Dynamics of the batch minority game with inhomogeneous decision noise

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We study the dynamics of a version of the batch minority game, with random external information and with different types of inhomogeneous decision noise (additive and multiplicative), using generating functional techniques à la De Dominicis. The control parameters in this model are the ratio $\alpha = p/N$ of the number p of possible values for the external information over the number N of trading agents, and the statistical properties of the agents' decision noise parameters. The presence of decision noise is found to have the general effect of damping macroscopic oscillations, which explains why in certain parameter regions it can effectively reduce the market volatility, as observed in earlier studies. In the limit $N \to \infty$ we (i) solve the first few time steps of the dynamics (for any α), (ii) calculate the location α_c of the phase transition (signaling the onset of anomalous response), and (iii) solve the statics for $\alpha > \alpha_c$. We find that α_c is not sensitive to additive decision noise, but we arrive at nontrivial phase diagrams in the case of multiplicative noise. Our theoretical results find excellent confirmation in numerical simulations.

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I. INTRODUCTION

One of the more recent application domains of equilibrium and nonequilibrium statistical mechanics is the analysis of simplified models describing large markets of competing traders (or agents). One such model, which in spite of its apparent simplicity was found to exhibit highly nontrivial behavior and has, therefore, attracted much attention, is the so-called minority game (MG) [1,2], which is a variation on the so-called El-Farol bar problem [3], which mimics, in a highly idealized fashion, a market of speculators attempting to profit by buying when most others wish to sell or selling when others wish to buy, without individual knowledge of their fellows but only of their collective consequences and external information available to all. An extensive overview of the literature on the MG and its many variations and extensions can be found in [4]. The striking feature of the MG, clearly observed in numerical simulations, is the nontrivial dependence of the market volatility (measuring global fluctuations) on the dimensionality of the information supplied to the agents (which is defined as the relative number α of different values which the information can take). For large α the volatility approaches the value corresponding to random trading, and the system is ergodic. As α is reduced, the volatility is also found to decrease beneath random, which is indicative of a more efficient market, where agents have "learned" to improve the effectiveness of their selection of trading strategies. A further decrease of α will at some critical point α_c force the system to undergo a phase transition to a highly nonergodic regime, where both a high-volatility state and a low-volatility state can emerge, dependent on initial conditions (this was only appreciated later).

In the original minority game, the information supplied to the agents consisted of the actual history of the market. However, it was soon realized [5] that the dynamics of the MG remains largely unaltered if, instead of the true history of the market, random information is supplied to the agents; given α, the only relevant condition is that all agents must be given the *same* information (whether sensible or otherwise). This led to a considerable simplification of theoretical approaches to the MG, since it reduced the process to a Markovian one. A further generalization of the game was the introduction of agents' decision noise [6], which was shown not only to improve worse than random behavior but also, more surprisingly, to be able to make it better than random for $\alpha < \alpha_c$.[1] The study [6] was followed by a number of papers aiming to develop a solvable statistical-mechanical theory, either by using decision noise to "regularize" the stochastic equations and replace these by deterministic ones (followed by an equilibrium analysis of the ergodic regime, built on the construction and exploitation of a Lyapunov function) [10,11], or by concentrating further on analysis of the stochastic equations themselves [12]. Since the MG process does not obey detailed balance, such studies (which also involved different implementations of the decision noise), proved to be hard, and their results partly controversial [13,14] (especially with regard to the questions of whether and when the stochastic MG equations can be replaced by suitable deterministic ones).

