in which the interactions are of the form  $J_{ij} = \sum_{\mu \leq p} f_{i\mu} \xi_j^\mu$ ), and with the following restriction on the number  $p$  of embedded patterns:  $\lim_{N \rightarrow \infty} p/\sqrt{N} = 0$  (as with sequential dynamics). For the bilinear models (18), the evolution in time of the overlap vector  $\mathbf{m}$  (which depends linearly on the  $\sigma_i$ ) is governed by (29), which now translates into the iterative map:

$$\mathbf{m}(t+1) = \langle \boldsymbol{\xi} \tanh[\beta \boldsymbol{\xi} \cdot \mathbf{A} \mathbf{m}(t)] \rangle_{\boldsymbol{\xi}} \quad (31)$$

with  $\rho(\boldsymbol{\xi})$  as defined in (20). Again symmetry of the synapses is not required. For parallel dynamics it is far more difficult than for sequential dynamics to construct Lyapunov functions, and prove that the macroscopic laws (31) for symmetric systems evolve towards a stable fixed-point (as one would expect), but it can still be done. For nonsymmetric systems the macroscopic laws (31) can in principle display all the interesting, but complicated, phenomena of nonconservative nonlinear systems. Nevertheless, it is also not uncommon that the Eq. (31) for nonsymmetric systems can be mapped by a time-dependent transformation onto the equations for related symmetric systems (mostly variants of the original Hopfield model).

As an example we show in Fig. 4 as functions of time the values of the overlaps  $\{m_\mu\}$  for  $p = 10$  and  $T = 0.5$ , resulting from numerical iteration of the macroscopic laws (31) for the model

$$J_{ij} = \frac{\nu}{N} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} + \frac{1-\nu}{N} \sum_{\mu} \xi_i^{\mu+1} \xi_j^{\mu} \quad (\mu : \text{mod } p)$$

i.e.  $A_{\lambda\rho} = \nu\delta_{\lambda\rho} + (1-\nu)\delta_{\lambda,\rho+1}$  ( $\lambda, \rho : \text{mod } p$ ), with randomly drawn pattern bits  $\xi_i^{\mu} \in \{-1, 1\}$ . The initial state is chosen to be the pure state  $m_{\mu} = \delta_{\mu,1}$ . At intervals of  $\Delta t = 20$  iterations the parameter  $\nu$  is reduced in  $\Delta\nu = 0.25$  steps from  $\nu = 1$  (where one recovers the symmetric Hopfield model) to  $\nu = 0$  (where one obtains a non-symmetric model which processes the  $p$  embedded patterns in strict sequential order as a period- $p$  limit-cycle). The analysis of Eq. (31) for the pure sequence processing case  $\nu = 0$  is greatly simplified by mapping the model onto the ordinary ( $\nu = 1$ ) Hopfield model, using the index permutation symmetries of the present pattern distribution, as follows (all pattern indices are periodic, mod  $p$ ). Define  $m_{\mu}(t) = M_{\mu-t}(t)$ , now

$$M_{\mu}(t+1) = \left\langle \xi_{\mu+t+1} \tanh \left[ \beta \sum_{\rho} \xi_{\rho+1} M_{\rho-t}(t) \right] \right\rangle_{\xi} = \langle \xi_{\mu} \tanh[\beta \xi \cdot \mathbf{M}(t)] \rangle_{\xi}.$$

We can now immediately infer, in particular, that to each stable macroscopic fixed-point attractor of the original Hopfield model corresponds a stable period- $p$  macroscopic limit-cycle attractor in the  $\nu = 1$  sequence processing model (e.g. pure states  $\leftrightarrow$  pure sequences, mixture states  $\leftrightarrow$  mixture sequences), with identical amplitude as a function of the noise level. Fig. 4 shows for  $\nu = 0$  (i.e.  $t > 80$ ) a relaxation towards such a pure sequence.

Finally we note that the fixed-points of the macroscopic Eqs. (14) and (20) (derived for sequential dynamics) are identical to those of (30) and (31) (derived for parallel dynamics). The stability properties of these fixed points, however, need not be the same, and have to be assessed on a case-by-case basis. For the Hopfield model, i.e. Eqs. (20) and (31) with  $A_{\mu\nu} = \delta_{\mu\nu}$ , they are found to be the same, but already for  $A_{\mu\nu} = -\delta_{\mu\nu}$  the two types of dynamics would behave differently.

### 3. Attractor neural networks with continuous neurons

#### 3.1. Closed macroscopic laws

##### 3.1.1. General derivation

We have seen in [1] that models of recurrent neural networks with continuous neural variables (e.g. graded response neurons or coupled oscillators) can often be described by a Fokker–Planck equation for the microscopic state probability density  $p_t(\boldsymbol{\sigma})$ :

$$\frac{d}{dt} p_t(\boldsymbol{\sigma}) = - \sum_i \frac{\partial}{\partial \sigma_i} [p_t(\boldsymbol{\sigma}) f_i(\boldsymbol{\sigma})] + T \sum_i \frac{\partial^2}{\partial \sigma_i^2} p_t(\boldsymbol{\sigma}). \quad (32)$$

Averages over  $p_t(\boldsymbol{\sigma})$  are denoted by  $\langle G \rangle = \int d\boldsymbol{\sigma} p_t(\boldsymbol{\sigma}) G(\boldsymbol{\sigma}, t)$ . From (32) one obtains directly (through integration by parts) an equation for the time derivative of averages:

$$\frac{d}{dt} \langle G \rangle = \left\langle \frac{\partial G}{\partial t} \right\rangle + \left\langle \sum_i \left[ f_i(\boldsymbol{\sigma}) + T \frac{\partial}{\partial \sigma_i} \right] \frac{\partial G}{\partial \sigma_i} \right\rangle \quad (33)$$

In particular, if we apply (33) to  $G(\boldsymbol{\sigma}, t) = \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]$ , for any set of macroscopic observables  $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = (\Omega_1(\boldsymbol{\sigma}), \dots, \Omega_n(\boldsymbol{\sigma}))$  (in the spirit of Section 2), we obtain a dynamic equation for the macroscopic probability density  $P_t(\boldsymbol{\Omega}) = \langle \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})] \rangle$ , which is again of the Fokker–Planck form:

$$\begin{aligned} \frac{d}{dt} P_t(\boldsymbol{\Omega}) = & - \sum_{\mu} \frac{\partial}{\partial \Omega_{\mu}} \left\{ P_t(\boldsymbol{\Omega}) \left\langle \sum_i \left[ f_i(\boldsymbol{\sigma}) + T \frac{\partial}{\partial \sigma_i} \right] \frac{\partial}{\partial \sigma_i} \Omega_{\mu}(\boldsymbol{\sigma}) \right\rangle_{\boldsymbol{\Omega}, t} \right\} \\ & + T \sum_{\mu\nu} \frac{\partial^2}{\partial \Omega_{\mu} \partial \Omega_{\nu}} \left\{ P_t(\boldsymbol{\Omega}) \left\langle \sum_i \left[ \frac{\partial}{\partial \sigma_i} \Omega_{\mu}(\boldsymbol{\sigma}) \right] \left[ \frac{\partial}{\partial \sigma_i} \Omega_{\nu}(\boldsymbol{\sigma}) \right] \right\rangle_{\boldsymbol{\Omega}, t} \right\} \end{aligned} \quad (34)$$

with the conditional (or subshell) averages:

$$\langle G(\boldsymbol{\sigma}) \rangle_{\boldsymbol{\Omega}, t} = \frac{\int d\boldsymbol{\sigma} p_t(\boldsymbol{\sigma}) \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})] G(\boldsymbol{\sigma})}{\int d\boldsymbol{\sigma} p_t(\boldsymbol{\sigma}) \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]}. \quad (35)$$

From (34) we infer that a sufficient condition for the observables  $\boldsymbol{\Omega}(\boldsymbol{\sigma})$  to evolve in time deterministically (i.e. for having vanishing diffusion matrix elements in (34)) in the limit  $N \rightarrow \infty$  is

$$\lim_{N \rightarrow \infty} \left\langle \sum_i \left[ \sum_{\mu} \left| \frac{\partial}{\partial \sigma_i} \Omega_{\mu}(\boldsymbol{\sigma}) \right| \right]^2 \right\rangle_{\boldsymbol{\Omega}, t} = 0. \quad (36)$$

If (36) holds, the macroscopic Fokker–Planck Eq. (34) reduces for  $N \rightarrow \infty$  to a Liouville equation, and the observables  $\boldsymbol{\Omega}(\boldsymbol{\sigma})$  will evolve in time according to the coupled deterministic equations:

$$\frac{d}{dt} \Omega_{\mu} = \lim_{N \rightarrow \infty} \left\langle \sum_i \left[ f_i(\boldsymbol{\sigma}) + T \frac{\partial}{\partial \sigma_i} \right] \frac{\partial}{\partial \sigma_i} \Omega_{\mu}(\boldsymbol{\sigma}) \right\rangle_{\boldsymbol{\Omega}, t}. \quad (37)$$

The deterministic macroscopic Eq. (37), together with its associated condition for validity (36) will form the basis for the subsequent analysis.

### 3.1.2. Closure: a toy model again.

The general derivation given above went smoothly. However, Eq. (37) are not yet closed. It turns out that to achieve closure even for simple continuous networks we can no longer get away with just a finite (small) number of macroscopic observables (as with binary neurons). This I will now illustrate with a simple toy network of graded response neurons:

$$\frac{d}{dt} u_i(t) = \sum_j J_{ij} g[u_j(t)] - u_i(t) + \eta_i(t) \quad (38)$$

with  $g[z] = \frac{1}{2}[\tanh(\gamma z) + 1]$  and with the standard Gaussian white noise  $\eta_i(t)$  (see [1]). In the language of (32) this means  $f_i(\mathbf{u}) = \sum_j J_{ij} g[u_j] - u_i$ . We choose uniform synapses  $J_{ij} = J/N$ , so  $f_i(\mathbf{u}) \rightarrow (J/N) \sum_j g[u_j] - u_i$ . If (36) were to hold, we would find the deterministic macroscopic laws

$$\frac{d}{dt} \Omega_\mu = \lim_{N \rightarrow \infty} \left\langle \sum_i \left[ \frac{J}{N} \sum_j g[u_j] - u_i + T \frac{\partial}{\partial u_i} \right] \frac{\partial}{\partial u_i} \Omega_\mu(\mathbf{u}) \right\rangle_{\Omega; t}. \quad (39)$$

In contrast to similar models with binary neurons, choosing as our macroscopic level of description  $\Omega(\mathbf{u})$  again simply the average  $m(\mathbf{u}) = N^{-1} \sum_i u_i$  now leads to an equation which fails to close:

$$\frac{d}{dt} m = \lim_{N \rightarrow \infty} J \left\langle \frac{1}{N} \sum_j g[u_j] \right\rangle_{m; t} - m.$$

The term  $N^{-1} \sum_j g[u_j]$  cannot be written as a function of  $N^{-1} \sum_i u_i$ . We might be tempted to try dealing with this problem by just including the offending term in our macroscopic set, and choose  $\Omega(\mathbf{u}) = (N^{-1} \sum_i u_i, N^{-1} \sum_i g[u_i])$ . This would indeed solve our closure problem for the  $m$ -equation, but we would now find a new closure problem in the equation for the newly introduced observable. The only way out is to choose an observable *function*, namely the distribution of potentials

$$\rho(u; \mathbf{u}) = \frac{1}{N} \sum_i \delta[u - u_i], \quad \rho(u) = \langle \rho(u; \mathbf{u}) \rangle = \left\langle \frac{1}{N} \sum_i \delta[u - u_i] \right\rangle. \quad (40)$$

This is to be done with care, in view of our restriction on the number of observables: we evaluate (40) at first only for  $n$  specific values  $u_\mu$  and take the limit  $n \rightarrow \infty$  only after the limit  $N \rightarrow \infty$ . Thus we define  $\Omega_\mu(\mathbf{u}) = \frac{1}{N} \sum_i \delta[u_\mu - u_i]$ , condition (36) reduces to the familiar expression  $\lim_{N \rightarrow \infty} n/\sqrt{N} = 0$ , and we get for  $N \rightarrow \infty$  and  $n \rightarrow \infty$  (taken in that order) from (39) a diffusion equation for the distribution of membrane potentials (describing a so-called ‘time-dependent Ornstein–Uhlenbeck process’ [9,10]):

$$\frac{d}{dt} \rho(u) = -\frac{\partial}{\partial u} \left\{ \rho(u) \left[ J \int du' \rho(u') g[u'] - u \right] \right\} + T \frac{\partial^2}{\partial u^2} \rho(u). \quad (41)$$

The natural<sup>5</sup> solution of (41) is the Gaussian distribution

$$\rho_t(u) = [2\pi\Sigma^2(t)]^{-\frac{1}{2}} e^{-\frac{1}{2}[u - \bar{u}(t)]^2 / \Sigma^2(t)} \quad (42)$$

in which  $\Sigma = [T + (\Sigma_0^2 - T) e^{-2\eta^2 t}]^{\frac{1}{2}}$ , and  $\bar{u}$  evolves in time according to

$$\frac{d}{dt} \bar{u} = J \int Dz g[\bar{u} + \Sigma z] - \bar{u} \quad (43)$$

<sup>5</sup> For non-Gaussian initial conditions  $\rho_0(u)$  the solution of (41) would in time converge towards the Gaussian solution.

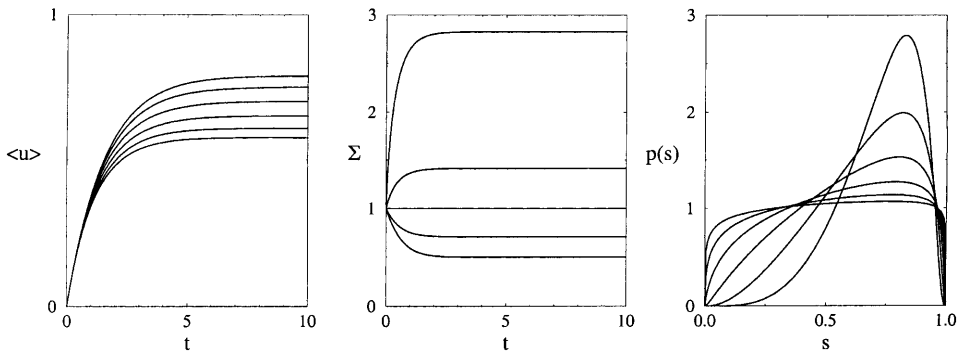


Fig. 5. Dynamics of a simple network of  $N$  graded response neurons (38) with synapses  $J_{ij} = J/N$  and nonlinearity  $g[z] = \frac{1}{2}[1 + \tanh(\gamma z)]$ , for  $N \rightarrow \infty$ ,  $\gamma = J = 1$ , and  $T \in \{0.25, 0.5, 1, 2, 4\}$ . Left: evolution of average membrane potential  $\langle u \rangle = \bar{u}$ , with noise levels  $T$  increasing from top graph ( $T = 0.25$ ) to bottom graph ( $T = 4$ ). Middle: evolution of the width  $\Sigma$  of the membrane potential distribution,  $\Sigma^2 = \langle u^2 \rangle - \langle u \rangle^2$ , with noise levels decreasing from top graph ( $T = 4$ ) to bottom graph ( $T = 0.25$ ). Right: asymptotic ( $t = \infty$ ) distribution of neural firing activities  $p(s) = \langle \delta[s - g[u]] \rangle$ , with noise levels increasing from the sharply peaked curve ( $T = 0.25$ ) to the almost flat curve ( $T = 4$ ).

(with  $Dz = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz$ ). We can now also calculate the distribution  $p(s)$  of neuronal firing activities  $s_i = g[u_i]$  at any time:

$$p(s) = \int du \rho(u) \delta[s - g[u]] = \frac{\rho(g^{\text{inv}}[s])}{\int_0^1 ds' \rho(g^{\text{inv}}[s'])}.$$

For our choice  $g[z] = \frac{1}{2} + \frac{1}{2} \tanh[\gamma z]$  we have  $g^{\text{inv}}[s] = \frac{1}{2\gamma} \log[s/(1-s)]$ , so in combination with (42):

$$0 < s < 1: \quad p(s) = \frac{\exp(-\frac{1}{2}[(2\gamma)^{-1} \log[s/(1-s)] - \bar{u}]^2 / \Sigma^2)}{\int_0^1 ds' \exp(-\frac{1}{2}[(2\gamma)^{-1} \log[s'/(1-s')] - \bar{u}]^2 / \Sigma^2)}. \quad (44)$$

The results of solving and integrating numerically (43) and (44) are shown in Fig. 5, for Gaussian initial conditions (42) with  $\bar{u}_0 = 0$  and  $\sigma_0 = 1$ , and with parameters  $\gamma = J = 1$  and different noise levels  $T$ . For low noise levels we find high average membrane potentials, low membrane potential variance, and high firing rates; for high noise levels the picture changes to lower average membrane potentials, higher potential variance, and uniformly distributed (noise-dominated) firing activities. The extreme cases  $T = 0$  and  $T = \infty$  are easily extracted from our equations. For  $T = 0$  one finds  $\Sigma(t) = \Sigma_0 e^{-t}$  and  $\frac{d}{dt} \bar{u} = Jg[\bar{u}] - \bar{u}$ . This leads to a final state where  $\bar{u} = \frac{1}{2}J + \frac{1}{2}J \tanh[\gamma \bar{u}]$  and where  $p(s) = \delta[s - \bar{u}/J]$ . For  $T = \infty$  one finds  $\Sigma = \infty$  (for any  $t > 0$ ) and  $\frac{d}{dt} \bar{u} = \frac{1}{2}J - \bar{u}$ . This leads to a final state where  $\bar{u} = \frac{1}{2}J$  and where  $p(s) = 1$  for all  $0 < s < 1$ .

None of the above results (not even those on the stationary state) could have been obtained within equilibrium statistical mechanics, since any network of connected graded response neurons will violate detailed balance [1]. Secondly, there

appears to be a qualitative difference between simple networks (e.g.  $J_{ij} = J/N$ ) of binary neurons versus those of continuous neurons, in terms of the types of macroscopic observables needed for deriving closed deterministic laws: a single number  $m = N^{-1} \sum_i \sigma_i$  versus a distribution  $\rho(\sigma) = N^{-1} \sum_i \delta[\sigma - \sigma_i]$ . Note, however, that in the binary case the latter distribution would in fact have been characterized fully by a single number: the average  $m$ , since  $\rho(\sigma) = \frac{1}{2}[1 + m]\delta[\sigma - 1] + \frac{1}{2}[1 - m]\delta[\sigma + 1]$ . In other words: there we were just lucky.

### 3.2. Application to graded response attractor networks

#### 3.2.1. Derivation of closed macroscopic laws

I will now turn to attractor networks with graded response neurons of the type (38), in which  $p$  binary patterns  $\xi^\mu = (\xi_1^\mu, \dots, \xi_N^\mu) \in \{-1, 1\}^N$  have been stored via separable Hebbian-type synapses (18):  $J_{ij} = (2/N) \sum_{\mu\nu=1}^p \xi_i^\mu A_{\mu\nu} \xi_j^\nu$  (the extra factor 2 is inserted for future convenience). Adding suitable thresholds  $\theta_i = -\frac{1}{2} \sum_j J_{ij}$  to the right-hand sides of (38), and choosing the nonlinearity  $g[z] = \frac{1}{2}(1 + \tanh[\gamma z])$  would then give us

$$\frac{d}{dt} u_i(t) = \sum_{\mu\nu} \xi_i^\mu A_{\mu\nu} \frac{1}{N} \sum_j \xi_j^\nu \tanh[\gamma u_j(t)] - u_i(t) + \eta_i(t)$$

so the deterministic forces are  $f_i(\mathbf{u}) = N^{-1} \sum_{\mu\nu} \xi_i^\mu A_{\mu\nu} \sum_j \xi_j^\nu \tanh[\gamma u_j] - u_i$ . Choosing our macroscopic observables  $\Omega(\mathbf{u})$  such that (36) holds, would lead to the deterministic macroscopic laws

$$\begin{aligned} \frac{d}{dt} \Omega_\mu &= \lim_{N \rightarrow \infty} \sum_{\mu\nu} A_{\mu\nu} \left\langle \left[ \frac{1}{N} \sum_j \xi_j^\nu \tanh[\gamma u_j] \right] \left[ \sum_i \xi_i^\mu \frac{\partial}{\partial u_i} \Omega_\mu(\mathbf{u}) \right] \right\rangle_{\Omega, t} \\ &+ \lim_{N \rightarrow \infty} \left\langle \sum_i \left[ T \frac{\partial}{\partial u_i} - u_i \right] \frac{\partial}{\partial u_i} \Omega_\mu(\mathbf{u}) \right\rangle_{\Omega, t}. \end{aligned} \quad (45)$$

As with the uniform synapses case, the main problem to be dealt with is how to choose the  $\Omega_\mu(\mathbf{u})$  such that (45) closes. It turns out that the canonical choice is to turn to the distributions of membrane potentials within each of the  $2^p$  sublattices, as introduced in (12):

$$I_\eta = \{i | \xi_i = \eta\} : \quad \rho_\eta(u; \mathbf{u}) = \frac{1}{|I_\eta|} \sum_{i \in I_\eta} \delta[u - u_i], \quad \rho_\eta(u) = \langle \rho_\eta(u; \mathbf{u}) \rangle \quad (46)$$

with  $\eta \in \{-1, 1\}^p$  and  $\lim_{N \rightarrow \infty} |I_\eta|/N = p_\eta$ . Again we evaluate the distributions in (46) at first only for  $n$  specific values  $u_\mu$  and send  $n \rightarrow \infty$  after  $N \rightarrow \infty$ . Now condition (36) reduces to  $\lim_{N \rightarrow \infty} 2^p/\sqrt{N} = 0$ . We will keep  $p$  finite, for simplicity. Using identities such as  $\sum_i \dots = \sum_\eta \sum_{i \in I_\eta} \dots$  and

$$i \in I_\eta : \quad \frac{\partial}{\partial u_i} \rho_\eta(u; \mathbf{u}) = -|I_\eta|^{-1} \frac{\partial}{\partial u} \delta[u - u_i], \quad \frac{\partial^2}{\partial u_i^2} \rho_\eta(u; \mathbf{u}) = |I_\eta|^{-1} \frac{\partial^2}{\partial u^2} \delta[u - u_i],$$

we then obtain for  $N \rightarrow \infty$  and  $n \rightarrow \infty$  (taken in that order) from Eq. (45)  $2^p$  coupled diffusion equations for the distributions  $\rho_{\mathbf{\eta}}(u)$  of membrane potentials in each of the  $2^p$  sublattices  $I_{\mathbf{\eta}}$ :

$$\frac{d}{dt} \rho_{\mathbf{\eta}}(u) = - \frac{\partial}{\partial u} \left\{ \rho_{\mathbf{\eta}}(u) \left[ \sum_{\mu\nu=1}^p \eta_{\mu} A_{\mu\nu} \sum_{\mathbf{\eta}'} p_{\mathbf{\eta}'} \eta'_{\nu} \int du' \rho_{\mathbf{\eta}'}(u') \tanh[\gamma u'] - u \right] \right\} + T \frac{\partial^2}{\partial u^2} \rho_{\mathbf{\eta}}(u). \tag{47}$$

Eq. (47) is the basis for our further analysis. It can be simplified only if we make additional assumptions on the system’s initial conditions, such as  $\delta$ -distributed or Gaussian distributed  $\rho_{\mathbf{\eta}}(u)$  at  $t = 0$  (see below); otherwise it will have to be solved numerically.

3.2.2. *Reduction to the level of pattern overlaps*

It is clear that (47) is again of the time-dependent Ornstein–Uhlenbeck form, and will thus again have Gaussian solutions as the natural ones:

$$\rho_{t,\mathbf{\eta}}(u) = [2\pi\Sigma_{\mathbf{\eta}}^2(t)]^{-\frac{1}{2}} e^{-\frac{1}{2}[u-\bar{u}_{\mathbf{\eta}}(t)]^2/\Sigma_{\mathbf{\eta}}^2(t)} \tag{48}$$

in which  $\Sigma_{\mathbf{\eta}}(t) = [T + (\Sigma_{\mathbf{\eta}}^2(0) - T) e^{-2t}]^{\frac{1}{2}}$ , and with the  $\bar{u}_{\mathbf{\eta}}(t)$  evolving in time according to

$$\frac{d}{dt} \bar{u}_{\mathbf{\eta}} = \sum_{\mathbf{\eta}'} p_{\mathbf{\eta}'} (\mathbf{\eta} \cdot \mathbf{A} \mathbf{\eta}') \int Dz \tanh[\gamma(\bar{u}_{\mathbf{\eta}'} + \Sigma_{\mathbf{\eta}'} z)] - \bar{u}_{\mathbf{\eta}}. \tag{49}$$

Our problem has thus been reduced successfully to the study of the  $2^p$  coupled scalar Eqs. (49). We can also measure the correlation between the firing activities  $s_i(u_i) = \frac{1}{2}[1 + \tanh(\gamma u_i)]$  and the pattern components (similar to the overlaps in the case of binary neurons). If the pattern bits are drawn at random, i.e.  $\lim_{N \rightarrow \infty} |I_{\mathbf{\eta}}|/N = p_{\mathbf{\eta}} = 2^{-p}$  for all  $\mathbf{\eta}$ , we can define a ‘graded response’ equivalent  $m_{\mu}(\mathbf{u}) = 2N^{-1} \sum_i \xi_i^{\mu} s_i(u_i) \in [-1, 1]$  of the pattern overlaps:

$$\begin{aligned} m_{\mu}(\mathbf{u}) &= \frac{2}{N} \sum_i \xi_i^{\mu} s_i(\mathbf{u}) = \frac{1}{N} \sum_i \xi_i^{\mu} \tanh(\gamma u_i) + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= \sum_{\mathbf{\eta}} p_{\mathbf{\eta}} \eta_{\mu} \int du \rho_{\mathbf{\eta}}(u; \mathbf{u}) \tanh(\gamma u) + \mathcal{O}(N^{-\frac{1}{2}}) \end{aligned} \tag{50}$$

Full recall of pattern  $\mu$  implies  $s_i(u_i) = \frac{1}{2}[\xi_i^{\mu} + 1]$ , giving  $m_{\mu}(\mathbf{u}) = 1$ . Since the distributions  $\rho_{\mathbf{\eta}}(u)$  obey deterministic laws for  $N \rightarrow \infty$ , the same will be true for the overlaps  $\mathbf{m} = (m_1, \dots, m_p)$ . For the Gaussian solutions (49) of (47) we can now proceed to replace the  $2^p$  macroscopic laws (49), which reduce to  $\frac{d}{dt} \bar{u}_{\mathbf{\eta}} = \mathbf{\eta} \cdot \mathbf{A} \mathbf{m} - \bar{u}_{\mathbf{\eta}}$  and give  $\bar{u}_{\mathbf{\eta}} = \bar{u}_{\mathbf{\eta}}(0)e^{-t} + \mathbf{\eta} \cdot \mathbf{A} \int_0^t ds e^{s-t} \mathbf{m}(s)$ , by  $p$  integral equations in terms of overlaps only:



$$m_\mu(t) = \sum_{\mathbf{\eta}} p_{\mathbf{\eta}} \eta_\mu \int \mathbf{Dz} \tanh \left[ \gamma \left( \bar{u}_{\mathbf{\eta}}(0) e^{-t} + \mathbf{\eta} \cdot \mathbf{A} \int_0^t ds e^{s-t} \mathbf{m}(s) + z \sqrt{T + (\Sigma_{\mathbf{\eta}}^2(0) - T) e^{-2t}} \right) \right] \quad (51)$$

with  $\mathbf{Dz} = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz$ . Here the sublattices only come in via the initial conditions.

### 3.2.3. Extracting the physics from the macroscopic laws

The equations describing the asymptotic (stationary) state can be written entirely without sublattices, by taking the  $t \rightarrow \infty$  limit in (51), using  $\bar{u}_{\mathbf{\eta}} \rightarrow \mathbf{\eta} \cdot \mathbf{A} \mathbf{m}$ ,  $\Sigma_{\mathbf{\eta}} \rightarrow \sqrt{T}$ , and the familiar notation  $\langle g(\xi) \rangle_{\xi} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i g(\xi_i) = 2^{-p} \sum_{\xi \in \{-1, 1\}^p} g(\xi)$ :

$$m_\mu = \left\langle \xi_\mu \int \mathbf{Dz} \tanh \left[ \gamma \left( \xi \cdot \mathbf{A} \mathbf{m} + z \sqrt{T} \right) \right] \right\rangle_{\xi} \quad \rho_{\mathbf{\eta}}(u) = [2\pi T]^{-\frac{1}{2}} e^{-\frac{1}{2}(u - \mathbf{\eta} \cdot \mathbf{A} \mathbf{m})^2 / T}. \quad (52)$$

Note the appealing similarity with previous results on networks with binary neurons in equilibrium [1]. For  $T = 0$  the overlap Eq. (52) becomes identical to those found for attractor networks with binary neurons and finite  $p$  (hence our choice to insert an extra factor 2 in defining the synapses), with  $\gamma$  replacing the inverse noise level  $\beta$  in the former.

For the simplest nontrivial choice  $A_{\mu\nu} = \delta_{\mu\nu}$  (i.e.  $J_{ij} = (2/N) \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu}$ , as in the Hopfield [2] model) Eq. (52) yields the familiar pure and mixture state solutions. For  $T = 0$  we find a continuous phase transition from nonrecall to pure states of the form  $m_\mu = m \delta_{\mu\nu}$  (for some  $\nu$ ) at  $\gamma_c = 1$ . For  $T > 0$  we have in (52) an additional Gaussian noise, absent in the models with binary neurons. Again the pure states are the first nontrivial solutions to enter the stage. Substituting  $m_\mu = m \delta_{\mu\nu}$  into (52) gives

$$m = \int \mathbf{Dz} \tanh[\gamma(m + z\sqrt{T})]. \quad (53)$$

Writing (53) as  $m^2 = \gamma m \int_0^m dk [1 - \int \mathbf{Dz} \tanh^2[\gamma(k + z\sqrt{T})]] \leq \gamma m^2$ , reveals that  $m = 0$  as soon as  $\gamma < 1$ . A continuous transition to an  $m > 0$  state occurs when  $\gamma^{-1} = 1 - \int \mathbf{Dz} \tanh^2[\gamma z \sqrt{T}]$ . A parametrization of this transition line in the  $(\gamma, T)$ -plane is given by

$$\gamma^{-1}(x) = 1 - \int \mathbf{Dz} \tanh^2(zx), \quad T(x) = x^2 / \gamma^2(x), \quad x \geq 0. \quad (54)$$

Discontinuous transitions away from  $m = 0$  (for which there is no evidence) would have to be calculated numerically. For  $\gamma = \infty$  we get the equation  $m = \text{erf}[m/\sqrt{2T}]$ , giving a continuous transition to  $m > 0$  at  $T_c = 2/\pi \approx 0.637$ . Alternatively, the latter number can also be found by taking  $\lim_{x \rightarrow \infty} T(x)$  in the above parametrization:

$$\begin{aligned} T_c(\gamma = \infty) &= \lim_{x \rightarrow \infty} x^2 \left[ 1 - \int \mathbf{Dz} \tanh^2(zx) \right]^2 \\ &= \lim_{x \rightarrow \infty} \left[ \int \mathbf{Dz} \frac{d}{dz} \tanh(zx) \right]^2 = \left[ 2 \int \mathbf{Dz} \delta(z) \right]^2 = 2/\pi. \end{aligned}$$

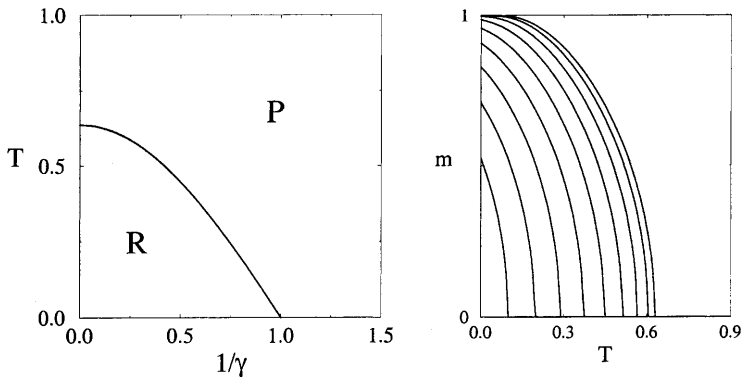


Fig. 6. Left: phase diagram of the Hopfield model with graded-response neurons and  $J_{ij} = (2/N) \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu}$ , away from saturation. P: paramagnetic phase, no recall. R: pattern recall phase. Solid line: separation of the above phases, marked by a continuous transition. Right: asymptotic recall amplitudes  $m = (2/N) \sum_i \xi_i^{\mu} s_i$  of pure states (defined such that full recall corresponds to  $m = 1$ ), as functions of the noise level  $T$ , for  $\gamma^{-1} \in \{0.1, 0.2, \dots, 0.8, 0.9\}$  (from top to bottom).

The resulting picture of the network's stationary state properties is illustrated in Fig. 6, which shows the phase diagram and the stationary recall overlaps of the pure states, obtained by numerical calculation and solution of Eqs. (54) and (53).

Let us now turn to dynamics. It follows from (52) that the 'natural' initial conditions for  $\bar{u}_{\eta}$  and  $\Sigma_{\eta}$  are of the form:  $\bar{u}_{\eta}(0) = \boldsymbol{\eta} \cdot \mathbf{k}_0$  and  $\Sigma_{\eta}(0) = \Sigma_0$  for all  $\boldsymbol{\eta}$ . Equivalently:

$$t = 0: \quad \rho_{\boldsymbol{\eta}}(u) = [2\pi\Sigma_0^2]^{-\frac{1}{2}} e^{-\frac{1}{2}[u - \boldsymbol{\eta} \cdot \mathbf{k}_0]^2 / \Sigma_0^2}, \quad \mathbf{k}_0 \in \mathfrak{R}^p, \quad \Sigma_0 \in \mathfrak{R}.$$

These would also be the typical and natural statistics if we were to prepare an initial firing state  $\{s_i\}$  by hand, via manipulation of the potentials  $\{u_i\}$ . For such initial conditions we can simplify the dynamical Eq. (51) to

$$m_{\mu}(t) = \left\langle \xi_{\mu} \int \mathbf{D}z \tanh \left[ \gamma \left( \boldsymbol{\xi} \cdot \mathbf{k}_0 e^{-t} + \mathbf{A} \int_0^t ds e^{s-t} \mathbf{m}(s) \right) + z \sqrt{T + (\Sigma_0^2 - T) e^{-2t}} \right] \right\rangle_{\xi}. \quad (55)$$

For the special case of the Hopfield synapses, i.e.  $A_{\mu\nu} = \delta_{\mu\nu}$ , it follows from (55) that recall of a given pattern  $\nu$  is triggered upon choosing  $k_{0,\mu} = k_0 \delta_{\mu\nu}$  (with  $k_0 > 0$ ), since then Eq. (55) generates  $m_{\mu}(t) = m(t) \delta_{\mu\nu}$  at any time, with the amplitude  $m(t)$  following from

$$m(t) = \int \mathbf{D}z \tanh \left[ \gamma \left[ k_0 e^{-t} + \int_0^t ds e^{s-t} m(s) + z \sqrt{T + (\Sigma_0^2 - T) e^{-2t}} \right] \right] \quad (56)$$

which is the dynamical counterpart of Eq. (53) (to which indeed it reduces for  $t \rightarrow \infty$ ).

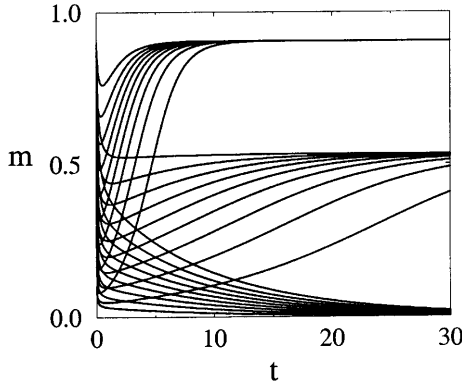


Fig. 7. Overlap evolution in the Hopfield model with graded-response neurons and  $J_{ij} = (2/N) \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu}$ , away from saturation. Gain parameter:  $\gamma = 4$ . Initial conditions:  $\rho_{\eta}(u) = \delta[u - k_0 \eta_v]$  (i.e. triggering recall of pattern  $v$ , with uniform membrane potentials within sublattices). Lines: recall amplitudes  $m = (2/N) \sum_i \xi_i^v s_i$  of pure state  $v$  as functions of time, for  $T = 0.25$  (upper set),  $T = 0.5$  (middle set) and  $T = 0.75$  (lower set), following different initial overlaps  $m_0 \in \{0.1, 0.2, \dots, 0.8, 0.9\}$ .

We finally specialize further to the case where our Gaussian initial conditions are not only chosen to trigger recall of a single pattern  $\xi^v$ , but in addition describe uniform membrane potentials within the sublattices, i.e.  $k_{0,\mu} = k_0 \delta_{\mu v}$  and  $\Sigma_0 = 0$ , so  $\rho_{\eta}(u) = \delta[u - k_0 \eta_v]$ . Here we can derive from (56) at  $t = 0$  the identity  $m_0 = \tanh[\gamma k_0]$ , which enables us to express  $k_0$  as  $k_0 = (2\gamma)^{-1} \log[(1 + m_0)/(1 - m_0)]$ , and find (56) reducing to

$$m(t) = \int Dz \tanh \left[ e^{-t} \log \left[ \frac{1 + m_0}{1 - m_0} \right]^{\frac{1}{2}} + \gamma \left[ \int_0^t ds e^{s-t} m(s) + z \sqrt{T(1 - e^{-2t})} \right] \right]. \quad (57)$$

Solving this equation numerically leads to graphs such as those shown in Fig. 7 for the choice  $\gamma = 4$  and  $T \in \{0.25, 0.5, 0.75\}$ . Compared to the overlap evolution in large networks of binary networks (away from saturation) one immediately observes richer behavior, e.g. nonmonotonicity.

The analysis and results described in this section, which can be done and derived in a similar fashion for other networks with continuous units (such as coupled oscillators), are somewhat difficult to find in research papers. There are two reasons for this. Firstly, nonequilibrium statistical mechanical studies only started being carried out around 1988, and obviously concentrated at first on the (simpler) networks with binary variables. Secondly, due to the absence of detailed balance in networks of graded response networks, the latter appear to have been suspected of consequently having highly complicated dynamics, and analysis terminated with pseudo-equilibrium studies [11]. In retrospect that turns out to have been too pessimistic a view on the power of nonequilibrium statistical mechanics: one finds that

dynamical tools can be applied without serious technical problems (although the calculations are somewhat more involved), and again yield interesting and explicit results in the form of phase diagrams and dynamical curves for macroscopic observables, with sensible physical interpretations.

#### 4. Correlation and response functions

We now turn to correlation functions  $C_{ij}(t, t')$  and response functions  $G_{ij}(t, t')$ . These will become the language in which the generating functional methods are formulated, which will enable us to solve the dynamics of recurrent networks in the (complex) regime near saturation (we take  $t > t'$ ):

$$C_{ij}(t, t') = \langle \sigma_i(t) \sigma_j(t') \rangle, \quad G_{ij}(t, t') = \partial \langle \sigma_i(t) \rangle / \partial \theta_j(t'). \tag{58}$$

The  $\{\sigma_i\}$  evolve in time according to equations of the form (1) (binary neurons, sequential updates), (22) (binary neurons, parallel updates) or (32) (continuous neurons). The  $\theta_i$  represent thresholds and/or external stimuli, which are added to the local fields in the cases (1) and (22), or added to the deterministic forces in the case of a Fokker–Planck Eq. (32). We retain  $\theta_i(t) = \theta_i$ , except for a perturbation  $\delta\theta_j(t')$  applied at time  $t'$  in defining the response function. Calculating averages such as (58) requires determining joint probability distributions involving neuron states at different times.

#### 4.1. Fluctuation–dissipation theorems

##### 4.1.1. Networks of binary neurons

For networks of binary neurons with discrete time dynamics of the form  $p_{\ell+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_{\ell}(\boldsymbol{\sigma}')$ , the probability of observing a given ‘path’  $\boldsymbol{\sigma}(\ell') \rightarrow \boldsymbol{\sigma}(\ell' + 1) \rightarrow \dots \rightarrow \boldsymbol{\sigma}(\ell - 1) \rightarrow \boldsymbol{\sigma}(\ell)$  of successive configurations between step  $\ell'$  and step  $\ell$  is given by the product of the corresponding transition matrix elements (without summation):

$$\begin{aligned} \text{Prob}[\boldsymbol{\sigma}(\ell'), \dots, \boldsymbol{\sigma}(\ell)] &= W[\boldsymbol{\sigma}(\ell); \boldsymbol{\sigma}(\ell - 1)] W[\boldsymbol{\sigma}(\ell - 1); \boldsymbol{\sigma}(\ell - 2)] \dots \\ &\times W[\boldsymbol{\sigma}(\ell' + 1); \boldsymbol{\sigma}(\ell')] p_{\ell'}(\boldsymbol{\sigma}(\ell')). \end{aligned}$$

This allows us to write

$$C_{ij}(\ell, \ell') = \sum_{\boldsymbol{\sigma}(\ell')} \dots \sum_{\boldsymbol{\sigma}(\ell)} \text{Prob}[\boldsymbol{\sigma}(\ell'), \dots, \boldsymbol{\sigma}(\ell)] \sigma_i(\ell) \sigma_j(\ell') = \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \sigma_i \sigma'_j W^{\ell - \ell'}[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_{\ell'}(\boldsymbol{\sigma}'), \tag{59}$$

$$G_{ij}(\ell, \ell') = \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}''} \sigma_i W^{\ell - \ell' - 1}[\boldsymbol{\sigma}; \boldsymbol{\sigma}''] \left[ \frac{\partial}{\partial \theta_j} W[\boldsymbol{\sigma}''; \boldsymbol{\sigma}'] \right] p_{\ell'}(\boldsymbol{\sigma}'). \tag{60}$$

From (59) and (60) it follows that both  $C_{ij}(\ell, \ell')$  and  $G_{ij}(\ell, \ell')$  will in the stationary state, i.e. upon substituting  $p_{\ell'}(\boldsymbol{\sigma}') = p_{\infty}(\boldsymbol{\sigma}')$ , only depend on  $\ell - \ell'$ :  $C_{ij}(\ell, \ell') \rightarrow C_{ij}(\ell - \ell')$  and  $G_{ij}(\ell, \ell') \rightarrow G_{ij}(\ell - \ell')$ . For this we do not require detailed bal-

ance. Detailed balance, however, leads to a simple relation between the response function  $G_{ij}(\tau)$  and the temporal derivative of the correlation function  $C_{ij}(\tau)$ .