More recently, in [15] the analysis of the MG was approached from a different angle: all problems and debates regarding microscopic determinism were simply avoided by re-defining the MG dynamics directly in the form of discrete-time deterministic equations, without decision noise (the so-called "batch minority game"). This allowed for an exact solution of the model using generating functional techniques à la De Dominicis [16], which was found to be in excellent agreement with numerical simulations, and which (due to it's dynamical nature) even applied to the nonergodic regime. The present study, which can be regarded as a natural followup on [15], achieves the following objectives. We gener-

[1]Using a phenomenological theory for the volatility, based on so-called "crowd-anticrowd" cancellations [7], this effect was later partially explained in [8,9].

alize the "thermal minority game" such as to allow different agents to have different levels of decision noise. This introduces inhomogeneity into the agent population, as in, e.g., [9], which leads to interesting new phenomena and phase diagrams. We generalize and apply the (exact) formalism of [15] (which was developed for the deterministic MG) to the case of having inhomogeneous decision noise, within the context of the discrete-time deterministic ("batch") equations. All our theoretical results are shown to find excellent confirmation in extensive numerical simulations.

II. MODEL DEFINITIONS

The minority game involves N agents, labeled with Latin indices i,j,k, etc. At each round l of the game, all agents act on the basis of the same piece of external information $I(l)$. In the original model [1] the history of the actual market was used as the information given to the agents. In view of the observation in [5] that random information is equally efficacious, we consider here that at each round l the agents are given the information $I(l)=I_{\mu(l)}$, where for each l the label $\mu(l)$ is chosen randomly and independently from $p=\alpha N$ possible values, i.e., $\mu(l)\in\{1,\ldots,\alpha N\}$. To determine how to convert the external information into a trading decision, each agent i has at his or her disposal S strategies $\boldsymbol{R}_{ia}=(R_{ia}^1,\ldots,R_{ia}^{\alpha N})\in\{-1,1\}^{\alpha N}$; $a\in\{1,\ldots,S\}$. Each component R_{ia}^{μ} is selected randomly and independently from $\{-1,1\}$ before the start of the game, with uniform probabilities, and remains fixed throughout the game. The strategies introduce quenched disorder into the model. Each strategy $\boldsymbol{R}_{ia}$ of every agent i is given an initial valuation or point score $p_{ia}(0)$. In the deterministic version of the game, given a choice $\mu(l)$ made for the information presented at round l, every agent i selects the strategy which for trader i has the highest valuation at that point in time, i.e., the strategy with label $\tilde{a}_i(l)=\arg\max p_{ia}(l)$, and subsequently makes a binary bid $b_i(l)=R_{i\tilde{a}_i(l)}^{\mu(l)}$. The (rescaled) total bid at stage l is defined as $A(l)=N^{-1/2}\Sigma_i b_i(l)$. Each agent subsequently updates the payoff values of each of his or her strategies a on the basis of comparing the bid which would have resulted from playing that strategy with the actual outcome:

$$p_{ia}(l+1)=p_{ia}(l)-R_{ia}^{\mu(l)}A(l). \tag{1}$$

The minus sign in this expression ensures that strategies that would have produced a minority decision are rewarded. Since the qualitative behavior of the market fluctuations was found to be very much the same for all nonextensive numbers of strategies per agent larger than one [1,2], we restrict our discussion to the $S=2$ model, where the equations can be simplified upon introducing for each agent the instantaneous difference between the two strategy valuations, $q_i(l)=[p_{i1}(l)-p_{i2}(l)]/2$, as well as the average strategy $\boldsymbol{\omega}_i=(\boldsymbol{R}_{i1}+\boldsymbol{R}_{i2})/2$ and the difference between the strategies $\boldsymbol{\xi}_i=(\boldsymbol{R}_{i1}-\boldsymbol{R}_{i2})/2$. The actually selected strategy in round l can now be written explicitly as a function of a binary variable $s_i(l)=\pm 1$, which in the original model takes the value $s_i(l)=\mathrm{sgn}[q_i(l)]$, that is $\boldsymbol{R}_{i\tilde{a}_i(l)}=\boldsymbol{\omega}_i+s_i(l)\boldsymbol{\xi}_i$, and the evolution of the difference will now be given by

$$q_i(l+1)=q_i(l)-\xi_i^{\mu(l)}\left[\Omega^{\mu(l)}+\frac{1}{\sqrt{N}}\sum_j \xi_j^{\mu(l)}s_j(l)\right], \tag{2}$$

with $\boldsymbol{\Omega}=N^{-1/2}\Sigma_j\boldsymbol{\omega}_j\in\mathbb{R}^{\alpha N}$.