We now turn to equilibrium systems, i.e. networks with symmetric synapses (and with all  $J_{ii} = 0$  in the case of sequential dynamics). We calculate the derivative of the transition matrix that occurs in (60) by differentiating the equilibrium condition  $p_{\text{eq}}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] p_{\text{eq}}(\boldsymbol{\sigma}')$  with respect to external fields:

$$\frac{\partial}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} \left\{ \frac{\partial W[\boldsymbol{\sigma}; \boldsymbol{\sigma}']}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}') + W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] \frac{\partial}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}') \right\}.$$

Detailed balance implies  $p_{\text{eq}}(\boldsymbol{\sigma}) = Z^{-1} e^{-\beta H(\boldsymbol{\sigma})}$  (in the parallel case we simply substitute the appropriate Hamiltonian  $H \rightarrow \tilde{H}$ ), giving  $\partial p_{\text{eq}}(\boldsymbol{\sigma}) / \partial \theta_j = -[Z^{-1} \partial Z / \partial \theta_j + \beta \partial H(\boldsymbol{\sigma}) / \partial \theta_j] p_{\text{eq}}(\boldsymbol{\sigma})$ , so that

$$\sum_{\boldsymbol{\sigma}'} \frac{\partial W[\boldsymbol{\sigma}; \boldsymbol{\sigma}']}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}') = \beta \left\{ \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] \frac{\partial H(\boldsymbol{\sigma}')}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}') - \frac{\partial H(\boldsymbol{\sigma})}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}) \right\}$$

(the term containing  $Z$  drops out). We now obtain for the response function (60) in equilibrium:

$$G_{ij}(\ell) = \beta \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \sigma_i W^{\ell-1}[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] \left\{ \sum_{\boldsymbol{\sigma}''} W[\boldsymbol{\sigma}'; \boldsymbol{\sigma}''] \frac{\partial H(\boldsymbol{\sigma}'')}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}'') - \frac{\partial H(\boldsymbol{\sigma}')}{\partial \theta_j} p_{\text{eq}}(\boldsymbol{\sigma}') \right\}. \quad (61)$$

The structure of (61) is similar to what follows upon calculating the evolution of the equilibrium correlation function (59) in a single iteration step:

$$C_{ij}(\ell) - C_{ij}(\ell - 1) = \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \sigma_i W^{\ell-1}[\boldsymbol{\sigma}; \boldsymbol{\sigma}'] \left\{ \sum_{\boldsymbol{\sigma}''} W[\boldsymbol{\sigma}'; \boldsymbol{\sigma}''] \sigma_j'' p_{\text{eq}}(\boldsymbol{\sigma}'') - \sigma_j' p_{\text{eq}}(\boldsymbol{\sigma}') \right\}. \quad (62)$$

Finally we calculate the relevant derivatives of the two Hamiltonians  $H(\boldsymbol{\sigma}) = -\sum_{i<j} J_{ij} \sigma_i \sigma_j - \sum_i \theta_i \sigma_i$  and  $\tilde{H}(\boldsymbol{\sigma}) = -\sum_i \theta_i \sigma_i - \beta^{-1} \sum_i \log 2 \cosh[\beta h_i(\boldsymbol{\sigma})]$  (with  $h_i(\boldsymbol{\sigma}) = \sum_j J_{ij} \sigma_j + \theta_i$ ), see [1]:

$$\partial H(\boldsymbol{\sigma}) / \partial \theta_j = -\sigma_j, \quad \partial \tilde{H}(\boldsymbol{\sigma}) / \partial \theta_j = -\sigma_j - \tanh[\beta h_j(\boldsymbol{\sigma})].$$

For sequential dynamics we hereby arrive directly at a FDT. For parallel dynamics we need one more identity (which follows from the definition of the transition matrix in (22) and the detailed balance property) to transform the  $\tanh$  occurring in the derivative of  $\tilde{H}$ :

$$\tanh[\beta h_j(\boldsymbol{\sigma}')] p_{\text{eq}}(\boldsymbol{\sigma}') = \sum_{\boldsymbol{\sigma}''} \sigma_j'' W[\boldsymbol{\sigma}''; \boldsymbol{\sigma}'] p_{\text{eq}}(\boldsymbol{\sigma}') = \sum_{\boldsymbol{\sigma}''} W[\boldsymbol{\sigma}'; \boldsymbol{\sigma}''] \sigma_j'' p_{\text{eq}}(\boldsymbol{\sigma}'').$$

For parallel dynamics  $\ell$  and  $\ell'$  are the real time labels  $t$  and  $t'$ , and we obtain, with  $\tau = t - t'$ :

Binary & Parallel:

$$G_{ij}(\tau > 0) = -\beta[C_{ij}(\tau + 1) - C_{ij}(\tau - 1)], \quad G_{ij}(\tau \leq 0) = 0. \tag{63}$$

For the continuous-time version (1) of sequential dynamics the time  $t$  is defined as  $t = \ell/N$ , and the difference equation (62) becomes a differential equation. For perturbations at time  $t'$  in the definition of the response function (60) to retain a nonvanishing effect at (re-scaled) time  $t$  in the limit  $N \rightarrow \infty$ , they will have to be re-scaled as well:  $\delta\theta_j(t') \rightarrow N\delta\theta_j(t')$ . As a result:

Binary & Sequential:

$$G_{ij}(\tau) = -\beta\theta(\tau) \frac{d}{d\tau} C_{ij}(\tau). \tag{64}$$

The need to re-scale perturbations in making the transition from discrete to continuous times has the same origin as the need to re-scale the random forces in the derivation of the continuous-time Langevin equation from a discrete-time process. Going from ordinary derivatives to functional derivatives (which is what happens in the continuous-time limit), implies replacing Kronecker delta's  $\delta_{t,t'}$  by Dirac delta-functions according to  $\delta_{t,t'} \rightarrow \Delta\delta(t - t')$ , where  $\Delta$  is the average duration of an iteration step. Eqs. (63) and (64) are examples of so-called fluctuation-dissipation theorems (FDT).

#### 4.1.2. Networks with continuous neurons

For systems described by a Fokker–Planck Eq. (32) the simplest way to calculate correlation and response functions is by first returning to the underlying discrete-time system and leaving the continuous time limit  $\Delta \rightarrow 0$  until the end. In [1] we saw that for small but finite time-steps  $\Delta$  the underlying discrete-time process is described by

$$t = \ell\Delta, \quad p_{\ell\Delta+\Delta}(\boldsymbol{\sigma}) = [1 + \Delta\mathcal{L}_{\boldsymbol{\sigma}} + \mathcal{O}(\Delta^{\frac{3}{2}})]p_{\ell\Delta}(\boldsymbol{\sigma})$$

with  $\ell = 0, 1, 2, \dots$  and with the differential operator

$$\mathcal{L}_{\boldsymbol{\sigma}} = - \sum_i \frac{\partial}{\partial \sigma_i} \left[ f_i(\boldsymbol{\sigma}) - T \frac{\partial}{\partial \sigma_i} \right]. \tag{65}$$

From this it follows that the conditional probability density  $p_{\ell\Delta}(\boldsymbol{\sigma}|\boldsymbol{\sigma}', \ell'\Delta)$  for finding state  $\boldsymbol{\sigma}$  at time  $\ell\Delta$ , given the system was in state  $\boldsymbol{\sigma}'$  at time  $\ell'\Delta$ , must be

$$p_{\ell\Delta}(\boldsymbol{\sigma}|\boldsymbol{\sigma}', \ell'\Delta) = [1 + \Delta\mathcal{L}_{\boldsymbol{\sigma}} + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell-\ell'} \delta[\boldsymbol{\sigma} - \boldsymbol{\sigma}']. \tag{66}$$

Eq. (66) will be our main building block. Firstly, we will calculate the correlations:

$$\begin{aligned} C_{ij}(\ell\Delta, \ell'\Delta) &= \langle \sigma_i(\ell\Delta) \sigma_j(\ell'\Delta) \rangle = \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' \sigma_i \sigma'_j p_{\ell\Delta}(\boldsymbol{\sigma}|\boldsymbol{\sigma}', \ell'\Delta) p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= \int d\boldsymbol{\sigma} \sigma_i [1 + \Delta\mathcal{L}_{\boldsymbol{\sigma}} + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell-\ell'} \int d\boldsymbol{\sigma}' \sigma'_j \delta[\boldsymbol{\sigma} - \boldsymbol{\sigma}'] p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= \int d\boldsymbol{\sigma} \sigma_i [1 + \Delta\mathcal{L}_{\boldsymbol{\sigma}} + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell-\ell'} [\sigma_j p_{\ell'\Delta}(\boldsymbol{\sigma})]. \end{aligned}$$

At this stage we can take the limits  $\Delta \rightarrow 0$  and  $\ell, \ell' \rightarrow \infty$ , with  $t = \ell\Delta$  and  $t' = \ell'\Delta$  finite, using  $\lim_{\Delta \rightarrow 0} [1 + \Delta A]^{k/\Delta} = e^{kA}$ :

$$C_{ij}(t, t') = \int d\boldsymbol{\sigma} \sigma_i e^{(t-t')\mathcal{L}_\sigma} [\sigma_j p_{t'}(\boldsymbol{\sigma})]. \quad (67)$$

Next we turn to the response function. A perturbation applied at time  $t' = \ell'\Delta$  to the Langevin forces  $f_i(\boldsymbol{\sigma})$  comes in at the transition  $\boldsymbol{\sigma}(\ell'\Delta) \rightarrow \boldsymbol{\sigma}(\ell'\Delta + \Delta)$ . As with sequential dynamics binary networks, the perturbation is re-scaled with the step size  $\Delta$  to retain significance as  $\Delta \rightarrow 0$ :

$$\begin{aligned} G_{ij}(\ell\Delta, \ell'\Delta) &= \frac{\partial \langle \sigma_i(\ell\Delta) \rangle}{\Delta \partial \theta_j(\ell'\Delta)} = \frac{\partial}{\Delta \partial \theta_j(\ell'\Delta)} \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' \sigma_i p_{\ell\Delta}(\boldsymbol{\sigma} | \boldsymbol{\sigma}', \ell'\Delta) p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' d\boldsymbol{\sigma}'' \sigma_i p_{\ell\Delta}(\boldsymbol{\sigma} | \boldsymbol{\sigma}'', \ell'\Delta + \Delta) \left[ \frac{\partial p_{\ell'\Delta + \Delta}(\boldsymbol{\sigma}'', \ell'\Delta)}{\Delta \partial \theta_j} \right] p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' d\boldsymbol{\sigma}'' \sigma_i [1 + \Delta \mathcal{L}_\sigma + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell - \ell' - 1} \delta[\boldsymbol{\sigma} - \boldsymbol{\sigma}''] \\ &\quad \times \left[ \frac{1}{\Delta} \frac{\partial}{\partial \theta_j} [1 + \Delta \mathcal{L}_{\boldsymbol{\sigma}'} + \mathcal{O}(\Delta^{\frac{3}{2}})] \delta[\boldsymbol{\sigma}' - \boldsymbol{\sigma}'] \right] p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= - \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' d\boldsymbol{\sigma}'' \sigma_i [1 + \Delta \mathcal{L}_\sigma + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell - \ell' - 1} \delta[\boldsymbol{\sigma} - \boldsymbol{\sigma}''] \delta[\boldsymbol{\sigma}' - \boldsymbol{\sigma}'] \\ &\quad \times \left[ \frac{\partial}{\partial \sigma_j'} + \mathcal{O}(\Delta^{\frac{1}{2}}) \right] p_{\ell'\Delta}(\boldsymbol{\sigma}') \\ &= - \int d\boldsymbol{\sigma} \sigma_i [1 + \Delta \mathcal{L}_\sigma + \mathcal{O}(\Delta^{\frac{3}{2}})]^{\ell - \ell' - 1} \left[ \frac{\partial}{\partial \sigma_j} + \mathcal{O}(\Delta^{\frac{1}{2}}) \right] p_{\ell'\Delta}(\boldsymbol{\sigma}). \end{aligned}$$

We take the limits  $\Delta \rightarrow 0$  and  $\ell, \ell' \rightarrow \infty$ , with  $t = \ell\Delta$  and  $t' = \ell'\Delta$  finite:

$$G_{ij}(t, t') = - \int d\boldsymbol{\sigma} \sigma_i e^{(t-t')\mathcal{L}_\sigma} \frac{\partial}{\partial \sigma_j} p_{t'}(\boldsymbol{\sigma}). \quad (68)$$

Eqs. (67) and (68) apply to arbitrary systems described by Fokker–Planck equations. In the case of conservative forces, i.e.  $f_i(\boldsymbol{\sigma}) = -\partial H(\boldsymbol{\sigma})/\partial \sigma_i$ , and when the system is in an equilibrium state at time  $t'$  so that  $C_{ij}(t, t') = C_{ij}(t - t')$  and  $G_{ij}(t, t') = G_{ij}(t - t')$ , we can take a further step using  $p_{t'}(\boldsymbol{\sigma}) = p_{\text{eq}}(\boldsymbol{\sigma}) = Z^{-1} e^{-\beta H(\boldsymbol{\sigma})}$ . In that case, taking the time derivative of expression (67) gives

$$\frac{\partial}{\partial \tau} C_{ij}(\tau) = \int d\boldsymbol{\sigma} \sigma_i e^{\tau \mathcal{L}_\sigma} \mathcal{L}_\sigma [\sigma_j p_{\text{eq}}(\boldsymbol{\sigma})].$$

Working out the key term in this expression gives

$$\begin{aligned} \mathcal{L}_\sigma [\sigma_j p_{\text{eq}}(\boldsymbol{\sigma})] &= - \sum_i \frac{\partial}{\partial \sigma_i} \left[ f_i(\boldsymbol{\sigma}) - T \frac{\partial}{\partial \sigma_i} \right] [\sigma_j p_{\text{eq}}(\boldsymbol{\sigma})] = T \frac{\partial}{\partial \sigma_j} p_{\text{eq}}(\boldsymbol{\sigma}) \\ &\quad - \sum_i \frac{\partial}{\partial \sigma_i} [\sigma_j J_i(\boldsymbol{\sigma})] \end{aligned}$$

with the components of the probability current density  $J_i(\boldsymbol{\sigma}) = [f_i(\boldsymbol{\sigma}) - T \frac{\partial}{\partial \sigma_i}] p_{\text{eq}}(\boldsymbol{\sigma})$ . In equilibrium, however, the current is zero by definition, so only the first term in the above expression survives. Insertion into our previous equation for  $\partial C_{ij}(\tau)/\partial \tau$ , and comparison with (68) leads to the FDT for continuous systems:

Continuous:

$$G_{ij}(\tau) = -\beta \theta(\tau) \frac{d}{d\tau} C_{ij}(\tau). \quad (69)$$

We will now calculate the correlation and response functions explicitly, and verify the validity or otherwise of the FDT relations, for attractor networks away from saturation.

#### 4.2. Example: simple attractor networks with binary neurons

##### 4.2.1. Correlation and response functions for sequential dynamics

We will consider the continuous time version (1) of the sequential dynamics, with the local fields  $h_i(\boldsymbol{\sigma}) = \sum_j J_{ij} \sigma_j + \theta_i$ , and the separable interaction matrix (18). We already solved the dynamics of this model for the case with zero external fields and away from saturation (i.e.  $p \ll \sqrt{N}$ ). Having nonzero, or even time-dependent, external fields does not affect the calculation much; one adds the external fields to the internal ones and finds the macroscopic laws (2) for the overlaps with the stored patterns being replaced by

$$\frac{d}{dt} \mathbf{m}(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \xi_i \tanh[\beta \xi_i \cdot \mathbf{A} \mathbf{m}(t) + \theta_i(t)] - \mathbf{m}(t). \quad (70)$$

Fluctuations in the local fields are of vanishing order in  $N$  (since the fluctuations in  $\mathbf{m}$  are), so that one can easily derive from the master Eq. (1) the following expressions for spin averages:

$$\frac{d}{dt} \langle \sigma_i(t) \rangle = \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t) + \theta_i(t)] - \langle \sigma_i(t) \rangle \quad (71)$$

$$i \neq j: \quad \frac{d}{dt} \langle \sigma_i(t) \sigma_j(t) \rangle = \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t) + \theta_i(t)] \langle \sigma_j(t) \rangle + \tanh \beta [\xi_j \cdot \mathbf{A} \mathbf{m}(t) + \theta_j(t)] \langle \sigma_i(t) \rangle - 2 \langle \sigma_i(t) \sigma_j(t) \rangle. \quad (72)$$

Correlations at different times are calculated by applying (71) to situations where the microscopic state at time  $t'$  is known exactly, i.e. where  $p_{\boldsymbol{\sigma}}(\boldsymbol{\sigma}) = \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'}$  for some  $\boldsymbol{\sigma}'$ :

$$\langle \sigma_i(t) \rangle |_{\boldsymbol{\sigma}(t')=\boldsymbol{\sigma}'} = \sigma'_i e^{-(t-t')} + \int_{t'}^t ds e^{s-t} \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(s; \boldsymbol{\sigma}', t') + \theta_i(s)] \quad (73)$$

with  $\mathbf{m}(s; \boldsymbol{\sigma}', t')$  denoting the solution of (70) following initial condition  $\mathbf{m}(t') = \frac{1}{N} \sum_i \sigma'_i \xi_i$ . If we multiply both sides of (73) by  $\sigma'_j$  and average over all possible states  $\boldsymbol{\sigma}'$  at time  $t'$  we obtain in leading order in  $N$ :



$$\begin{aligned} \langle \sigma_i(t) \sigma_j(t') \rangle &= \langle \sigma_i(t') \sigma_j(t') \rangle e^{-(t-t')} \\ &+ \int_{t'}^t ds e^{s-t} \langle \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(s; \boldsymbol{\sigma}(t'), t') + \theta_i(s)] \sigma_j(t') \rangle. \end{aligned}$$

Because of the existence of deterministic laws for the overlaps  $\mathbf{m}$  in the  $N \rightarrow \infty$  limit, we know with probability one that during the stochastic process the actual value  $\mathbf{m}(\boldsymbol{\sigma}(t'))$  must be given by the solution of (70), evaluated at time  $t'$ . As a result we obtain, with  $C_{ij}(t, t') = \langle \sigma_i(t) \sigma_j(t') \rangle$ :

$$C_{ij}(t, t') = C_{ij}(t', t') e^{-(t-t')} + \int_{t'}^t ds e^{s-t} \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(s) + \theta_i(s)] \langle \sigma_j(t') \rangle. \quad (74)$$

Similarly we obtain from the solution of (71) an equation for the leading order in  $N$  of the response functions, by derivation with respect to external fields:

$$\begin{aligned} \frac{\partial \langle \sigma_i(t) \rangle}{\partial \theta_j(t')} &= \beta \theta(t-t') \int_{-\infty}^t ds e^{s-t} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(s) + \theta_i(s)]] \\ &\times \left[ \frac{1}{N} \sum_k (\xi_i \cdot \mathbf{A} \xi_k) \frac{\partial \langle \sigma_k(s) \rangle}{\partial \theta_j(t')} + \delta_{ij} \delta(s-t') \right] \end{aligned}$$

or

$$\begin{aligned} G_{ij}(t, t') &= \beta \delta_{ij} \theta(t-t') e^{-(t-t')} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t') + \theta_i(t')]] \\ &+ \beta \theta(t-t') \int_{t'}^t ds e^{s-t} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(s) + \theta_i(s)]] \\ &\times \frac{1}{N} \sum_k (\xi_i \cdot \mathbf{A} \xi_k) G_{kj}(s, t'). \end{aligned} \quad (75)$$

For  $t = t'$  we retain in leading order in  $N$  only the instantaneous single site contribution

$$\lim_{t \uparrow t'} G_{ij}(t, t') = \beta \delta_{ij} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t) + \theta_i(t)]]. \quad (76)$$

This leads to the following ansatz for the scaling with  $N$  of the  $G_{ij}(t, t')$ , which can be shown to be correct by insertion into (75), in combination with the correctness at  $t = t'$  following from (76):

$$i = j: \quad G_{ii}(t, t') = \mathcal{O}(1), \quad i \neq j: \quad G_{ij}(t, t') = \mathcal{O}(N^{-1}).$$

Note that this implies  $\frac{1}{N} \sum_k (\xi_i \cdot \mathbf{A} \xi_k) G_{kj}(s, t') = \mathcal{O}(\frac{1}{N})$ . In leading order in  $N$  we now find

$$G_{ij}(t, t') = \beta \delta_{ij} \theta(t-t') e^{-(t-t')} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t') + \theta_i(t')]]. \quad (77)$$

For those cases where the macroscopic laws (70) describe evolution to a stationary state  $\mathbf{m}$ , obviously requiring stationary external fields  $\theta_i(t) = \theta_i$ , we can take the limit  $t \rightarrow \infty$ , with  $t - t' = \tau$  fixed, in the two results (74) and (77). Using the  $t \rightarrow \infty$  limits of (71) and (72) we subsequently find time translation invariant expressions:  $\lim_{t \rightarrow \infty} C_{ij}(t, t - \tau) = C_{ij}(\tau)$  and  $\lim_{t \rightarrow \infty} G_{ij}(t, t - \tau) = G_{ij}(\tau)$ , with in leading order in  $N$

$$C_{ij}(\tau) = \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m} + \theta_i] \tanh \beta[\xi_j \cdot \mathbf{A}\mathbf{m} + \theta_j] + \delta_{ij} e^{-\tau} [1 - \tanh^2 \beta[\xi_i \cdot \mathbf{A}\mathbf{m} + \theta_i]] \quad (78)$$

$$G_{ij}(\tau) = \beta \delta_{ij} \theta(\tau) e^{-\tau} [1 - \tanh^2 \beta[\xi_i \cdot \mathbf{A}\mathbf{m} + \theta_i]] \quad (79)$$

for which indeed the FDT (64) holds:  $G_{ij}(\tau) = -\beta \theta(\tau) \frac{d}{d\tau} C_{ij}(\tau)$ .