In the so-called thermal minority game [6,12], the deterministic decision rule $s_j(l)=\mathrm{sgn}[q_j(l)]$ is replaced by a stochastic recipe. Two different choices were proposed. In [6] the probability of the choice $s_i(l)=\pm 1$ was taken proportional to $[1\mp e^{\beta q_i(l)}]^{-1}$, whereas in [12] also the alternative choice $[1\mp e^{\beta\,\mathrm{sgn}[q_i(l)]}]^{-1}$ was considered; as we shall show later, these are, respectively, examples of additive and multiplicative noise. In both cases the stochasticity is parametrized by a single control parameter $\beta=T^{-1}$, the "inverse temperature." In both cases a nonzero T was shown in simulations to lead to a reduction in the volatility for $\alpha<\alpha_c$, to a value lower than that for purely random decision choices. For $\alpha>\alpha_c$ the additive noise choice was shown to have no consequence for the long-time volatility behavior [12–14], reducing to the noiseless value, whereas multiplicative noise still had an effect (increasing the volatility) [12].

Here we generalize this idea further by allowing different traders to have different levels of stochasticity in their decision making (see also [9], where the MG with two such levels was studied). We will consider decision noise of the general form

$$s_j(l)=\sigma[q_j(l),z_j(l)|T_j], \tag{3}$$

in which the $z_j(l)$ are independent and zero-average random numbers, described by some symmetric distribution $P(z)$ which is normalized according to $\int dz\ P(z)=\int dz P(z)z^2=1$. The function $\sigma[q,z|T]\in\{-1,1\}$ is chosen to interpolate smoothly via a single control parameter T between $\sigma[q,z|0]=\mathrm{sgn}[q]$ for $T=0$ and $\sigma[q,z|\infty]=\pm 1$ (randomly, with equal probabilities) for $T=\infty$, so that T provides a measure of the degree of stochasticity in the traders' decision making (with random choice in the case $q=0$). Typical examples are additive and multiplicative noise definitions such as

$$\sigma[q,z|T]=\mathrm{sgn}[q+Tz],\quad \text{additive}, \tag{4}$$

$$\sigma[q,z|T]=\mathrm{sgn}[q]\ \ \mathrm{sgn}[1+Tz],\quad \text{multiplicative}. \tag{5}$$

In the first case (4) the noise has the potential to be overruled by the so-called "frozen" agents [17], who have $q_i(t)\sim\tilde{q}_i t$ for $t\to\infty$ [13–15]. In the second case the decision noise will even retain its effect for frozen agents (if they exist). The above definitions represent situations, where for $T_i>0$, a trader i need not always use his or her "best" strategy; for $T_i\to 0$ we revert back to the deterministic model. The impact of the multiplicative noise (5) can be characterized by the monotonic function

$$\lambda(T)=\int dz P(z)\mathrm{sgn}[1+Tz], \tag{6}$$

with $\lambda(0)=1$ and $\lambda(\infty)=0$. For example, for a Gaussian $P(z)$ one has $\lambda(T)=\text{erf}[1/\sqrt{2}T]$. The two versions of the minority game studied in [6,12] correspond to the forms (4) and (5) with $P(z)=\frac{1}{2}K[1-\tanh^2(Kz)]$ and $T_i=T$ for all i.