#### 4.2.2. Correlation and response functions for parallel dynamics

We now turn to the parallel dynamical rules (22), with the local fields  $h_i(\boldsymbol{\sigma}) = \sum_j J_{ij} \sigma_j + \theta_i$ , and the interaction matrix (18). As before, having time-dependent external fields amounts simply to adding these fields to the internal ones, and the dynamic laws (31) are found to be replaced by

$$\mathbf{m}(t+1) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \xi_i \tanh[\beta \xi_i \cdot \mathbf{A}\mathbf{m}(t) + \theta_i(t)]. \quad (80)$$

Fluctuations in the local fields are again of vanishing order in  $N$ , and the parallel dynamics versions of Eqs. (71) and (72), to be derived from (22), are found to be

$$\langle \sigma_i(t+1) \rangle = \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m}(t) + \theta_i(t)], \quad (81)$$

$$i \neq j: \quad \langle \sigma_i(t+1) \sigma_j(t+1) \rangle = \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m}(t) + \theta_i(t)] \tanh \beta[\xi_j \cdot \mathbf{A}\mathbf{m}(t) + \theta_j(t)]. \quad (82)$$

With  $\mathbf{m}(t; \boldsymbol{\sigma}', t')$  denoting the solution of the map (80) following initial condition  $\mathbf{m}(t') = \frac{1}{N} \sum_i \sigma'_i \xi_i$ , we immediately obtain from Eqs. (81) and (82), the correlation functions:

$$C_{ij}(t, t) = \delta_{ij} + [1 - \delta_{ij}] \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m}(t-1) + \theta_i(t-1)] \times \tanh \beta[\xi_j \cdot \mathbf{A}\mathbf{m}(t-1) + \theta_j(t-1)], \quad (83)$$

$$t > t': \quad C_{ij}(t, t') = \langle \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m}(t-1; \boldsymbol{\sigma}(t'), t') + \theta_i(t-1)] \sigma_j(t') \rangle = \tanh \beta[\xi_i \cdot \mathbf{A}\mathbf{m}(t-1) + \theta_i(t-1)] \times \tanh \beta[\xi_j \cdot \mathbf{A}\mathbf{m}(t'-1) + \theta_j(t'-1)]. \quad (84)$$

From (81) also follow equations determining the leading order in  $N$  of the response functions  $G_{ij}(t, t')$ , by derivation with respect to the external fields  $\theta_j(t')$ :

$$\begin{aligned}
 t' > t - 1: \quad G_{ij}(t, t') &= 0, \\
 t' = t - 1: \quad G_{ij}(t, t') &= \beta \delta_{ij} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t - 1) + \theta_i(t - 1)]], \\
 t' < t - 1: \quad G_{ij}(t, t') &= \beta [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t - 1) + \theta_i(t - 1)]] \\
 &\quad \times \frac{1}{N} \sum_k (\xi_i \cdot \mathbf{A} \xi_k) G_{kj}(t - 1, t').
 \end{aligned} \tag{85}$$

It now follows iteratively that all off-diagonal elements must be of vanishing order in  $N$ :  $G_{ij}(t, t - 1) = \delta_{ij} G_{ii}(t, t - 1) \rightarrow G_{ij}(t, t - 2) = \delta_{ij} G_{ii}(t, t - 2) \rightarrow \dots$ , so that in leading order

$$G_{ij}(t, t') = \beta \delta_{ij} \delta_{t, t'+1} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m}(t') + \theta_i(t')]]. \tag{86}$$

For those cases where the macroscopic laws (80) describe evolution to a stationary state  $\mathbf{m}$ , with stationary external fields, we can take the limit  $t \rightarrow \infty$ , with  $t - t' = \tau$  fixed, in (83), (84) and (86). We find time translation invariant expressions:  $\lim_{t \rightarrow \infty} C_{ij}(t, t - \tau) = C_{ij}(\tau)$  and  $\lim_{t \rightarrow \infty} G_{ij}(t, t - \tau) = G_{ij}(\tau)$ , with in leading order in  $N$ :

$$C_{ij}(\tau) = \tanh \beta [\xi_i \cdot \mathbf{A} \mathbf{m} + \theta_i] \tanh \beta [\xi_j \cdot \mathbf{A} \mathbf{m} + \theta_j] + \delta_{ij} \delta_{\tau, 0} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m} + \theta_i]] \tag{87}$$

$$G_{ij}(\tau) = \beta \delta_{ij} \delta_{\tau, 1} [1 - \tanh^2 \beta [\xi_i \cdot \mathbf{A} \mathbf{m} + \theta_i]] \tag{88}$$

obeying the FDT (63):  $G_{ij}(\tau > 0) = -\beta [C_{ij}(\tau + 1) - C_{ij}(\tau - 1)]$ .

### 4.3. Example: graded response neurons with uniform synapses

Let us finally find out how to calculate correlation and response function for the simple network (38) of graded response neurons, with (possibly time-dependent) external forces  $\theta_i(t)$ , and with uniform synapses  $J_{ij} = J/N$ :

$$\frac{d}{dt} u_i(t) = \frac{J}{N} \sum_j g[\gamma u_j(t)] - u_i(t) + \theta_i(t) + \eta_i(t). \tag{89}$$

For a given realization of the external forces and the Gaussian noise variables  $\{\eta_i(t)\}$  we can formally integrate (89) and find

$$u_i(t) = u_i(0) e^{-t} + \int_0^t ds e^{s-t} \left[ J \int du \rho(u; \mathbf{u}(s)) g[\gamma u] + \theta_i(s) + \eta_i(s) \right] \tag{90}$$

with the distribution of membrane potentials  $\rho(u; \mathbf{u}) = N^{-1} \sum_i \delta[u - u_i]$ . The correlation function  $C_{ij}(t, t') = \langle u_i(t) u_j(t') \rangle$  immediately follows from (90). Without loss of generality we can define  $t \geq t'$ . For absent external forces (which we only need to define the response function), and upon using  $\langle \eta_i(s) \rangle = 0$  and  $\langle \eta_i(s) \eta_j(s') \rangle = 2T \delta_{ij} \delta(s - s')$ , we arrive at

$$C_{ij}(t, t') = T\delta_{ij}(e^{t-t'} - e^{-t-t'}) + \left\langle \left[ u_i(0) e^{-t} + J \int du g[\gamma u] \int_0^t ds e^{s-t} \rho(u; \mathbf{u}(s)) \right] \times \left[ u_j(0) e^{-t'} + J \int du g[\gamma u] \int_0^{t'} ds' e^{s'-t'} \rho(u; \mathbf{u}(s')) \right] \right\rangle.$$

For  $N \rightarrow \infty$ , however, we know the distribution of potentials to evolve deterministically:  $\rho(u; \mathbf{u}(s)) \rightarrow \rho_s(u)$  where  $\rho_s(u)$  is the solution of (41). This allows us to simplify the above expression to

$$N \rightarrow \infty : \quad C_{ij}(t, t') = T\delta_{ij}(e^{t-t'} - e^{-t-t'}) + \left\langle \left[ u_i(0) e^{-t} + J \int du g[\gamma u] \int_0^t ds e^{s-t} \rho_s(u) \right] \times \left[ u_j(0) e^{-t'} + J \int du g[\gamma u] \int_0^{t'} ds' e^{s'-t'} \rho_{s'}(u) \right] \right\rangle. \tag{91}$$

Next we turn to the response function  $G_{ij}(t, t') = \delta \langle u_i(t) \rangle / \delta \xi_j(t')$  (its definition involves functional rather than scalar differentiation, since time is continuous). After this differentiation the forces  $\{\theta_i(s)\}$  can be put to zero. Functional differentiation of (90), followed by averaging, then leads us to

$$G_{ij}(t, t') = \theta(t-t')\delta_{ij} e^{t-t'} - J \int du g[\gamma u] \frac{\partial}{\partial u} \int_0^t ds e^{s-t} \frac{1}{N} \sum_k \lim_{\theta \rightarrow 0} \left\langle \delta[u - u_k(s)] \frac{\delta u_k(s)}{\delta \theta_j(t')} \right\rangle.$$

In view of (90) we make the self-consistent ansatz  $\delta u_k(s) / \delta \xi_j(s') = \mathcal{O}(N^{-1})$  for  $k \neq j$ . This produces

$$N \rightarrow \infty : \quad G_{ij}(t, t') = \theta(t-t')\delta_{ij} e^{t-t'}. \tag{92}$$

Since Eq. (41) evolves towards a stationary state, we can also take the limit  $t \rightarrow \infty$ , with  $t - t' = \tau$  fixed, in (91). Assuming nonpathological decay of the distribution of potentials allows us to put  $\lim_{t \rightarrow \infty} \int_0^t ds e^{s-t} \rho_s(u) = \rho(u)$  (the stationary solution of (41)), with which we find not only (92) but also (91) reducing to time translation invariant expressions for  $N \rightarrow \infty$ ,  $\lim_{t \rightarrow \infty} C_{ij}(t, t - \tau) = C_{ij}(\tau)$  and  $\lim_{t \rightarrow \infty} G_{ij}(t, t - \tau) = G_{ij}(\tau)$ , in which

$$C_{ij}(\tau) = T\delta_{ij} e^{-\tau} + J^2 \left\{ \int du \rho(u) g[\gamma u] \right\}^2, \quad G_{ij}(\tau) = \theta(\tau) \delta_{ij} e^{-\tau}. \tag{93}$$

Clearly the leading orders in  $N$  of these two functions obey the FDT (69):  $G_{ij}(\tau) = -\beta\theta(\tau) \frac{d}{d\tau} C_{ij}(\tau)$ . As with the binary neuron attractor networks for which we calculated the correlation and response functions earlier, the impact of detailed balance violation (occurring when  $A_{\mu\nu} \neq A_{\nu\mu}$  in networks with binary neurons and synapses (18), and in all networks with graded response neurons [1]) on the validity of the FDTs, vanishes for  $N \rightarrow \infty$ , provided our networks are relatively simple and

evolve to a stationary state in terms of the macroscopic observables (the latter need not necessarily happen, see e.g. Figs. 1 and 4). Detailed balance violation, however, would be noticed in the finite size effects [12].

### 5. Dynamics in the complex regime

The approach we followed so far to derive closed macroscopic laws from the microscopic equations fails when the number of attractors is no longer small compared to the number  $N$  of microscopic neuronal variables. In statics we have seen [1] that, at the work floor level, the fingerprint of complexity is the need to use replica theory, rather than the relatively simple and straightforward methods based on (or equivalent to) calculating the density of states for given realizations of the macroscopic observables. This is caused by the presence of a number of ‘disorder’ variables per degree of freedom which is proportional to  $N$ , over which we are forced to average the macroscopic laws. One finds that in dynamics this situation is reflected in the inability to find an exact set of closed equations for a finite number of observables (or densities). We will see that the natural dynamical counterpart of equilibrium replica theory is generating functional analysis.

#### 5.1. Overview of methods and theories

Let us return to the simplest setting in which to study the problem: single pattern recall in an attractor neural network with  $N$  binary neurons and  $p = \alpha N$  stored patterns in the nontrivial regime, where  $\alpha > 0$ . We choose parallel dynamics, i.e. (22), with Hebbian-type synapses of the form (18) with  $A_{\mu\nu} = \delta_{\mu\nu}$ , i.e.  $J_{ij} = N^{-1} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu}$ , giving us the parallel dynamics version of the Hopfield model [2]. Our interest is in the recall overlap  $m(\boldsymbol{\sigma}) = N^{-1} \sum_i \sigma_i \xi_i^1$  between system state and pattern one. We saw in [1] that for  $N \rightarrow \infty$  the fluctuations in the values of the recall overlap  $m$  will vanish, and that for initial states where all  $\sigma_i(0)$  are drawn independently the overlap  $m$  will obey

$$\begin{aligned}
 m(t+1) &= \int dz P_t(z) \tanh[\beta(m(t) + z)], \\
 P_t(z) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \left\langle \delta \left[ z - \frac{1}{N} \sum_{\mu > 1} \xi_i^{\mu} \xi_i^{\mu} \sum_{j \neq i} \xi_j^{\mu} \sigma_j(t) \right] \right\rangle
 \end{aligned}
 \tag{94}$$

and that all complications in a dynamical analysis of the  $\alpha > 0$  regime are concentrated in the calculation of the distribution  $P_t(z)$  of the (generally nontrivial) interference noise.

##### 5.1.1. Gaussian approximations

As a simple approximation one could just assume [13] that the  $\sigma_i$  remain uncorrelated at all times, i.e.  $\text{Prob}[\sigma_i(t) = \pm \xi_i^1] = \frac{1}{2} [1 \pm m(t)]$  for all  $t \geq 0$ , such that the argument given in [1] for  $t = 0$  (leading to a Gaussian  $P(z)$ ) would hold generally, and where the mapping (94) would describe the overlap evolution at all times:

$$P_t(z) = [2\pi\alpha]^{-\frac{1}{2}} e^{-\frac{1}{2}z^2/\alpha} : \quad m(t+1) = \int Dz \tanh[\beta(m(t) + z\sqrt{\alpha})] \quad (95)$$

with the Gaussian measure  $Dz = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz$ . This equation, however, must be generally incorrect. Firstly, Fig. 5 in [1] shows that knowledge of  $m(t)$  *only* does not permit prediction of  $m(t+1)$ . Secondly, expansion of the right-hand side of (95) for small  $m(t)$  shows that (95) predicts a critical noise level (at  $\alpha = 0$ ) of  $T_c = \beta_c^{-1} = 1$ , and a storage capacity (at  $T = 0$ ) of  $\alpha_c = 2/\pi \approx 0.637$ , whereas both numerical simulations and equilibrium statistical mechanical calculations [1] point to  $\alpha_c \approx 0.139$ . Rather than taking all  $\sigma_i$  to be independent, a weaker assumption would be to just assume the interference noise distribution  $P_t(z)$  to be a zero-average Gaussian one, at any time, with statistically independent noise variables  $z$  at different times. One can then derive (for  $N \rightarrow \infty$  and fully connected networks) an evolution equation for the width  $\Sigma(t)$ , giving [14,15]:

$$P_t(z) = [2\pi\Sigma^2(t)]^{-\frac{1}{2}} e^{-\frac{1}{2}z^2/\Sigma^2(t)} : \quad m(t+1) = \int Dz \tanh[\beta(m(t) + z\Sigma(t))]$$

$$\Sigma^2(t+1) = \alpha + 2\alpha m(t+1)m(t)h[m(t), \Sigma(t)] + \Sigma^2(t)h^2[m(t), \Sigma(t)]$$

with  $h[m, \Sigma] = \beta[1 - \int Dz \tanh^2[\beta(m + z\Sigma)]]$ . These equations describe correctly the qualitative features of recall dynamics, and are found to work well when retrieval actually occurs. For nonretrieval trajectories, however, they appear to underestimate the impact of interference noise: they predict  $T_c = 1$  (at  $\alpha = 0$ ) and a storage capacity (at  $T = 0$ ) of  $\alpha_c \approx 0.1597$  (which should have been about 0.139). A final refinement of the Gaussian approach [16] consisted in allowing for correlations between the noise variables  $z$  at different times (while still describing them by Gaussian distributions). This results in a hierarchy of macroscopic equations, which improve upon the previous Gaussian theories and even predict the correct stationary state and phase diagrams, but still fail to be correct at intermediate times. The fundamental problem with all Gaussian theories, however sophisticated, is clearly illustrated in Fig. 6 of [1]: the interference noise distribution is generally *not* of a Gaussian shape.  $P_t(z)$  is only approximately Gaussian when pattern recall occurs. Hence the successes of Gaussian theories in describing recall trajectories, and their perpetual problems in describing the non-recall ones.

### 5.1.2. Non-Gaussian approximations

In view of the non-Gaussian shape of the interference noise distribution, several attempts have been made at constructing non-Gaussian approximations. In all cases the aim is to arrive at a theory involving only macroscopic observables with a *single* time argument. Fig. 6 of [1] suggests that for a fully connected network with binary neurons and parallel dynamics a more accurate ansatz for  $P_t(z)$  would be the sum of two Gaussians. In [17] the following choice was proposed, guided by the structure of the exact formalism to be described later:

$$P_t(z) = P_t^+(z) + P_t^-(z),$$

$$P_t^\pm(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \delta_{\sigma_i(t), \pm \xi_i^1} \left\langle \delta \left[ z - \frac{1}{N} \sum_{\mu > 1} \xi_i^1 \xi_i^\mu \sum_{j \neq i} \xi_j^\mu \sigma_j(t) \right] \right\rangle,$$

$$P_t^\pm(z) = \frac{1 \pm m(t)}{2 \Sigma(t) \sqrt{2\pi}} e^{-\frac{1}{2} [z \mp d(t)]^2 / \Sigma^2(t)}$$

followed by a self-consistent calculation of  $d(t)$  (representing an effective ‘retarded self-interaction’, since it has an effect equivalent to adding  $h_i(\boldsymbol{\sigma}(t)) \rightarrow h_i(\boldsymbol{\sigma}(t)) + d(t)\sigma_i(t)$ ), and of the width  $\Sigma(t)$  of the two distributions  $P_t^\pm(z)$ , together with

$$\begin{aligned} m(t+1) &= \frac{1}{2} [1 + m(t)] \int Dz \tanh[\beta(m(t) + d(t) + z\Sigma(t))] \\ &\quad + \frac{1}{2} [1 - m(t)] \int Dz \tanh[\beta(m(t) - d(t) + z\Sigma(t))]. \end{aligned}$$

The resulting three-parameter theory, in the form of closed dynamic equations for  $\{m, d, \Sigma\}$ , is found to give a nice (but not perfect) agreement with numerical simulations.

A different philosophy was followed in [18] (for sequential dynamics). First (as yet exact) equations are derived for the evolution of the two macroscopic observables  $m(\boldsymbol{\sigma}) = m_1(\boldsymbol{\sigma})$  and  $r(\boldsymbol{\sigma}) = \alpha^{-1} \sum_{\mu > 1} m_\mu^2(\boldsymbol{\sigma})$ , with  $m_\mu(\boldsymbol{\sigma}) = N^{-1} \sum_i \xi_i^\mu \sigma_i$ , which are both found to involve  $P_t(z)$ :

$$\frac{d}{dt} m = \int dz P_t(z) \tanh[\beta(m + z)], \quad \frac{d}{dt} r = \frac{1}{\alpha} \int dz P_t(z) z \tanh[\beta(m + z)] + 1 - r.$$

Next one closes these equations *by hand*, using a maximum-entropy (or ‘Occam’s Razor’) argument: instead of calculating  $P_t(z)$  from (94) with the real (unknown) microscopic distribution  $p_t(\boldsymbol{\sigma})$ , it is calculated upon assigning equal probabilities to all states  $\boldsymbol{\sigma}$  with  $m(\boldsymbol{\sigma}) = m$  and  $r(\boldsymbol{\sigma}) = r$ , followed by averaging over all realizations of the stored patterns with  $\mu > 1$ . In order words: one assumes (i) that the microscopic states visited by the system are ‘typical’ within the appropriate  $(m, r)$  subshells of state space, and (ii) that one can average over the disorder. Assumption (ii) is harmless, the most important step is (i). This procedure results in an explicit (non-Gaussian) expression for the noise distribution in terms of  $(m, r)$  only, a closed two-parameter theory which is exact for short times and in equilibrium, accurate predictions of the macroscopic flow in the  $(m, r)$ -plane (such as that shown in Fig. 5 of [1]), but (again) deviations in predicted time dependencies at intermediate times. This theory, and its performance, was later improved by applying the same ideas to a derivation of a dynamic equation for the function  $P_t(z)$  itself (rather than for  $m$  and  $r$  only) [19]; research is still under way with the aim to construct a theory along these lines which is fully exact.

### 5.1.3. Exact results: generating functional analysis

The only fully exact procedure available at present is known under various names, such as ‘generating functional analysis’, ‘path integral formalism’ or ‘dynamic mean-field theory’, and is based on a philosophy different from those described so far. Rather than working with the probability  $p_t(\boldsymbol{\sigma})$  of finding a microscopic state  $\boldsymbol{\sigma}$  at time  $t$  in order to calculate the statistics of a set of macroscopic observables  $\boldsymbol{\Omega}(\boldsymbol{\sigma})$  at time  $t$ , one here turns to the probability  $\text{Prob}[\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(t_m)]$  of finding a microscopic *path*  $\boldsymbol{\sigma}(0) \rightarrow \boldsymbol{\sigma}(1) \rightarrow \dots \rightarrow \boldsymbol{\sigma}(t_m)$ . One also adds time-dependent external sources to the local fields,  $h_i(\boldsymbol{\sigma}) \rightarrow h_i(\boldsymbol{\sigma}) + \theta_i(t)$ , in order to probe the networks via perturbations and define a response function. The idea is to concentrate on the moment generating function  $Z[\boldsymbol{\Psi}]$ , which, like  $\text{Prob}[\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(t_m)]$ , fully captures the statistics of paths:

$$Z[\boldsymbol{\Psi}] = \langle e^{-i \sum_i \sum_{t=0}^{t_m} \Psi_i(t) \sigma_i(t)} \rangle. \quad (96)$$

It generates averages of the relevant observables, including those involving neuron states at different times, such as correlation functions  $C_{ij}(t, t') = \langle \sigma_i(t) \sigma_j(t') \rangle$  and response functions  $G_{ij}(t, t') = \partial \langle \sigma_i(t) \rangle / \partial \theta_j(t')$ , upon differentiation with respect to the dummy variables  $\{\Psi_i(t)\}$ :

$$\begin{aligned} \langle \sigma_i(t) \rangle &= i \lim_{\Psi \rightarrow 0} \frac{\partial Z[\boldsymbol{\Psi}]}{\partial \Psi_i(t)}, & C_{ij}(t, t') &= - \lim_{\Psi \rightarrow 0} \frac{\partial^2 Z[\boldsymbol{\Psi}]}{\partial \Psi_i(t) \partial \Psi_j(t')}, \\ G_{ij}(t, t') &= i \lim_{\Psi \rightarrow 0} \frac{\partial^2 Z[\boldsymbol{\Psi}]}{\partial \Psi_i(t) \partial \theta_j(t')}. \end{aligned} \quad (97)$$

Next one assumes (correctly) that for  $N \rightarrow \infty$  only the statistical properties of the stored patterns will influence the macroscopic quantities, so that the generating function  $Z[\boldsymbol{\Psi}]$  can be averaged over all pattern realizations, i.e.  $Z[\boldsymbol{\Psi}] \rightarrow \overline{Z[\boldsymbol{\Psi}]}$ . As in replica theories (the canonical tool to deal with complexity in equilibrium) one carries out the disorder average *before* the average over the statistics of the neuron states, resulting for  $N \rightarrow \infty$  in what can be interpreted as a theory describing a single ‘effective’ binary neuron  $\sigma(t)$ , with an effective local field  $h(t)$  and the dynamics  $\text{Prob}[\sigma(t+1) = \pm 1] = \frac{1}{2} [1 \pm \tanh[\beta h(t)]]$ . However, this effective local field is found to generally depend on past states of the neuron, and on zero-average but temporally correlated Gaussian noise contributions  $\phi(t)$ :

$$h(t|\{\boldsymbol{\sigma}\}, \{\boldsymbol{\phi}\}) = m(t) + \theta(t) + \alpha \sum_{t' < t} R(t, t') \sigma(t') + \sqrt{\alpha} \phi(t). \quad (98)$$