There has been much discussion on the derivation of the "correct" continuous microdynamics (see, e.g., [12–14,18]). Here we circumvent that controversy and debate by directly employing a generating functional method [16] for the dynamics and by discussing the "batch" version of the problem [15] from the outset, rather than the original "on-line" version. In the batch version, rather than modifying the $\{q_i\}$ after every observation of an individual piece of external information, they are modified according to the *average* effect of the possible choices for the external information:

$$q_i(l+1)=q_i(l)-\frac{1}{p}\sum_{\mu=1}^{p}\xi_i^\mu\left[\Omega^\mu+\frac{1}{\sqrt{N}}\sum_j \xi_j^\mu s_j(l)\right], \tag{7}$$

giving

$$q_i(t+1)=q_i(t)-h_i-\sum_j J_{ij}\sigma[q_j(t),z_j(t)|T_j], \tag{8}$$

where $J_{ij}=2N^{-1}\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j$ and $h_i=2N^{-\frac{1}{2}}\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}$. The specific choice of time scaling in Eq. (8) has been made for later convenience. The batch dynamics (8) has the advantage of being sufficiently simple and transparent to allow for a straightforward exact dynamical solution of the model, using generating functional techniques [15]. The process (8) is not exactly equivalent to Eq. (2), not even for $N\to\infty$ (see [19] for the generating functional analysis of the on-line dynamics and its relation to the batch alternative), but it does present qualitatively similar features [12].

The magnitude of the market fluctuations, or volatility, is given by

$$\sigma^2=\left\langle\frac{1}{p}\sum_\mu (A^\mu)^2\right\rangle_{\boldsymbol{z}}-\left\langle\frac{1}{p}\sum_\mu A^\mu\right\rangle_{\boldsymbol{z}}^2, \tag{9}$$

where $A^\mu=N^{-\frac{1}{2}}\Sigma_i[\omega_i^\mu+s_i\xi_i^\mu]$ and where $\langle\ \dots\ \rangle_{\boldsymbol{z}}$ denotes an average over the random numbers $\{z_i\}$. One easily derives

$$\left\langle\frac{1}{p}\sum_\mu A^\mu\right\rangle_{\boldsymbol{z}}=\frac{1}{\alpha N\sqrt{N}}\sum_i \langle s_i\rangle_{\boldsymbol{z}}\sum_\mu \xi_i^\mu+O(N^{-1/2}), \tag{10}$$

$$\left\langle\frac{1}{p}\sum_\mu (A^\mu)^2\right\rangle_{\boldsymbol{z}}=\frac{1}{2}+\frac{1}{\alpha N}\left[\sum_i h_i\langle s_i\rangle_{\boldsymbol{z}}+\frac{1}{2}\sum_{ij}J_{ij}\langle s_is_j\rangle_{\boldsymbol{z}}\right]+\mathcal{O}(N^{-1/2}). \tag{11}$$

Purely random trading corresponds to $\langle p^{-1}\Sigma_\mu A^\mu\rangle_{\boldsymbol{z}}=0$ and $\sigma^2=1$. Following [15] we also define the volatility matrix $\Xi_{tt'}$:

$$\Xi_{tt'}=\left\langle\frac{1}{p}\sum_\mu\left[A_t^\mu-\left\langle\frac{1}{p}\sum_\nu A_t^\nu\right\rangle_{\boldsymbol{z}}\right]\left[A_{t'}^\mu-\left\langle\frac{1}{p}\sum_\nu A_{t'}^\nu\right\rangle_{\boldsymbol{z}}\right]\right\rangle_{\boldsymbol{z}}, \tag{12}$$

which measures the temporal correlations of the market fluctuations. Note that $\sigma_t^2=\Xi_{tt}$. In the case where the average bid $\langle A\rangle$ is zero (as in the present model), the volatility measures the efficiency of the market.