The first comprehensive neural network studies along these lines, dealing with fully connected networks, were carried out in [20,21], followed by applications to a-symmetrically and symmetrically extremely diluted networks [22,23] (we will come back to those later). More recent applications include sequence processing networks [24].<sup>6</sup> For  $N \rightarrow \infty$  the differences between different models are found to show up only in the

<sup>6</sup> In the case of sequence recall the overlap  $m$  is defined with respect to the ‘moving’ target, i.e.  $m(t) = \frac{1}{N} \sum_i \sigma_i(t) \xi_i^t$ .



actual form taken by the effective local field (98), i.e. in the dependence of the ‘retarded self-interaction’ kernel  $R(t, t')$  and the covariance matrix  $\langle \phi(t)\phi(t') \rangle$  of the interference-induced Gaussian noise on the macroscopic objects  $\mathbf{C} = \{C(s, s') = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i C_{ii}(s, s')\}$  and  $\mathbf{G} = \{G(s, s') = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i G_{ii}(s, s')\}$ . For instance:<sup>7</sup>

Model	Synapses $J_{ij}$	$R(t, t')$	$\langle \phi(t)\phi(t') \rangle$
<i>Fully connected, static patterns</i>	$\frac{1}{N} \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu$	$[(\mathbf{I} - \mathbf{G})^{-1} \mathbf{G}](t, t')$	$[(\mathbf{I} - \mathbf{G})^{-1} \mathbf{C} (\mathbf{I} - \mathbf{G}^\dagger)^{-1}](t, t')$
<i>Fully connected, pattern sequence</i>	$\frac{1}{N} \sum_{\mu=1}^{\alpha N} \xi_i^{\mu+1} \xi_j^\mu$	0	$\sum_{n \geq 0} [(\mathbf{G}^\dagger)^n \mathbf{C} \mathbf{G}^n](t, t')$
<i>Symm extr diluted, static patterns</i>	$\frac{c_{ij}}{c} \sum_{\mu=1}^{\alpha c} \xi_i^\mu \xi_j^\mu$	$G(t, t')$	$C(t, t')$
<i>Asymm extr diluted, static patterns</i>	$\frac{c_{ij}}{c} \sum_{\mu=1}^{\alpha c} \xi_i^\mu \xi_j^\mu$	0	$C(t, t')$

with the  $c_{ij}$  drawn at random according to  $P(c_{ij}) = \frac{c}{N} \delta_{c_{ij}, 1} + (1 - \frac{c}{N}) \delta_{c_{ij}, 0}$  (either symmetrically, i.e.  $c_{ij} = c_{ji}$ , or independently) and where  $c_{ii} = 0$ ,  $\lim_{N \rightarrow \infty} c/N = 0$ , and  $c \rightarrow \infty$ . In all cases the observables (overlaps and correlation and response functions) are to be solved from the following closed equations, involving the statistics of the single effective neuron experiencing the field (98):

$$m(t) = \langle \sigma(t) \rangle, \quad C(t, t') = \langle \sigma(t)\sigma(t') \rangle, \quad G(t, t') = \partial \langle \sigma(t) \rangle / \partial \theta(t'). \quad (99)$$

It is now clear that Gaussian theories can at most produce exact results for asymmetric networks. Any degree of symmetry in the synapses is found to induce a nonzero retarded self-interaction, via the kernel  $K(t, t')$ , which constitutes a non-Gaussian contribution to the local fields. Exact closed macroscopic theories apparently require a number of macroscopic observables which grows as  $\mathcal{O}(t^2)$  in order to predict the dynamics up to time  $t$ . In the case of sequential dynamics the picture is found to be very similar to the one above; instead of discrete time labels  $t \in \{0, 1, \dots, t_m\}$ , path summations and matrices, there one has a real time variable  $t \in [0, t_m]$ , path-integrals and integral operators. The remainder of this paper is devoted to the derivation of the above results and their implications.

## 5.2. Generating functional analysis for binary neurons

### 5.2.1. General definitions

I will now show more explicitly how the generating functional formalism works for networks of binary neurons. We define parallel dynamics, i.e. (22), driven as usual by local fields of the form  $h_i(\boldsymbol{\sigma}; t) = \sum_j J_{ij} \sigma_j + \theta_i(t)$ , but with a more general choice of Hebbian-type synapses, in which we allow for a possible random dilution (to reduce repetition in our subsequent derivations):

$$J_{ij} = \frac{c_{ij}}{c} \sum_{\mu=1}^p \xi_i^\mu \xi_j^\mu, \quad p = \alpha c. \quad (100)$$

<sup>7</sup> In the case of extremely diluted models the structure variables are also treated as disorder, and thus averaged out.

Architectural properties are reflected in the variables  $c_{ij} \in \{0, 1\}$ , whereas information storage is to be effected by the remainder in (100), involving  $p$  randomly and independently drawn patterns  $\xi^\mu = (\xi_1^\mu, \dots, \xi_N^\mu) \in \{-1, 1\}^N$ . I will deal both with symmetric and with asymmetric architectures (always putting  $c_{ii} = 0$ ), in which the variables  $c_{ij}$  are drawn randomly according to

Symmetric:

$$c_{ij} = c_{ji}, \quad \forall i < j \quad P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0}. \tag{101}$$

Asymmetric:

$$\forall i \neq j \quad P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0} \tag{102}$$

(one could also study intermediate degrees of symmetry; this would involve only simple adaptations). Thus  $c_{kl}$  is statistically independent of  $c_{ij}$  as soon as  $(k, l) \notin \{(i, j), (j, i)\}$ . In leading order in  $N$  one has  $\langle \sum_j c_{ij} \rangle = c$  for all  $i$ , so  $c$  gives the average number of neurons contributing to the field of any given neuron. In view of this, the number  $p$  of patterns to be stored can be expected to scale as  $p = \alpha c$ . The connectivity parameter  $c$  is chosen to diverge with  $N$ , i.e.  $\lim_{N \rightarrow \infty} c^{-1} = 0$ . If  $c = N$  we obtain the fully connected (parallel dynamics) Hopfield model. Extremely diluted networks are obtained when  $\lim_{N \rightarrow \infty} c/N = 0$ .

For simplicity we make the so-called ‘condensed ansatz’: we assume that the system state has an  $\mathcal{O}(N^0)$  overlap only with a single pattern, say  $\mu = 1$ . This situation is induced by initial conditions: we take a randomly drawn  $\sigma(0)$ , generated by

$$p(\sigma(0)) = \prod_i \left\{ \frac{1}{2} [1 + m_0] \delta_{\sigma_i(0), \xi_i^1} + \frac{1}{2} [1 - m_0] \delta_{\sigma_i(0), -\xi_i^1} \right\} \tag{103}$$

$$\text{so } \frac{1}{N} \sum_i \xi_i^1 \langle \sigma_i(0) \rangle = m_0.$$

The patterns  $\mu > 1$ , as well as the architecture variables  $c_{ij}$ , are viewed as disorder. One assumes that for  $N \rightarrow \infty$  the macroscopic behaviour of the system is ‘self-averaging’, i.e. only dependent on the statistical properties of the disorder (rather than on its microscopic realisation). Averages over the disorder are written as  $\overline{\dots}$ . We next define the disorder-averaged generating function:

$$\overline{Z[\Psi]} = \overline{\langle e^{-i \sum_i \sum_t \Psi_i(t) \sigma_i(t)} \rangle} \tag{104}$$

in which the time  $t$  runs from  $t = 0$  to some (finite) upper limit  $t_m$ . Note that  $\overline{Z[0]} = 1$ . With a modest amount of foresight we define the macroscopic site-averaged and disorder-averaged objects  $m(t) = N^{-1} \sum_i \xi_i^1 \overline{\langle \sigma_i(t) \rangle}$ ,  $C(t, t') = N^{-1} \sum_i \overline{\langle \sigma_i(t) \sigma_i(t') \rangle}$  and  $G(t, t') = N^{-1} \sum_i \overline{\partial \langle \sigma_i(t) \rangle / \partial \theta_i(t')}$ . According to (97) they can be obtained from (104) as follows:

$$m(t) = \lim_{\Psi \rightarrow 0} \frac{i}{N} \sum_j \xi_j^1 \frac{\partial \overline{Z[\Psi]}}{\partial \Psi_j(t)}, \tag{105}$$

$$C(t, t') = - \lim_{\Psi \rightarrow 0} \frac{1}{N} \sum_j \frac{\partial^2 \overline{Z[\Psi]}}{\partial \Psi_j(t) \partial \Psi_j(t')}, \quad G(t, t') = \lim_{\Psi \rightarrow 0} \frac{i}{N} \sum_j \frac{\partial^2 \overline{Z[\Psi]}}{\partial \Psi_j(t) \partial \theta_j(t')}. \quad (106)$$

So far we have only reduced our problem to the calculation of the function  $\overline{Z[\Psi]}$  in (104), which will play a part similar to that of the disorder-averaged free energy in equilibrium calculations (see [1]).

### 5.2.2. Evaluation of the disorder-averaged generating function

As in equilibrium replica calculations, the hope is that progress can be made by carrying out the disorder averages first. In equilibrium calculations we use the replica trick to convert our disorder averages into feasible ones; here the idea is to isolate the local fields at different times and different sites by inserting appropriate  $\delta$ -distributions:

$$\begin{aligned} 1 &= \prod_{it} \int dh_i(t) \delta[h_i(t) - \sum_j J_{ij} \sigma_j(t) - \theta_i(t)] \\ &= \int \{d\mathbf{h} d\hat{\mathbf{h}}\} \exp\left(i \sum_{it} \hat{h}_i(t) \left[h_i(t) - \sum_j J_{ij} \sigma_j(t) - \theta_i(t)\right]\right) \end{aligned}$$

with  $\{d\mathbf{h} d\hat{\mathbf{h}}\} = \prod_{it} [d\hat{h}_i(t) dh_i(t) / 2\pi]$ , giving

$$\overline{Z[\Psi]} = \int \{d\mathbf{h} d\hat{\mathbf{h}}\} e^{i \sum_{it} \hat{h}_i(t) [h_i(t) - \theta_i(t)]} \left\langle e^{-i \sum_{it} \Psi_i(t) \sigma_i(t)} \overline{\left[ e^{-i \sum_{it} \hat{h}_i(t) \sum_j J_{ij} \sigma_j(t)} \right]} \right\rangle_{\text{pf}}$$

in which  $\langle \dots \rangle_{\text{pf}}$  refers to averages over a constrained stochastic process of the type (22), but with prescribed fields  $\{h_i(t)\}$  at all sites and at all times. Note that with such prescribed fields the probability of generating a path  $\{\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(t_m)\}$  is given by

$$\begin{aligned} &\text{Prob}[\boldsymbol{\sigma}(0), \dots, \boldsymbol{\sigma}(t_m) | \{h_i(t)\}] \\ &= p(\boldsymbol{\sigma}(0)) \exp\left(\sum_{it} [\beta \sigma_i(t+1) h_i(t) - \log 2 \cosh[\beta h_i(t)]]\right) \end{aligned}$$

so

$$\begin{aligned} \overline{Z[\Psi]} &= \int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\boldsymbol{\sigma}(0)} \dots \sum_{\boldsymbol{\sigma}(t_m)} p(\boldsymbol{\sigma}(0)) e^{N \mathcal{F}[\{\boldsymbol{\sigma}\}, \{\hat{\mathbf{h}}\}]} \prod_{it} \exp(i \hat{h}_i(t) [h_i(t) - \theta_i(t)] \\ &\quad - i \Psi_i(t) \sigma_i(t) + \beta \sigma_i(t+1) h_i(t) - \log 2 \cosh[\beta h_i(t)]) \end{aligned} \quad (107)$$

with

$$\mathcal{F}[\{\boldsymbol{\sigma}\}, \{\hat{\mathbf{h}}\}] = \frac{1}{N} \log \overline{\left[ e^{-i \sum_{it} \hat{h}_i(t) \sum_j J_{ij} \sigma_j(t)} \right]}. \quad (108)$$

We concentrate on the term  $\mathcal{F}[\dots]$  (with the disorder), of which we need only know the limit  $N \rightarrow \infty$ , since only terms inside  $\overline{Z[\Psi]}$  which are exponential in  $N$  will retain statistical relevance. In the disorder-average of (108) every site  $i$  plays an equivalent role, so the leading order in  $N$  of (108) should depend only on site-averaged functions of the  $\{\sigma_i(t), \hat{h}_i(t)\}$ , with no reference to any special direction except the one defined by pattern  $\xi^1$ . The simplest such functions with a single time variable are

$$a(t; \{\boldsymbol{\sigma}\}) = \frac{1}{N} \sum_i \xi_i^1 \sigma_i(t), \quad k(t; \{\hat{\mathbf{h}}\}) = \frac{1}{N} \sum_i \xi_i^1 \hat{h}_i(t), \tag{109}$$

whereas the simplest ones with two time variables would appear to be

$$q(t, t'; \{\boldsymbol{\sigma}\}) = \frac{1}{N} \sum_i \sigma_i(t) \sigma_i(t'), \quad \mathcal{Q}(t, t'; \{\hat{\mathbf{h}}\}) = \frac{1}{N} \sum_i \hat{h}_i(t) \hat{h}_i(t'), \tag{110}$$

$$K(t, t'; \{\boldsymbol{\sigma}, \hat{\mathbf{h}}\}) = \frac{1}{N} \sum_i \hat{h}_i(t) \sigma_i(t'). \tag{111}$$

It will turn out that all models of the type (100), with either (101) or (102), have the crucial property that (109–111) are in fact the *only* functions to appear in the leading order of (108):

$$\mathcal{F}[\dots] = \Phi[\{a(t; \dots), k(t; \dots), q(t, t'; \dots), \mathcal{Q}(t, t'; \dots), K(t, t'; \dots)\}] + \dots (N \rightarrow \infty) \tag{112}$$

for some as yet unknown function  $\Phi[\dots]$ . This allows us to proceed with the evaluation of (107). We can achieve site factorization in (107) if we isolate the macroscopic objects (109–111) by introducing suitable  $\delta$ -distributions (taking care that all exponents scale linearly with  $N$ , to secure statistical relevance). Thus we insert

$$\begin{aligned} 1 &= \prod_{t=0}^{t_m} \int da(t) \delta[a(t) - a(t; \{\boldsymbol{\sigma}\})] \\ &= \left[ \frac{N}{2\pi} \right]^{t_m+1} \int d\mathbf{a} d\hat{\mathbf{a}} \exp \left( iN \sum_t \hat{a}(t) \left[ a(t) - \frac{1}{N} \sum_j \xi_j^1 \sigma_j(t) \right] \right), \\ 1 &= \prod_{t=0}^{t_m} \int dk(t) \delta[k(t) - k(t; \{\hat{\mathbf{h}}\})] \\ &= \left[ \frac{N}{2\pi} \right]^{t_m+1} \int d\mathbf{k} d\hat{\mathbf{k}} \exp \left( iN \sum_t \hat{k}(t) \left[ k(t) - \frac{1}{N} \sum_j \xi_j^1 \hat{h}_j(t) \right] \right), \\ 1 &= \prod_{t, t'=0}^{t_m} \int dq(t, t') \delta[q(t, t') - q(t, t'; \{\boldsymbol{\sigma}\})] \\ &= \left[ \frac{N}{2\pi} \right]^{(t_m+1)^2} \int d\mathbf{q} d\hat{\mathbf{q}} \exp \left( iN \sum_{t, t'} \hat{q}(t, t') \left[ q(t, t') - \frac{1}{N} \sum_j \sigma_j(t) \sigma_j(t') \right] \right), \end{aligned}$$

$$\begin{aligned}
 1 &= \prod_{t,t'=0}^{t_m} \int dQ(t,t') \delta[Q(t,t') - Q(t,t'; \{\hat{\mathbf{h}}\})] \\
 &= \left[ \frac{N}{2\pi} \right]^{(t_m+1)^2} \int d\mathbf{Q} d\hat{\mathbf{Q}} \exp \left( iN \sum_{t,t'} \hat{Q}(t,t') \left[ Q(t,t') - \frac{1}{N} \sum_j \hat{h}_j(t) \hat{h}_j(t') \right] \right), \\
 1 &= \prod_{t,t'=0}^{t_m} \int dK(t,t') \delta[K(t,t') - K(t,t'; \{\boldsymbol{\sigma}, \hat{\mathbf{h}}\})] \\
 &= \left[ \frac{N}{2\pi} \right]^{(t_m+1)^2} \int d\mathbf{K} d\hat{\mathbf{K}} \exp \left( iN \sum_{t,t'} \hat{K}(t,t') \left[ K(t,t') - \frac{1}{N} \sum_j \hat{h}_j(t) \sigma_j(t') \right] \right).
 \end{aligned}$$

Insertion of these integrals into (107), followed by insertion of (112) and usage of the shorthand

$$\begin{aligned}
 \Psi[\mathbf{a}, \hat{\mathbf{a}}, \mathbf{k}, \hat{\mathbf{k}}, \mathbf{q}, \hat{\mathbf{q}}, \mathbf{Q}, \hat{\mathbf{Q}}, \mathbf{K}, \hat{\mathbf{K}}] &= i \sum_t [\hat{a}(t)a(t) + \hat{k}(t)k(t)] \\
 &+ i \sum_{t,t'} [\hat{q}(t,t')q(t,t') + \hat{Q}(t,t')Q(t,t') + \hat{K}(t,t')K(t,t')]
 \end{aligned} \tag{113}$$

then leads us to

$$\begin{aligned}
 \overline{Z[\Psi]} &= \int d\mathbf{a} d\hat{\mathbf{a}} d\mathbf{k} d\hat{\mathbf{k}} d\mathbf{q} d\hat{\mathbf{q}} d\mathbf{Q} d\hat{\mathbf{Q}} d\mathbf{K} d\hat{\mathbf{K}} \exp(N\Psi[\mathbf{a}, \hat{\mathbf{a}}, \mathbf{k}, \hat{\mathbf{k}}, \mathbf{q}, \hat{\mathbf{q}}, \mathbf{Q}, \hat{\mathbf{Q}}, \mathbf{K}, \hat{\mathbf{K}}]) \\
 &+ N\Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] + \mathcal{O}(\dots) \int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\boldsymbol{\sigma}(0)} \cdots \sum_{\boldsymbol{\sigma}(t_m)} p(\boldsymbol{\sigma}(0)) \\
 &\times \prod_{it} \exp(i\hat{h}_i(t)[h_i(t) - \theta_i(t)] - i\psi_i(t)\sigma_i(t) + \beta\sigma_i(t+1)h_i(t) - \log 2 \cosh[\beta h_i(t)]) \\
 &\times \prod_i \exp \left( -i\xi_i^1 \sum_t [\hat{a}(t)\sigma_i(t) + \hat{k}(t)\hat{h}_i(t)] - i \sum_{t,t'} [\hat{q}(t,t')\sigma_i(t)\sigma_i(t') \right. \\
 &\quad \left. + \hat{Q}(t,t')\hat{h}_i(t)\hat{h}_i(t') + \hat{K}(t,t')\hat{h}_i(t)\sigma_i(t') \right)
 \end{aligned} \tag{114}$$

in which the term denoted as  $\mathcal{O}(\dots)$  covers both the nondominant orders in (108) and the  $\mathcal{O}(\log N)$  relics of the various pre-factors  $[N/2\pi]$  in the above integral representations of the  $\delta$ -distributions (note:  $t_m$  was assumed fixed). We now see explicitly in (114) that the summations and integrations over neuron states and local fields fully factorize over the  $N$  sites. A simple transformation  $\{\sigma_i(t), h_i(t), \hat{h}_i(t)\} \rightarrow \{\xi_i^1 \sigma_i(t), \xi_i^1 h_i(t), \xi_i^1 \hat{h}_i(t)\}$  brings the result into the form

$$\begin{aligned}
 & \int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\boldsymbol{\sigma}(0)} \cdots \sum_{\boldsymbol{\sigma}(t_m)} p(\boldsymbol{\sigma}(0)) \\
 & \times \prod_{it} \exp(i\hat{h}_i(t)[h_i(t) - \xi_i^1 \theta_i(t)] - i\xi_i^1 \psi_i(t) \sigma_i(t) \\
 & \quad + \beta \sigma_i(t+1) h_i(t) - \log 2 \cosh[\beta h_i(t)]) \\
 & \times \prod_i \exp\left(-i\xi_i^1 \sum_t [\hat{a}(t) \sigma_i(t) + \hat{k}(t) \hat{h}_i(t)] \right. \\
 & \quad \left. - i \sum_{t,t'} [\hat{q}(t,t') \sigma_i(t) \sigma_i(t') + \hat{Q}(t,t') \hat{h}_i(t) \hat{h}_i(t') + K(t,t') \hat{h}_i(t) \sigma_i(t')] \right) \\
 & = e^{N\Xi[\hat{\mathbf{a}}, \hat{\mathbf{k}}, \hat{\mathbf{q}}, \hat{\mathbf{Q}}, \hat{\mathbf{K}}]}
 \end{aligned}$$

with

$$\begin{aligned}
 \Xi[\hat{\mathbf{a}}, \hat{\mathbf{k}}, \hat{\mathbf{q}}, \hat{\mathbf{Q}}, \hat{\mathbf{K}}] &= \frac{1}{N} \sum_i \log \int \{dh d\hat{h}\} \sum_{\boldsymbol{\sigma}(0) \cdots \boldsymbol{\sigma}(t_m)} \pi_0(\boldsymbol{\sigma}(0)) \\
 & \times \exp\left(\sum_t \{i\hat{h}(t)[h(t) - \xi_i^1 \theta_i(t)] - i\xi_i^1 \psi_i(t) \sigma(t)\}\right) \\
 & \times \exp\left(\sum_t \{\beta \sigma(t+1) h(t) - \log 2 \cosh[\beta h(t)]\} \right. \\
 & \quad \left. - i \sum_t [\hat{a}(t) \sigma(t) + \hat{k}(t) \hat{h}(t)] - i \sum_{t,t'} [\hat{q}(t,t') \sigma(t) \sigma(t') \right. \\
 & \quad \left. + \hat{Q}(t,t') \hat{h}(t) \hat{h}(t') + \hat{K}(t,t') \hat{h}(t) \sigma(t')\}\right) \tag{115}
 \end{aligned}$$

in which  $\{dh d\hat{h}\} = \prod_t [dh(t) d\hat{h}(t) / 2\pi]$  and  $\pi_0(\boldsymbol{\sigma}) = \frac{1}{2} [1 + m_0] \delta_{\boldsymbol{\sigma}, 1} + \frac{1}{2} [1 - m_0] \delta_{\boldsymbol{\sigma}, -1}$ . At this stage (114) acquires the form of an integral to be evaluated via the saddle-point (or ‘steepest descent’) method:

$$\overline{Z[\{\psi(t)\}]} = \int d\mathbf{a} d\hat{\mathbf{a}} d\mathbf{k} d\hat{\mathbf{k}} d\mathbf{q} d\hat{\mathbf{q}} d\mathbf{Q} d\hat{\mathbf{Q}} d\mathbf{K} d\hat{\mathbf{K}} e^{N\{\Psi[\dots] + \Phi[\dots] + \Xi[\dots]\} + \mathcal{O}(\dots)} \tag{116}$$

in which the functions  $\Psi[\dots]$ ,  $\Phi[\dots]$  and  $\Xi[\dots]$  are defined by (112), (113) and (115).

### 5.2.3. The saddle-point problem

The disorder-averaged generating function (116) is for  $N \rightarrow \infty$  dominated by the physical saddle-point of the macroscopic surface

$$\Psi[\mathbf{a}, \hat{\mathbf{a}}, \mathbf{k}, \hat{\mathbf{k}}, \mathbf{q}, \hat{\mathbf{q}}, \mathbf{Q}, \hat{\mathbf{Q}}, \mathbf{K}, \hat{\mathbf{K}}] + \Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] + \Xi[\hat{\mathbf{a}}, \hat{\mathbf{k}}, \hat{\mathbf{q}}, \hat{\mathbf{Q}}, \hat{\mathbf{K}}] \tag{117}$$

with the three contributions defined in (112), (113) and (115). It will be advantageous at this stage to define the following effective measure (which will be further simplified later):

$$\begin{aligned}
 & \langle f[\{\sigma\}, \{h\}, \{\hat{h}\}]_* \rangle \\
 &= \frac{1}{N} \sum_i \left\{ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_i[\{\sigma\}, \{h\}, \{\hat{h}\}] f[\{\sigma\}, \{h\}, \{\hat{h}\}]}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_i[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right\} \quad (118)
 \end{aligned}$$

with

$$\begin{aligned}
 & M_i[\{\sigma\}, \{h\}, \{\hat{h}\}] = \pi_0(\sigma(0)) \\
 & \times \exp \left( \sum_t \{i\hat{h}(t)[h(t) - \xi_i^1 \theta_i(t)] - i\xi_i^1 \psi_i(t)\sigma(t) \right. \\
 & \quad \left. + \beta\sigma(t+1)h(t) - \log 2 \cosh[\beta h(t)] \right) \\
 & \times \exp \left( -i \sum_t [\hat{a}(t)\sigma(t) + \hat{k}(t)\hat{h}(t)] - i \sum_{t,t'} [\hat{q}(t,t')\sigma(t)\sigma(t') \right. \\
 & \quad \left. + \hat{Q}(t,t')\hat{h}(t)\hat{h}(t') + \hat{K}(t,t')\hat{h}(t)\sigma(t')] \right)
 \end{aligned}$$

in which the values to be inserted for  $\{\hat{m}(t), \hat{k}(t), \hat{q}(t, t'), \hat{Q}(t, t'), \hat{K}(t, t')\}$  are given by the saddle-point of (117). Variation of (117) with respect to all the original macroscopic objects occurring as arguments (those without the ‘hats’) gives the following set of saddle-point equations:

$$\hat{a}(t) = i\partial\Phi/\partial a(t), \quad \hat{k}(t) = i\partial\Phi/\partial k(t), \quad (119)$$

$$\hat{q}(t, t') = i\partial\Phi/\partial q(t, t'), \quad \hat{Q}(t, t') = i\partial\Phi/\partial Q(t, t'), \quad \hat{K}(t, t') = i\partial\Phi/\partial K(t, t'). \quad (120)$$

Variation of (117) with respect to the conjugate macroscopic objects (those with the ‘hats’), in turn, and usage of our newly introduced short-hand notation  $\langle \dots \rangle_*$ , gives:

$$a(t) = \langle \sigma(t) \rangle_*, \quad k(t) = \langle \hat{h}(t) \rangle_*, \quad (121)$$

$$q(t, t') = \langle \sigma(t)\sigma(t') \rangle_*, \quad Q(t, t') = \langle \hat{h}(t)\hat{h}(t') \rangle_*, \quad K(t, t') = \langle \hat{h}(t)\sigma(t') \rangle_*. \quad (122)$$

The coupled equations (119)–(122) are to be solved simultaneously, once we have calculated the term  $\Phi[\dots]$  (112) which depends on the synapses. This appears to be a formidable task; it can, however, be simplified considerably upon first deriving the physical meaning of the above macroscopic quantities. We apply (105)–(116), using identities such as

$$\begin{aligned}
 \frac{\partial \Xi[\dots]}{\partial \Psi_j(t)} &= -\frac{i}{N} \xi_j^1 \left[ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}] \sigma(t)}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right] \\
 \frac{\partial \Xi[\dots]}{\partial \Theta_j(t)} &= -\frac{i}{N} \xi_j^1 \left[ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}] \hat{h}(t)}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right]
 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \Xi[\dots]}{\partial \Psi_j(t) \partial \Psi_j(t')} &= -\frac{1}{N} \left[ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}] \sigma(t) \sigma(t')}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right] \\ &\quad - N \left[ \frac{\partial \Xi[\dots]}{\partial \Psi_j(t)} \right] \left[ \frac{\partial \Xi[\dots]}{\partial \Psi_j(t')} \right] \\ \frac{\partial^2 \Xi[\dots]}{\partial \theta_j(t) \partial \theta_j(t')} &= -\frac{1}{N} \left[ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}] \hat{h}(t) \hat{h}(t')}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right] \\ &\quad - N \left[ \frac{\partial \Xi[\dots]}{\partial \theta_j(t)} \right] \left[ \frac{\partial \Xi[\dots]}{\partial \theta_j(t')} \right] \\ \frac{\partial^2 \Xi[\dots]}{\partial \Psi_j(t) \partial \theta_j(t')} &= -\frac{i}{N} \left[ \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}] \sigma(t) \hat{h}(t')}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0)\dots\sigma(t_m)} M_j[\{\sigma\}, \{h\}, \{\hat{h}\}]} \right] \\ &\quad - N \left[ \frac{\partial \Xi[\dots]}{\partial \Psi_j(t)} \right] \left[ \frac{\partial \Xi[\dots]}{\partial \theta_j(t')} \right] \end{aligned}$$

and using the short-hand notation (118) wherever possible. Note that the external fields  $\{\psi_i(t), \theta_i(t)\}$  occur only in the function  $\Xi[\dots]$ , not in  $\Psi[\dots]$  or  $\Phi[\dots]$ , and that overall constants in  $\overline{Z[\Psi]}$  can always be recovered *a posteriori*, using  $\overline{Z[\mathbf{0}]} = 1$ :

$$\begin{aligned} m(t) &= \lim_{\Psi \rightarrow \mathbf{0}} \frac{i}{N} \sum_i \xi_i^1 \frac{\int d\mathbf{a} \dots d\hat{\mathbf{K}} \left[ \frac{N \partial \Xi}{\partial \Psi_i(t)} \right] e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}}{\int d\mathbf{a} \dots d\hat{\mathbf{K}} e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}} = \lim_{\Psi \rightarrow \mathbf{0}} \langle \sigma(t) \rangle_*, \\ C(t, t') &= -\lim_{\Psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_i \frac{\int d\mathbf{a} \dots d\hat{\mathbf{K}} \left[ \frac{N \partial^2 \Xi}{\partial \Psi_i(t) \partial \Psi_i(t')} + \frac{N \partial \Xi}{\partial \Psi_i(t)} \frac{N \partial \Xi}{\partial \Psi_i(t')} \right] e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}}{\int d\mathbf{a} \dots d\hat{\mathbf{K}} e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}} \\ &= \lim_{\Psi \rightarrow \mathbf{0}} \langle \sigma(t) \sigma(t') \rangle_*, \\ iG(t, t') &= -\lim_{\Psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_i \frac{\int d\mathbf{a} \dots d\hat{\mathbf{K}} \left[ \frac{N \partial^2 \Xi}{\partial \Psi_i(t) \partial \theta_i(t')} + \frac{N \partial \Xi}{\partial \Psi_i(t)} \frac{N \partial \Xi}{\partial \theta_i(t')} \right] e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}}{\int d\mathbf{a} \dots d\hat{\mathbf{K}} e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}} \\ &= \lim_{\Psi \rightarrow \mathbf{0}} \langle \sigma(t) \hat{h}(t') \rangle_*. \end{aligned}$$

Finally we obtain useful identities from the seemingly trivial statements  $N^{-1} \sum_i \xi_i^1 \partial \overline{Z[\mathbf{0}]} / \partial \theta_i(t) = 0$  and  $N^{-1} \sum_i \partial^2 \overline{Z[\mathbf{0}]} / \partial \theta_i(t) \partial \theta_i(t') = 0$ :

$$0 = \lim_{\Psi \rightarrow \mathbf{0}} \frac{i}{N} \sum_i \xi_i^1 \frac{\int d\mathbf{a} \dots d\hat{\mathbf{K}} \left[ \frac{N \partial \Xi}{\partial \theta_i(t)} \right] e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}}{\int d\mathbf{a} \dots d\hat{\mathbf{K}} e^{N[\Psi + \Phi + \Xi] + \mathcal{L}(\dots)}} = \lim_{\Psi \rightarrow \mathbf{0}} \langle \hat{h}(t) \rangle_*,$$



$$\begin{aligned}
 0 &= - \lim_{\psi \rightarrow 0} \frac{1}{N} \sum_i \frac{\int \mathbf{d}\mathbf{a} \dots \mathbf{d}\hat{\mathbf{K}} \left[ \frac{N \partial^2 \Xi}{\partial \theta_i(t) \partial \theta_i(t')} + \frac{N \partial \Xi}{\partial \theta_i(t)} \frac{N \partial \Xi}{\partial \theta_i(t')} \right] e^{N[\Psi + \Phi + \Xi] + \mathcal{O}(\dots)}}{\int \mathbf{d}\mathbf{a} \dots \mathbf{d}\hat{\mathbf{K}} e^{N[\Psi + \Phi + \Xi] + \mathcal{O}(\dots)}} \\
 &= \lim_{\psi \rightarrow 0} \langle \hat{h}(t) \hat{h}(t') \rangle_*.
 \end{aligned}$$

In combination with (121) and (122), the above five identities simplify our problem considerably. The dummy fields  $\psi_i(t)$  have served their purpose and will now be put to zero, as a result we can now identify our macroscopic observables *at the relevant saddle-point* as:

$$a(t) = m(t), \quad k(t) = 0, \quad q(t, t') = C(t, t'), \quad Q(t, t') = 0, \quad K(t, t') = iG(t', t). \quad (123)$$

Finally we make a convenient choice for the external fields,  $\theta_i(t) = \xi_i^1 \theta(t)$ , with which the effective measure  $\langle \dots \rangle_*$  of (124) simplifies to

$$\langle f[\{\sigma\}, \{h\}, \{\hat{h}\}] \rangle_* = \frac{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0) \dots \sigma(t_m)} M[\{\sigma\}, \{h\}, \{\hat{h}\}] f[\{\sigma\}, \{h\}, \{\hat{h}\}]}{\int \{d\mathbf{h} d\hat{\mathbf{h}}\} \sum_{\sigma(0) \dots \sigma(t_m)} M[\{\sigma\}, \{h\}, \{\hat{h}\}]} \quad (124)$$

with

$$\begin{aligned}
 M[\{\sigma\}, \{h\}, \{\hat{h}\}] &= \pi_0(\sigma(0)) \\
 &\times \exp \left( \sum_t \{i\hat{h}(t)[h(t) - \theta(t)] + \beta\sigma(t+1)h(t) \right. \\
 &\quad \left. - \log 2 \cosh[\beta h(t)] \} - i \sum_t [\hat{a}(t)\sigma(t) + \hat{k}(t)\hat{h}(t)] \right) \\
 &\times \exp \left( -i \sum_{t, t'} [\hat{q}(t, t')\sigma(t)\sigma(t') + \hat{Q}(t, t')\hat{h}(t)\hat{h}(t') + \hat{K}(t, t')\hat{h}(t)\sigma(t')] \right).
 \end{aligned}$$

In summary our saddle-point equations are given by (119)–(122), and the physical meaning of the macroscopic quantities is given by (123) (apparently many of them must be zero). Our final task is finding (112), i.e. calculating the leading order of

$$\mathcal{F}[\{\sigma\}, \{\hat{\mathbf{h}}\}] = \frac{1}{N} \log \left[ e^{-i \sum_{ii} \hat{h}_i(t) \sum_j J_{ij} \sigma_j(t)} \right] \quad (125)$$

which is where the properties of the synapses (100) come in.

### 5.3. Parallel dynamics Hopfield model near saturation

#### 5.3.1. The disorder average

The fully connected Hopfield [2] network (here with parallel dynamics) is obtained upon choosing  $c = N$  in the recipe (100), i.e.  $c_{ij} = 1 - \delta_{ij}$  and  $p = \alpha N$ . The disorder

average thus involves only the patterns with  $\mu > 1$ . In view of our objective to write (125) in the form (112), we will substitute the observables defined in (109)–(111) whenever possible. Now (125) gives

$$\begin{aligned} \mathcal{F}[\dots] &= \frac{1}{N} \log \left[ \overline{\exp(-iN^{-1} \sum_t \sum_{\mu} \sum_{i \neq j} \xi_i^{\mu} \xi_j^{\mu} \hat{h}_i(t) \sigma_j(t))} \right] \\ &= i\alpha \sum_t K(t, t; \{\sigma, \hat{h}\}) - i \sum_t a(t)k(t) \\ &\quad + \alpha \log \left[ \overline{\exp\left(-i \sum_t \left[ \sum_i \xi_i \hat{h}_i(t) / \sqrt{N} \right] \left[ \sum_i \xi_i \sigma_i(t) / \sqrt{N} \right] \right)} \right] + \mathcal{O}(N^{-1}). \end{aligned} \tag{126}$$

We concentrate on the last term:

$$\begin{aligned} &\left[ \overline{\exp\left(-i \sum_t \left[ \sum_i \xi_i \hat{h}_i(t) / \sqrt{N} \right] \left[ \sum_i \xi_i \sigma_i(t) / \sqrt{N} \right] \right)} \right] \\ &= \int d\mathbf{x} d\mathbf{y} e^{-i\mathbf{x}\cdot\mathbf{y}} \prod_t \left\{ \delta\left[x(t) - \frac{\sum_i \xi_i \sigma_i(t)}{\sqrt{N}}\right] \delta\left[y(t) - \frac{\sum_i \xi_i \hat{h}_i(t)}{\sqrt{N}}\right] \right\} \\ &= \int \frac{d\mathbf{x} d\mathbf{y} d\hat{\mathbf{x}} d\hat{\mathbf{y}}}{(2\pi)^{2(t_m+1)}} e^{i[\hat{\mathbf{x}}\cdot\mathbf{x} + \hat{\mathbf{y}}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y}]} \left[ \overline{\exp\left(-\frac{i}{\sqrt{N}} \sum_i \xi_i \sum_t [\hat{x}(t)\sigma_i(t) + \hat{y}(t)\hat{h}_i(t)]\right)} \right] \\ &= \int \frac{d\mathbf{x} d\mathbf{y} d\hat{\mathbf{x}} d\hat{\mathbf{y}}}{(2\pi)^{2(t_m+1)}} \exp\left(i[\hat{\mathbf{x}}\cdot\mathbf{x} + \hat{\mathbf{y}}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y}] \right. \\ &\quad \left. + \sum_i \log \cos\left[\frac{1}{\sqrt{N}} \sum_t [\hat{x}(t)\sigma_i(t) + \hat{y}(t)\hat{h}_i(t)]\right] \right) \\ &= \int \frac{d\mathbf{x} d\mathbf{y} d\hat{\mathbf{x}} d\hat{\mathbf{y}}}{(2\pi)^{2(t_m+1)}} \exp\left(i[\hat{\mathbf{x}}\cdot\mathbf{x} + \hat{\mathbf{y}}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y}] \right. \\ &\quad \left. - \frac{1}{2N} \sum_i \left\{ \sum_t [\hat{x}(t)\sigma_i(t) + \hat{y}(t)\hat{h}_i(t)] \right\}^2 + \mathcal{O}(N^{-1}) \right) \\ &= \int \frac{d\mathbf{x} d\mathbf{y} d\hat{\mathbf{x}} d\hat{\mathbf{y}}}{(2\pi)^{2(t_m+1)}} \exp\left(i[\hat{\mathbf{x}}\cdot\mathbf{x} + \hat{\mathbf{y}}\cdot\mathbf{y} - \mathbf{x}\cdot\mathbf{y}] - \frac{1}{2} \sum_{t,t'} [\hat{x}(t)\hat{x}(t')q(t,t') \right. \\ &\quad \left. + 2\hat{x}(t)\hat{y}(t')K(t',t) + \hat{y}(t)\hat{y}(t')Q(t,t')] + \mathcal{O}(N^{-1}) \right). \end{aligned}$$

Together with (126) we have now shown that the disorder average (125) is indeed, in leading order in  $N$ , of the form (112) (as claimed), with

$$\begin{aligned}
 \Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] &= i\alpha \sum_t K(t, t) - i\mathbf{a} \cdot \mathbf{k} + \alpha \log \int \frac{d\mathbf{x} d\mathbf{y} d\hat{\mathbf{x}} d\hat{\mathbf{y}}}{(2\pi)^{2(t_m+1)}} \\
 &\quad \times \exp\left(i[\hat{\mathbf{x}} \cdot \mathbf{x} + \hat{\mathbf{y}} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}] - \frac{1}{2}[\hat{\mathbf{x}} \cdot \mathbf{q}\hat{\mathbf{x}} + 2\hat{\mathbf{y}} \cdot \mathbf{K}\hat{\mathbf{x}} + \hat{\mathbf{y}} \cdot \mathbf{Q}\hat{\mathbf{y}}]\right) \\
 &= i\alpha \sum_t K(t, t) - i\mathbf{a} \cdot \mathbf{k} \\
 &\quad + \alpha \log \int \frac{d\mathbf{u} d\mathbf{v}}{(2\pi)^{t_m+1}} \exp\left(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}]\right)
 \end{aligned} \tag{127}$$

(which, of course, can be simplified further).

### 5.3.2. Simplification of the saddle-point equations

We are now in a position to work out Eqs. (119) and (120). For the single-time observables this gives  $\hat{a}(t) = k(t)$  and  $\hat{k}(t) = a(t)$ , and for the two-time ones:

$$\begin{aligned}
 \hat{q}(t, t') &= -\frac{1}{2}\alpha i \frac{\int d\mathbf{u} d\mathbf{v} u(t)u(t') \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])}{\int d\mathbf{u} d\mathbf{v} \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])}, \\
 \hat{Q}(t, t') &= -\frac{1}{2}\alpha i \frac{\int d\mathbf{u} d\mathbf{v} v(t)v(t') \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])}{\int d\mathbf{u} d\mathbf{v} \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])}, \\
 \hat{K}(t, t') &= -\alpha i \frac{\int d\mathbf{u} d\mathbf{v} v(t)u(t') \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])}{\int d\mathbf{u} d\mathbf{v} \exp(-\frac{1}{2}[\mathbf{u} \cdot \mathbf{q}\mathbf{u} + 2\mathbf{v} \cdot \mathbf{K}\mathbf{u} - 2i\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{Q}\mathbf{v}])} - \alpha \delta_{t,t'}.
 \end{aligned}$$

At the physical saddle-point we can use (123) to express all nonzero objects in terms of the observables  $m(t)$ ,  $C(t, t')$  and  $G(t, t')$ , with a clear physical meaning. Thus we find  $\hat{a}(t) = 0$ ,  $\hat{k}(t) = m(t)$ , and

$$\hat{q}(t, t') = -\frac{1}{2}\alpha i \frac{\int d\mathbf{u} d\mathbf{v} u(t)u(t') e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} \int d\mathbf{u} d\mathbf{v} e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} = 0 \tag{128}$$

$$\hat{Q}(t, t') = -\frac{1}{2}\alpha i \frac{\int d\mathbf{u} d\mathbf{v} v(t)v(t') e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} \int d\mathbf{u} d\mathbf{v} e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} = -\frac{1}{2}\alpha i \left[ (\mathbf{1} - \mathbf{G})^{-1} \mathbf{C} (\mathbf{1} - \mathbf{G}^\dagger)^{-1} \right] (t, t') \tag{129}$$

$$\hat{K}(t, t') + \alpha \delta_{t,t'} = -\alpha i \frac{\int d\mathbf{u} d\mathbf{v} v(t)u(t') e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} \int d\mathbf{u} d\mathbf{v} e^{-\frac{1}{2}[\mathbf{u} \cdot \mathbf{C}\mathbf{u} - 2i\mathbf{u} \cdot [\mathbf{1} - \mathbf{G}]\mathbf{v}]} = \alpha (\mathbf{1} - \mathbf{G})^{-1} (t, t') \tag{130}$$

(with  $G^\dagger(t, t') = G(t', t)$ , and using standard manipulations of Gaussian integrals). Note that we can use the identity  $(\mathbf{1} - \mathbf{G})^{-1} - \mathbf{1} = \sum_{\ell \geq 0} \mathbf{G}^\ell - \mathbf{1} = \sum_{\ell > 0} \mathbf{G}^\ell = \mathbf{G}(\mathbf{1} - \mathbf{G})^{-1}$  to compactify (130) to

$$\hat{K}(t, t') = \alpha[\mathbf{G}(\mathbf{1} - \mathbf{G})^{-1}](t, t'). \quad (131)$$

We have now expressed all our objects in terms of the disorder-averaged recall overlap  $\mathbf{m} = \{m(t)\}$  and the disorder-averaged single-site correlation and response functions  $\mathbf{C} = \{C(t, t')\}$  and  $\mathbf{G} = \{G(t, t')\}$ . We can next simplify the effective measure (124), which plays a crucial role in the remaining saddle-point equations. Inserting  $\hat{a}(t) = \hat{q}(t, t') = 0$  and  $\hat{k}(t) = m(t)$  into (124), first of all, gives us

$$M[\{\sigma\}, \{h\}, \{\hat{h}\}] = \pi_0(\sigma(0)) \exp \left( \sum_t \{i\hat{h}(t) \left[ h(t) - m(t) - \theta(t) - \sum_{t'} \hat{K}(t, t') \sigma(t') \right] \right. \\ \left. + \beta \sigma(t+1) h(t) - \log 2 \cosh[\beta h(t)] \right) - i \sum_{t, t'} \hat{Q}(t, t') \hat{h}(t) \hat{h}(t'). \quad (132)$$

Secondly, causality ensures that  $G(t, t') = 0$  for  $t \leq t'$ , from which, in combination with (131), it follows that the same must be true for the kernel  $\hat{K}(t, t')$ , since

$$\hat{K}(t, t') = \alpha[\mathbf{G}(\mathbf{1} - \mathbf{G})^{-1}](t, t') = \alpha\{\mathbf{G} + \mathbf{G}^2 + \mathbf{G}^3 + \dots\}(t, t').$$

This, in turn, guarantees that the function  $M[\dots]$  in (132) is already normalized:

$$\int \{dh d\hat{h}\} \sum_{\sigma(0) \dots \sigma(t_m)} M[\{\sigma\}, \{h\}, \{\hat{h}\}] = 1.$$

One can prove this iteratively. After summation over  $\sigma(t_m)$  (which due to causality cannot occur in the term with the kernel  $\hat{K}(t, t')$ ) one is left with just a single occurrence of the field  $h(t_m)$  in the exponent, integration over which reduces to  $\delta[\hat{h}(t_m)]$ , which then eliminates the conjugate field  $\hat{h}(t_m)$ . This cycle of operations is next applied to the variables at time  $t_m - 1$ , etc. The effective measure (124) can now be written simply as

$$\langle f[\{\sigma\}, \{h\}, \{\hat{h}\}] \rangle_* = \sum_{\sigma(0) \dots \sigma(t_m)} \int \{dh d\hat{h}\} M[\{\sigma\}, \{h\}, \{\hat{h}\}] f[\{\sigma\}, \{h\}, \{\hat{h}\}]$$

with  $M[\dots]$  as given in (132). The remaining saddle-point equations to be solved, which can be slightly simplified by using the identity  $\langle \sigma(t) \hat{h}(t') \rangle_* = i \partial \langle \sigma(t) \rangle_* / \partial \theta(t')$ , are

$$m(t) = \langle \sigma(t) \rangle_*, \quad C(t, t') = \langle \sigma(t) \sigma(t') \rangle_*, \quad G(t, t') = \partial \langle \sigma(t) \rangle_* / \partial \theta(t'). \quad (133)$$

### 5.3.3. Extracting the physics from the saddle-point equations

At this stage we observe in (133) that we only need to insert functions of spin states into the effective measure  $\langle \dots \rangle_*$  (rather than fields or conjugate fields), so the effective measure can again be simplified. Upon inserting (129) and (131) into the function (132) we obtain  $\langle f[\{\sigma\}] \rangle_* = \sum_{\sigma(0) \dots \sigma(t_m)} \text{Prob}[\{\sigma\}] f[\{\sigma\}]$ , with

$$\text{Prob}\{\{\sigma\}\} = \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \prod_t \left[ \frac{1}{2} [1 + \sigma(t+1) \tanh[\beta h(t|\{\sigma\}, \{\phi\})]] \right] \quad (134)$$

in which  $\pi_0(\sigma(0)) = \frac{1}{2} [1 + \sigma(0)m_0]$ , and

$$h(t|\{\sigma\}, \{\phi\}) = m(t) + \theta(t) + \alpha \sum_{t' < t} [\mathbf{G}(\mathbf{1} - \mathbf{G})^{-1}](t, t') \sigma(t') + \alpha^{\frac{1}{2}} \phi(t) \quad (135)$$

$$P[\{\phi\}] = \frac{\exp\left(-\frac{1}{2} \sum_{t, t'} \phi(t) \left[ (\mathbf{1} - \mathbf{G}^\dagger) \mathbf{C}^{-1} (\mathbf{1} - \mathbf{G}) \right] (t, t') \phi(t')\right)}{(2\pi)^{(t_m+1)/2} \det^{-\frac{1}{2}} \left[ (\mathbf{1} - \mathbf{G}^\dagger) \mathbf{C}^{-1} (\mathbf{1} - \mathbf{G}) \right]} \quad (136)$$

(note: to predict neuron states up until time  $t_m$  we only need the fields up until time  $t_m - 1$ ). We recognize (134) as describing an effective single neuron, with the usual dynamics  $\text{Prob}[\sigma(t+1) = \pm 1] = \frac{1}{2} [1 \pm \tanh[\beta h(t)]]$ , but with the fields (135). This result is indeed of the form (98), with a retarded self-interaction kernel  $R(t, t')$  and covariance matrix  $\langle \phi(t) \phi(t') \rangle$  of the Gaussian  $\phi(t)$  given by

$$R(t, t') = [\mathbf{G}(\mathbf{1} - \mathbf{G})^{-1}](t, t'), \quad \langle \phi(t) \phi(t') \rangle = [(\mathbf{1} - \mathbf{G})^{-1} \mathbf{C} (\mathbf{1} - \mathbf{G}^\dagger)^{-1}](t, t'). \quad (137)$$

For  $\alpha \rightarrow 0$  we lose all the complicated terms in the local fields, and recover the type of simple expression we found earlier for finite  $p$ :  $m(t+1) = \tanh[\beta(m(t) + \theta(t))]$ .

It can be shown [25] (space limitations prevent a demonstration in this paper) that the equilibrium solutions obtained via replica theory in replica-symmetric ansatz [26] can be recovered as those time-translation invariant solutions<sup>8</sup> of the above dynamic equations which (i) obey the parallel dynamics FDT, and (ii) obey  $\lim_{\tau \rightarrow \infty} G(\tau) = 0$ . It can also be shown that the AT [27] instability, where replica symmetry ceases to hold, corresponds to a dynamical instability in the present formalism, where so-called anomalous response sets in:  $\lim_{\tau \rightarrow \infty} G(\tau) \neq 0$ .

Before we calculate the solution explicitly for the first few time-steps, we first work out the relevant averages using (134). Note that always  $C(t, t) = \langle \sigma^2(t) \rangle_* = 1$  and  $G(t, t') = R(t, t') = 0$  for  $t \leq t'$ . As a result the covariance matrix of the Gaussian fields can be written as

$$\begin{aligned} \langle \phi(t) \phi(t') \rangle &= [(\mathbf{1} - \mathbf{G})^{-1} \mathbf{C} (\mathbf{1} - \mathbf{G}^\dagger)^{-1}](t, t') \\ &= \sum_{s, s' \geq 0} [\delta_{t,s} + R(t, s)] C(s, s') [\delta_{s', t'} + R(t', s')] \\ &= \sum_{s=0}^t \sum_{s'=0}^{t'} [\delta_{t,s} + R(t, s)] C(s, s') [\delta_{s', t'} + R(t', s')]. \end{aligned} \quad (138)$$

<sup>8</sup> i.e.  $m(t) = m$ ,  $C(t, t') = C(t - t')$  and  $G(t, t') = G(t - t')$ .

Considering arbitrary positive integer powers of the response function immediately shows that

$$(\mathbf{G}^\ell)(t, t') = 0 \quad \text{if } t' > t - \ell \tag{139}$$

which, in turn, gives

$$R(t, t') = \sum_{\ell > 0} (\mathbf{G}^\ell)(t, t') = \sum_{\ell=1}^{t-t'} (\mathbf{G}^\ell)(t, t'). \tag{140}$$

Similarly we obtain from  $(\mathbf{1} - \mathbf{G})^{-1} = \mathbf{1} + \mathbf{R}$  that for  $t' \geq t$ :  $(\mathbf{1} - \mathbf{G})^{-1}(t, t') = \delta_{t,t'}$ . To suppress notation we will simply put  $h(t|..)$  instead of  $h(t|\{\sigma\}, \{\phi\})$ ; this need not cause any ambiguity. We notice that summation over neuron variables  $\sigma(s)$  and integration over Gaussian variables  $\phi(s)$  with time arguments  $s$  higher than those occurring in the function to be averaged can always be carried out immediately, giving (for  $t > 0$  and  $t' < t$ ):

$$m(t) = \sum_{\sigma(0) \dots \sigma(t-1)} \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(t-1|..)] \times \prod_{s=0}^{t-2} \frac{1}{2} [1 + \sigma(s+1) \tanh[\beta h(s|..)]] \tag{141}$$

$$G(t, t') = \beta \left\{ C(t, t'+1) - \sum_{\sigma(0) \dots \sigma(t-1)} \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(t-1|..)] \times \tanh[\beta h(t'|..)] \prod_{s=0}^{t-2} \frac{1}{2} [1 + \sigma(s+1) \tanh[\beta h(s|..)]] \right\} \tag{142}$$

(which we obtain directly for  $t' = t - 1$ , and which follows for times  $t' < t - 1$  upon using the identity  $\sigma[1 - \tanh^2(x)] = [1 + \sigma \tanh(x)][\sigma - \tanh(x)]$ ). For the correlations we distinguish between  $t' = t - 1$  and  $t' < t - 1$ :

$$C(t, t-1) = \sum_{\sigma(0) \dots \sigma(t-2)} \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(t-1|..)] \times \tanh[\beta h(t-2|..)] \prod_{s=0}^{t-3} \frac{1}{2} [1 + \sigma(s+1) \tanh[\beta h(s|..)]], \tag{143}$$

whereas for  $t' < t - 1$  we have

$$C(t, t') = \sum_{\sigma(0) \dots \sigma(t-1)} \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(t-1|..)] \sigma(t') \times \prod_{s=0}^{t-2} \frac{1}{2} [1 + \sigma(s+1) \tanh[\beta h(s|..)]]. \tag{144}$$

Let us finally work out explicitly the final macroscopic laws (141)–(144), with (135) and (136), for the first few time steps. For arbitrary times our equations will have to be evaluated numerically; we will see below, however, that this can be done in an iterative (i.e. easy) manner. At  $t = 0$  we just have the two observables  $m(0) = m_0$  and  $C(0, 0) = 1$ .

#### 5.3.4. The first few time-steps

The field at  $t = 0$  is  $h(0|..) = m_0 + \theta(0) + \alpha^{\frac{1}{2}}\phi(0)$ , since the retarded self-interaction does not yet come into play. The distribution of  $\phi(0)$  is fully characterized by its variance, which (138) claims to be

$$\langle \phi^2(0) \rangle = C(0, 0) = 1.$$

Therefore, with  $Dz = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} dz$ , we immediately find (141)–(144) reducing to

$$m(1) = \int Dz \tanh[\beta(m_0 + \theta(0) + z\sqrt{\alpha})], \quad C(1, 0) = m_0 m(1), \quad (145)$$

$$G(1, 0) = \beta \left\{ 1 - \int Dz \tanh^2[\beta(m_0 + \theta(0) + z\sqrt{\alpha})] \right\}. \quad (146)$$

For the self-interaction kernel this implies, using (140), that  $R(1, 0) = G(1, 0)$ . We now move on to  $t = 2$ . Here Eqs. (141)–(144) give us

$$m(2) = \frac{1}{2} \sum_{\sigma(0)} \int d\phi(0) d\phi(1) P[\phi(0), \phi(1)] \tanh[\beta h(1|..)] [1 + \sigma(0)m_0],$$

$$C(2, 1) = \frac{1}{2} \sum_{\sigma(0)} \int d\phi(1) d\phi(0) P[\phi(0), \phi(1)] \tanh[\beta h(1|..)] \\ \times \tanh[\beta h(0|..)] [1 + \sigma(0)m_0],$$

$$C(2, 0) = \frac{1}{2} \sum_{\sigma(0)\sigma(1)} \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(1|..)] \sigma(0) \\ \times \frac{1}{2} [1 + \sigma(1) \tanh[\beta h(0|..)]] [1 + \sigma(0)m_0],$$

$$G(2, 1) = \beta \left\{ 1 - \frac{1}{2} \sum_{\sigma(0)} \int d\phi(0) d\phi(1) P[\phi(0), \phi(1)] \right. \\ \left. \times \tanh^2[\beta h(1|..)] [1 + \sigma(0)m_0] \right\},$$

$$G(2, 0) = \beta \left\{ C(2, 1) - \frac{1}{2} \sum_{\sigma(0)} \int d\phi(0) d\phi(1) P[\phi(0), \phi(1)] \tanh[\beta h(1|..)] \right. \\ \left. \times \tanh[\beta h(0|..)] [1 + \sigma(0)m_0] \right\} = 0.$$

We already know that  $\langle \phi^2(0) \rangle = 1$ ; the remaining two moments we need in order to determine  $P[\phi(0), \phi(1)]$  follow again from (138):

$$\begin{aligned} \langle \phi(1)\phi(0) \rangle &= \sum_{s=0}^1 [\delta_{1,s} + \delta_{0,s}R(1, 0)]C(s, 0) = C(1, 0) + G(1, 0), \\ \langle \phi^2(1) \rangle &= \sum_{s=0}^1 \sum_{s'=1}^1 [\delta_{1,s} + \delta_{0,s}R(1, 0)]C(s, s')[\delta_{s',1} + \delta_{s',0}R(1, 0)] \\ &= G^2(1, 0) + 2C(0, 1)G(1, 0) + 1. \end{aligned}$$

We now know  $P[\phi(0), \phi(1)]$  and can work out all macroscopic objects with  $t = 2$  explicitly, if we wish. I will not do this here in full, but only point at the emerging pattern of all calculations at a given time  $t$  depending only on macroscopic quantities that have been calculated at times  $t' < t$ , which allows for iterative solution. Let us just work out  $m(2)$  explicitly, in order to compare the first two recall overlaps  $m(1)$  and  $m(2)$  with the values found in simulations and in approximate theories. We note that calculating  $m(2)$  only requires the field  $\phi(1)$ , for which we found  $\langle \phi^2(1) \rangle = G^2(1, 0) + 2C(0, 1)G(1, 0) + 1$ :

$$\begin{aligned} m(2) &= \frac{1}{2} \sum_{\sigma(0)} \int d\phi(1) P[\phi(1)] \tanh[\beta(m(1) + \theta(1)) \\ &\quad + \alpha G(1, 0)\sigma(0) + \alpha^{\frac{1}{2}}\phi(1)][1 + \sigma(0)m_0] \\ &= \frac{1}{2} [1 + m_0] \int Dz \tanh[\beta(m(1) + \theta(1)) \\ &\quad + \alpha G(1, 0) + z\sqrt{\alpha[G^2(1, 0) + 2m_0m(1)G(1, 0) + 1]}] \\ &\quad + \frac{1}{2} [1 - m_0] \int Dz \tanh[\beta(m(1) + \theta(1) - \alpha G(1, 0)) \\ &\quad + z\sqrt{\alpha[G^2(1, 0) + 2m_0m(1)G(1, 0) + 1]}]. \end{aligned}$$

5.3.5. *Exact results versus simulations and gaussian approximations*

I close this section on the fully connected networks with a comparison of some of the approximate theories, the (exact) generating functional formalism, and numerical simulations, for the case  $\theta(t) = 0$  (no external stimuli at any time). The evolution of the recall overlap in the first two time-steps has been described as follows:

*Naive Gaussian Approximation:*

$$\begin{aligned} m(1) &= \int Dz \tanh[\beta(m(0) + z\sqrt{\alpha})], \\ m(2) &= \int Dz \tanh[\beta(m(1) + z\sqrt{\alpha})]. \end{aligned}$$



*Amari–Maginu theory:*

$$\begin{aligned}
 m(1) &= \int \mathbf{D}z \tanh[\beta(m(0) + z\sqrt{\alpha})], \\
 m(2) &= \int \mathbf{D}z \tanh[\beta(m(1) + z\Sigma\sqrt{\alpha})], \\
 \Sigma^2 &= 1 + 2m(0)m(1)G + G^2, \\
 G &= \beta \left[ 1 - \int \mathbf{D}z \tanh^2[\beta(m(0) + z\sqrt{\alpha})] \right].
 \end{aligned}$$

*Exact solution:*

$$\begin{aligned}
 m(1) &= \int \mathbf{D}z \tanh[\beta(m(0) + z\sqrt{\alpha})], \\
 m(2) &= \frac{1}{2}[1 + m_0] \int \mathbf{D}z \tanh[\beta(m(1) + \alpha G + z\Sigma\sqrt{\alpha})] \\
 &\quad + \frac{1}{2}[1 - m_0] \int \mathbf{D}z \tanh[\beta(m(1) - \alpha G + z\Sigma\sqrt{\alpha})], \\
 \Sigma^2 &= 1 + 2m(0)m(1)G + G^2, \\
 G &= \beta \left[ 1 - \int \mathbf{D}z \tanh^2[\beta(m(0) + z\sqrt{\alpha})] \right].
 \end{aligned}$$

We can now appreciate why the more advanced Gaussian approximation (Amari–Maginu theory, [14]) works well when the system state is close to the target attractor. This theory gets the moments of the Gaussian part of the interference noise distribution at  $t = 1$  exactly right, but not the discrete part, whereas close to the attractor both the response function  $G(1, 0)$  and one of the two pre-factors  $\frac{1}{2}[1 \pm m_0]$  in the exact expression for  $m(2)$  will be very small, and the latter will therefore indeed approach a Gaussian shape. One can also see why the non-Gaussian approximation of [17] made sense: in the calculation of  $m(2)$  the interference noise distribution can indeed be written as the sum of two Gaussian ones (although for  $t > 2$  this will cease to be true). Numerical evaluation of these expressions result in explicit predictions which can be tested against numerical simulations. This is done in Fig. 8, which confirms the picture sketched above, and hints that the performance of the Gaussian approximations is indeed worse for those initial conditions which fail to trigger pattern recall.

#### 5.4. Extremely diluted attractor networks near saturation

Extremely diluted attractor networks are obtained upon choosing  $\lim_{N \rightarrow \infty} c/N = 0$  (while still  $c \rightarrow \infty$ ) in definition (100) of the Hebbian-type synapses. The disorder average now involves both the patterns with  $\mu > 1$  and the realization of the ‘wiring’ variables  $c_{ij} \in \{0, 1\}$ . Again, in working out the key function (125) we will show that for  $N \rightarrow \infty$  the outcome can be written in terms of the macroscopic quantities (109)–(111). We carry out the average over the spatial structure variables  $\{c_{ij}\}$  first:

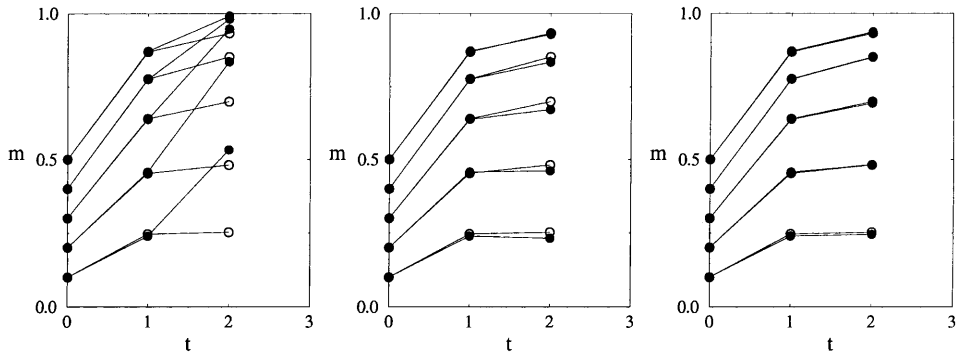


Fig. 8. The first few time steps in the evolution of the overlap  $m(\sigma) = N^{-1} \sum_i \sigma_i \xi_i^1$  in a parallel dynamics Hopfield model with  $\alpha = T = 0.1$  and random patterns, following initial states correlated with pattern one only. Left: simulations (○) versus naive Gaussian approximation (●). Middle: simulations (○) versus advanced Gaussian approximation (Amari–Maguin theory, ●). Right: simulations (○) versus (exact) generating functional theory (●). All simulations were done with  $N = 30,000$ .

$$\begin{aligned} \mathcal{F}[\dots] &= \frac{1}{N} \log \left[ \exp \left( -\frac{i}{c} \sum_{i \neq j} c_{ij} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t) \right) \right] \\ &= \frac{1}{N} \log \prod_{i < j} \exp \left( -\frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \left[ c_{ij} \sum_t \hat{h}_i(t) \sigma_j(t) + c_{ji} \sum_t \hat{h}_j(t) \sigma_i(t) \right] \right). \end{aligned}$$

At this stage we have to distinguish between symmetric and asymmetric dilutions.

5.4.1. The disorder average

First we deal with the case of symmetric dilution:  $c_{ij} = c_{ji}$  for all  $i \neq j$ . The average over the  $c_{ij}$ , with the distribution (101), is trivial:

$$\begin{aligned} &\overline{\prod_{i < j} \exp \left( -\frac{i}{c} c_{ij} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] \right)} \\ &= \overline{\prod_{i < j} \left\{ 1 + \frac{c}{N} \left[ \exp \left( -\frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] \right) - 1 \right] \right\}} \\ &= \overline{\prod_{i < j} \left\{ 1 - \frac{c}{N} \left[ \frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] \dots \right. \right.} \\ &\quad \left. \left. + \frac{1}{2c^2} \left[ \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] \right]^2 + \mathcal{O}(c^{-3/2}) \right] \right\}} \end{aligned}$$

$$= \overline{\prod_{i < j} \exp \left( -\frac{i}{N} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] \right)} \\ - \frac{1}{2cN} \left[ \overline{\sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)]} \right]^2 + \mathcal{O} \left( \frac{1}{N\sqrt{c}} \right) + \mathcal{O} \left( \frac{c}{N^2} \right).$$

We separate in the exponent the terms where  $\mu = \nu$  in the quadratic term (being of the form  $\sum_{\mu\nu} \dots$ ), and the terms with  $\mu = 1$ . Note:  $p = \alpha c$ . We also use the definitions (109)–(111) wherever we can:

$$\mathcal{F}[\dots] = -i \sum_t a(t)k(t) - \frac{1}{2} \alpha \sum_{st} [q(s,t)Q(s,t) + K(s,t)K(t,s)] + \mathcal{O}(c^{-\frac{1}{2}}) + \mathcal{O}(c/N) \\ + \frac{1}{N} \log \left\{ \overline{\exp \left( -\frac{i}{N} \sum_{\mu > 1} \sum_t \left[ \sum_i \xi_i^{\mu} \hat{h}_i(t) \right] \left[ \sum_j \xi_j^{\mu} \sigma_j(t) \right] \right)} \right. \\ \left. - \frac{1}{4cN} \overline{\sum_{i \neq j} \sum_{\mu \neq \nu} \sum_{st} \xi_i^{\mu} \xi_j^{\mu} \xi_i^{\nu} \xi_j^{\nu} [\hat{h}_i(s) \sigma_j(s) + \hat{h}_j(s) \sigma_i(s)] [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)]} \right\}$$

Our ‘condensed ansatz’ implies that for  $\mu > 1$ :  $N^{-\frac{1}{2}} \sum_i \xi_i^{\mu} \sigma_i(t) = \mathcal{O}(1)$  and  $N^{-\frac{1}{2}} \sum_i \xi_i^{\mu} \hat{h}_i(t) = \mathcal{O}(1)$ . Thus the first term in the exponent containing the disorder is  $\mathcal{O}(c)$ , contributing  $\mathcal{O}(c/N)$  to  $\mathcal{F}[\dots]$ . We therefore retain only the second term in the exponent. However, the same argument applies to the second term. There all contributions can be seen as uncorrelated in leading order, so that  $\sum_{i \neq j} \sum_{\mu \neq \nu} \dots = \mathcal{O}(Np)$ , giving a nonleading  $\mathcal{O}(N^{-1})$  cumulative contribution to  $\mathcal{F}[\dots]$ . Thus, provided  $\lim_{N \rightarrow \infty} c^{-1} = \lim_{N \rightarrow \infty} c/N = 0$  (which we assumed), we have shown that the disorder average (125) is again, in leading order in  $N$ , of the form (112) (as claimed), with

*Symmetric:*

$$\Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] = -i \mathbf{a} \cdot \mathbf{k} - \frac{1}{2} \alpha \sum_{st} [q(s,t)Q(s,t) + K(s,t)K(t,s)]. \quad (147)$$

Next we deal with the asymmetric case (102), where  $c_{ij}$  and  $c_{ji}$  are independent. Again the average over the  $c_{ij}$  is trivial; here it gives

$$\overline{\prod_{i < j} \left\{ e^{-\frac{i}{c} c_{ij} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t)} e^{-\frac{j}{c} c_{ji} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_j(t) \sigma_i(t)} \right\}} \\ = \overline{\prod_{i < j} \left\{ 1 + \frac{c}{N} \left[ e^{-\frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t)} - 1 \right] \right\} \left\{ 1 + \frac{c}{N} \left[ e^{-\frac{j}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_j(t) \sigma_i(t)} - 1 \right] \right\}}$$

$$= \prod_{i < j} \left\{ \overline{1 - \frac{c}{N} \left[ \frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t) + \frac{1}{2c^2} \left[ \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t) \right]^2 + \mathcal{O}(c^{-\frac{3}{2}}) \right]} \right\} \\ \times \left\{ \overline{1 - \frac{c}{N} \left[ \frac{i}{c} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_j(t) \sigma_i(t) + \frac{1}{2c^2} \left[ \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_j(t) \sigma_i(t) \right]^2 + \mathcal{O}(c^{-\frac{3}{2}}) \right]} \right\}$$

(in which the horizontal bars of the two constituent lines are to be read as connected)

$$= \prod_{i < j} \exp \left( \overline{-\frac{i}{N} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t [\hat{h}_i(t) \sigma_j(t) + \hat{h}_j(t) \sigma_i(t)] - \frac{1}{2cN} \left[ \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_i(t) \sigma_j(t) \right]^2} \right. \\ \left. - \frac{1}{2cN} \left[ \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \sum_t \hat{h}_j(t) \sigma_i(t) \right]^2 + \mathcal{O}\left(\frac{1}{N\sqrt{c}}\right) + \mathcal{O}\left(\frac{c}{N^2}\right) \right).$$

Again we separate in the exponent the terms where  $\mu = \nu$  in the quadratic term (being of the form  $\sum_{\mu\nu} \dots$ ), and the terms with  $\mu = 1$ , and use the definitions (109)–(111):

$$\mathcal{F}[\dots] = -i \sum_t a(t)k(t) - \frac{1}{2} \alpha \sum_{st} q(s,t)Q(s,t) + \mathcal{O}(c^{-\frac{1}{2}}) + \mathcal{O}(c/n) \\ + \frac{1}{N} \log \left\{ \overline{\exp \left( -\frac{i}{N} \sum_{\mu > 1} \sum_t \left[ \sum_i \xi_i^{\mu} \hat{h}_i(t) \right] \left[ \sum_j \xi_j^{\mu} \sigma_j(t) \right] \right)} \right. \\ \left. - \frac{1}{2cN} \sum_{i \neq j} \sum_{\mu \neq \nu} \xi_i^{\mu} \xi_j^{\mu} \xi_i^{\nu} \xi_j^{\nu} \sum_{st} \hat{h}_i(s) \sigma_j(s) \hat{h}_i(t) \sigma_j(t) \right\}.$$

The scaling arguments given in the symmetric case, based on our ‘condensed ansatz’, apply again, and tell us that the remaining terms with the disorder are of vanishing order in  $N$ . We have again shown that the disorder average (125) is, in leading order in  $N$ , of the form (112), with

*Asymmetric:*

$$\Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] = -i\mathbf{a} \cdot \mathbf{k} - \frac{1}{2} \alpha \sum_{st} q(s,t)Q(s,t). \tag{148}$$

5.4.2. *Extracting the physics from the saddle-point equations*

First we combine the above two results (147)–(148) in the following way (with  $\Delta = 1$  for symmetric dilution and  $\Delta = 0$  for asymmetric dilution):

$$\Phi[\mathbf{a}, \mathbf{k}, \mathbf{q}, \mathbf{Q}, \mathbf{K}] = -i\mathbf{a} \cdot \mathbf{k} - \frac{1}{2} \alpha \sum_{st} [q(s,t)Q(s,t) + \Delta K(s,t)K(t,s)]. \tag{149}$$

We can now work out Eqs. (119) and (120), and use (123) to express the result at the physical saddle-point in terms of the trio  $\{m(t), C(t, t'), G(t, t')\}$ . For the single-time observables this gives (as with the fully connected system)  $\hat{a}(t) = k(t)$  and  $\hat{k}(t) = a(t)$ ; for the two-time ones we find:

$$\hat{Q}(t, t') = -\frac{1}{2}i\alpha C(t, t'), \quad \hat{q}(t, t') = 0, \quad \hat{K}(t, t') = \alpha\Delta G(t, t').$$

We now observe that the remainder of the derivation followed for the fully connected network can be followed with only two minor adjustments to the terms generated by  $\hat{K}(t, t')$  and by  $\hat{Q}(t, t')$ :  $\alpha\mathbf{G}(\mathbf{1} - \mathbf{G})^{-1} \rightarrow \alpha\Delta\mathbf{G}$  in the retarded self-interaction, and  $(\mathbf{1} - \mathbf{G})^{-1}\mathbf{C}(\mathbf{1} - \mathbf{G}^\dagger)^{-1} \rightarrow \mathbf{C}$  in the covariance of the Gaussian noise in the effective single neuron problem. This results in the familiar saddle-point equations (133) for an effective single neuron problem, with state probabilities (134) equivalent to the dynamics  $\text{Prob}[\sigma(t+1) = \pm 1] = \frac{1}{2}[1 \pm \tanh[\beta h(t)]]$ , and in which  $\pi_0(\sigma(0)) = \frac{1}{2}[1 + \sigma(0)m_0]$  and

$$\begin{aligned} h(t|\{\sigma\}, \{\phi\}) &= m(t) + \theta(t) + \alpha\Delta \sum_{t' < t} G(t, t')\sigma(t') + \alpha^{\frac{1}{2}}\phi(t), \\ P[\{\phi\}] &= \frac{e^{-\frac{1}{2}\sum_{t,t'}\phi(t)\mathbf{C}^{-1}(t,t')\phi(t')}}{(2\pi)^{(t_m+1)/2}\det^{\frac{1}{2}}\mathbf{C}}. \end{aligned} \quad (150)$$

#### 5.4.3. Physics of networks with asymmetric dilution

Asymmetric dilution corresponds to  $\Delta = 0$ , i.e. there is no retarded self-interaction, and the response function no longer plays a role. In (150) we now only retain  $h(t|\dots) = m(t) + \theta(t) + \alpha^{\frac{1}{2}}\phi(t)$ , with  $\langle\phi^2(t)\rangle = C(1, 1) = 1$ . We now find (141) simply giving

$$\begin{aligned} m(t+1) &= \sum_{\sigma(0)\dots\sigma(t)} \pi_0(\sigma(0)) \int \{d\phi\} P[\{\phi\}] \tanh[\beta h(t|\dots)] \\ &\quad \times \prod_{s=0}^{t-1} \frac{1}{2} [1 + \sigma(s+1) \tanh[\beta h(s|\dots)]] \\ &= \int \mathbf{D}z \tanh[\beta(m(t) + \theta(t) + z\sqrt{\alpha})]. \end{aligned} \quad (151)$$

Apparently this is the one case where the simple Gaussian dynamical law (95) is exact at all times. Similarly, for  $t > t'$  Eqs. (142)–(144) for correlation and response functions reduce to

$$\begin{aligned} C(t, t') &= \int \frac{d\phi_a d\phi_b \exp\left(-\frac{\frac{1}{2}\phi_a^2 + \phi_b^2 - 2C(t-1, t'-1)\phi_a\phi_b}{1 - C^2(t-1, t'-1)}\right)}{2\pi\sqrt{1 - C^2(t-1, t'-1)}} \tanh[\beta(m(t-1) + \theta(t-1) + \phi_a\sqrt{\alpha})] \\ &\quad \times \tanh[\beta(m(t'-1) + \theta(t'-1) + \phi_b\sqrt{\alpha})], \end{aligned} \quad (152)$$

$$G(t, t') = \beta \delta_{t, t'+1} \left\{ 1 - \int \mathbf{D}z \tanh^2[\beta(m(t-1) + \theta(t-1) + z\sqrt{\alpha})] \right\}. \tag{153}$$

Let us also inspect the stationary state  $m(t) = m$ , for  $\theta(t) = 0$ . One easily proves that  $m = 0$  as soon as  $T > 1$ , using  $m^2 = \beta m \int_0^m dk [1 - \int \mathbf{D}z \tanh^2[\beta(k + z\sqrt{\alpha})]] \leq \beta m^2$ . A continuous bifurcation occurs from the  $m = 0$  state to an  $m > 0$  state when  $T = 1 - \int \mathbf{D}z \tanh^2[\beta z\sqrt{\alpha}]$ . A parameterization of this transition line in the  $(\alpha, T)$ -plane is given by

$$T(x) = 1 - \int \mathbf{D}z \tanh^2(zx), \quad \alpha(x) = x^2 T^2(x), \quad x \geq 0.$$

For  $\alpha = 0$  we just get  $m = \tanh(\beta m)$  so  $T_c = 1$ . For  $T = 0$  we obtain the equation  $m = \text{erf}[m/\sqrt{2\alpha}]$ , giving a continuous transition to  $m > 0$  solutions at  $\alpha_c = 2/\pi \approx 0.637$ . The remaining question concerns the nature of the  $m = 0$  state. Inserting  $m(t) = \theta(t) = 0$  (for all  $t$ ) into (152) tells us that  $C(t, t') = f[C(t-1, t'-1)]$  for  $t > t' > 0$ , with ‘initial conditions’  $C(t, 0) = m(t)m_0$ , where

$$f[C] = \int \frac{d\phi_a d\phi_b}{2\pi\sqrt{1-C^2}} \exp\left(-\frac{1}{2} \frac{\phi_a^2 + \phi_b^2 - 2C\phi_a\phi_b}{1-C^2}\right) \tanh[\beta\sqrt{\alpha}\phi_a] \tanh[\beta\sqrt{\alpha}\phi_b].$$

In the  $m = 0$  regime we have  $C(t, 0) = 0$  for any  $t > 0$ , inducing  $C(t, t') = 0$  for any  $t > t'$ , due to  $f[0] = 0$ . Thus we conclude that  $C(t, t') = \delta_{t, t'}$  in the  $m = 0$  phase, i.e. this phase is paramagnetic rather than of a spin-glass type. The resulting phase diagram is given in Fig. 9, together with that of symmetric dilution (for comparison).

#### 5.4.4. Physics of networks with symmetric dilution

This is the more complicated situation. In spite of the extreme dilution, the interaction symmetry makes sure that the spins still have a sufficient number of common ancestors for complicated correlations to build up in finite time. We have

$$h(t|\{\sigma\}, \{\phi\}) = m(t) + \theta(t) + \alpha \sum_{t' < t} G(t, t') \sigma(t') + \alpha^{\frac{1}{2}} \phi(t), \tag{154}$$

$$P[\{\phi\}] = \frac{\exp(-\frac{1}{2} \sum_{t, t'} \phi(t) \mathbf{C}^{-1}(t, t') \phi(t'))}{(2\pi)^{(m+1)/2} \det^{\frac{1}{2}} \mathbf{C}}.$$

The effective single neuron problem (134) and (154) is found to be exactly of the form found also for the Gaussian model in [1] (which, in turn, maps onto the parallel dynamics SK model [28]) with the synapses  $J_{ij} = J_0 \xi_i \xi_j / N + J z_{ij} / \sqrt{N}$  (in which the  $z_{ij}$  are symmetric zero-average and unit-variance Gaussian variables, and  $J_{ii} = 0$  for all  $i$ ), with the identification:

$$J \rightarrow \sqrt{\alpha}, \quad J_0 \rightarrow 1$$

(this becomes clear upon applying the generating functional analysis to the Gaussian model, page limitations prevent me from explicit demonstration here). Since one can

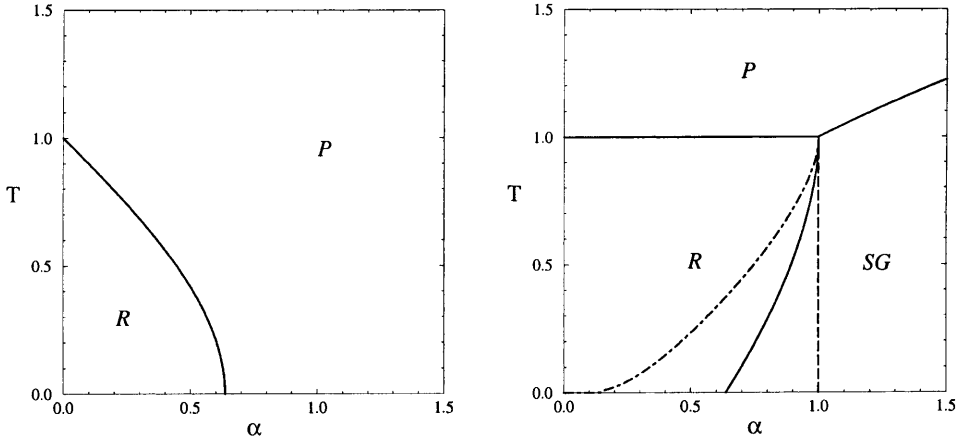


Fig. 9. Phase diagrams of extremely diluted attractor networks. Left: asymmetric dilution,  $c_{ij}$  and  $c_{ji}$  are statistically independent. Solid line: continuous transition, separating a non-recall (paramagnetic) region (P) from a recall region (R). The line reaches  $T = 0$  at  $\alpha_c = 2/\pi \approx 0.637$ . Right: symmetric dilution,  $c_{ij} = c_{ji}$  for all  $i, j$ . Solid lines: continuous transitions, separating a nonrecall region (P) from a recall region (R), for  $\alpha < 1$ , and from a spin-glass region (SG), for  $\alpha > 1$ . Dashed-dotted line: the AT instability. The R→SG line (calculated within RS) reaches  $T = 0$  at  $\alpha_c^{RS} = 2/\pi \approx 0.637$ . In RSB the latter is replaced by a new (dashed) line, giving a new storage capacity of  $\alpha_c^{RSB} = 1$ .

show that for  $J_0 > 0$  the parallel dynamics SK model gives the same equilibrium state as the sequential one, we can now immediately write down the stationary solution of our dynamic equations which corresponds to the FDT regime, with  $q = \lim_{\tau \rightarrow \infty} \lim_{t \rightarrow \infty} C(t, t + \tau)$ :

$$q = \int Dz \tanh^2[\beta(m + z\sqrt{\alpha q})], \quad m = \int Dz \tanh[\beta(m + z\sqrt{\alpha q})]. \quad (155)$$

These are neither identical to the equations for the fully connected Hopfield model, nor to those of the asymmetrically diluted model. Using the equivalence with the (sequential and parallel) SK model [28] we can immediately translate the phase transition lines as well, giving:

	<i>SK model</i>	<i>Symmetrically diluted model</i>
$P \rightarrow F$ :	$T = J_0$ for $J_0 > J$	$T = 1$ for $\alpha < 1$
$P \rightarrow SG$ :	$T = J$ for $J_0 < J$	$T = \sqrt{\alpha}$ for $\alpha > 1$
$F \rightarrow SG$ (in RS):	$T = J_0(1 - q)$ for $T < J_0$	$T = 1 - q$ for $T < 1$
$F \rightarrow SG$ (in RSB):	$J_0 = J$ for $T < J$	$\alpha = 1$ for $T < \sqrt{\alpha}$
AT -- line:	$T^2 = J^2 \int Dz \cosh^{-4} \beta[J_0 m + Jz\sqrt{q}]$	$T^2 = \alpha \int Dz \cosh^{-4} \beta[m + z\sqrt{\alpha q}]$

where  $q = \int Dz \tanh^2 \beta[m + z\sqrt{\alpha q}]$ . Note that for  $T = 0$  we have  $q = 1$ , so that the equation for  $m$  reduces to the one found for asymmetric dilution:  $m = \text{erf}[m/\sqrt{2\alpha}]$ . However, the phase diagram shows that the line  $F \rightarrow SG$  is entirely in the RSB

region and describes physically unrealistic re-entrance (as in the SK model), so that the true transition must be calculated using Parisi's replica-symmetry breaking (RSB) formalism (see e.g. [29]), giving here  $\alpha_c = 1$ .

The extremely diluted models analyzed here were first studied in [30] (asymmetric dilution) and [23] (symmetric dilution). We note that it is not extreme dilution which is responsible for a drastic simplification in the macroscopic dynamics in the complex regime (i.e. close to saturation), but rather the absence of synaptic symmetry. Any finite degree of synaptic symmetry, whether in a fully connected or in an extremely diluted attractor network, immediately generates an effective retarded self-interaction in the dynamics, which is ultimately responsible for highly nontrivial 'glassy' dynamics.

## 6. Epilogue

In this paper I have tried to explain how the techniques from nonequilibrium statistical mechanics can be used to solve the dynamics of recurrent neural networks. As in the companion paper on statics in this volume, I have restricted myself to relatively simple models, where one can most clearly see the potential and restrictions of these techniques, without being distracted by details. I have dealt with binary neurons and graded response neurons, and with fully connected and extremely diluted networks, with symmetric but also with nonsymmetric synapses. Similar calculations could have been done for neuron models which are not based on firing rates, such as coupled oscillators or integrate-and-fire type ones, see e.g. [31]. My hope is that bringing together methods and results that have so far been mostly scattered over research papers, and by presenting these in a uniform language to simplify comparison, I will have made the area somewhat more accessible to the interested outsider.

At another level I hope to have compensated somewhat for the incorrect view that has sometimes surfaced in the past that statistical mechanics applies only to recurrent networks with symmetric synapses, and is therefore not likely to have a lasting impact on neuro-biological modeling. This was indeed true for equilibrium statistical mechanics, but it is not true for nonequilibrium statistical mechanics. This does not mean that there are no practical restrictions in the latter; the golden rule of there not being any free lunches is obviously also valid here. Whenever we wish to incorporate more biological details in our models, we will have to reduce our ambition to obtain exact solutions, work much harder, and turn to our computer at an earlier stage. However, the practical restrictions in dynamics are of a quantitative nature (equations tend to become more lengthy and messy), rather than of a qualitative one (in statics the issue of detailed balance decides whether or not we can at all start a calculation). The main stumbling block that remains is the issue of spatial structure. Short-range models are extremely difficult to handle, and this is likely to remain so for a long time. In statistical mechanics the state of the art in short-range models is to be able to identify phase transitions, and calculate critical exponents, but this is generally not the type of information one is interested in when studying the operation of recurrent neural networks.



Yet, since dynamical techniques are still far less hampered by the need to impose biologically dubious (or even unacceptable) model constraints than equilibrium techniques, and since there are now well-established and efficient methods and techniques to obtain model solutions in the form of macroscopic laws for large systems (some are exact, some are useful approximations), the future in the statistical mechanical analysis of biologically more realistic recurrent neural networks is clearly in the nonequilibrium half of the statistical mechanics playing field.

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