III. GENERATING FUNCTIONAL ANALYSIS

The canonical tool to deal with the dynamics of the present problem is generating functional analysis à la De Dominicis [16], which allows one to carry out the disorder average (here the average over all strategies) and take the $N\to\infty$ limit exactly. The final result of the analysis is a set of closed equations, which can be interpreted as describing the dynamics of an effective "single agent" [16,21]. Due to the disorder in the process, this single agent will acquire an effective "memory," i.e., he or she will evolve according to a nontrivial non-Markovian stochastic process. Here we will follow closely the steps taken in [15], and we refer to the latter paper for full details of the calculation. In contrast to the situation in [15], for the present noisy version of the game one finds a microscopic transition probability density operator $W(\boldsymbol{q}|\boldsymbol{q}')$:

$$W(\boldsymbol{q}|\boldsymbol{q}')=\int\frac{d\hat{\boldsymbol{q}}}{(2\pi)^N}\times\left\langle\exp\left[\sum_i i\hat{q}_i\left(q_i-q_i'+h_i+\sum_j J_{ij}s_j'\right)\right]\right\rangle_{\boldsymbol{z}}, \tag{13}$$

with the short hand $s_j'=\sigma[q_j',z_j|T_j]$. The moment generating functional for a stochastic process of the present type is defined as

$$Z[\boldsymbol{\psi}]=\left\langle\exp\left[i\sum_t\sum_i \psi_i(t)q_i(t)\right]\right\rangle=\int\prod_t [d\boldsymbol{q}(t)W(\boldsymbol{q}(t+1)|\boldsymbol{q}(t))]\,p_0[\boldsymbol{q}(0)]\times\exp\left[i\sum_t\sum_i\psi_i(t)q_i(t)\right]. \tag{14}$$

Derivation of the generating functional with respect to the conjugate variables $\boldsymbol{\psi}$ generates all moments of $\boldsymbol{q}$ at arbitrary times. Upon introducing the two shorthands:

$$w_t^\mu = \frac{\sqrt{2}}{\sqrt{N}} \sum_i \hat{q}_i(t)\xi_i^\mu, \quad x_t^\mu = \frac{\sqrt{2}}{\sqrt{N}} \sum_i s_i(t)\xi_i^\mu, \tag{15}$$

as well as $D\boldsymbol{q} = \Pi_{it}[dq_i(t)/\sqrt{2\pi}]$, $D\boldsymbol{w} = \Pi_{\mu t}[dw_t^\mu/\sqrt{2\pi}]$, and $D\boldsymbol{x} = \Pi_{\mu t}[dx_t^\mu/\sqrt{2\pi}]$ (with similar definitions for $D\hat{\boldsymbol{q}}$, $D\hat{\boldsymbol{w}}$, and $D\hat{\boldsymbol{x}}$, respectively), the generating functional takes the following form:

$$Z[\boldsymbol{\psi}] = \int D\boldsymbol{w}\, D\hat{\boldsymbol{w}}\, D\boldsymbol{x}\, D\hat{\boldsymbol{x}} \times \exp\left\{ i \sum_{t\mu} [\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu + \sqrt{2} w_t^\mu (\Omega^\mu + x_t^\mu)] \right\} \times \int D\boldsymbol{q}\, D\hat{\boldsymbol{q}}\, p_0[\boldsymbol{q}(0)] \left\langle \exp\left[(-i\sqrt{2}/\sqrt{N}) \times \sum_{\mu i} \xi_i^\mu \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)] \right] \right\rangle_{\boldsymbol{z}} \times \exp\left\{ i \sum_{ti} [\hat{q}_i(t)[q_i(t+1) - q_i(t) - \theta_i(t)] + \psi_i(t) q_i(t)] \right\}, \tag{16}$$

where, as in [15], we introduced external "forces" $\theta_i(t)$ to generate response functions.

To describe typical behavior, and in view of the self-averaging character of the large N limit, at this stage we average over over the explicit choices of the quenched random parameters $\{\boldsymbol{R}\}$. These averages are not affected in any way by the introduction of the noise variables $\{z_i\}$ or the independent temperatures T_i, and the further procedure of [15] still applies here, generating the dynamical order parameters $C_{tt'} = N^{-1}\Sigma_i s_i(t) s_i(t')$, $K_{tt'} = N^{-1}\Sigma_i s_i(t)\hat{q}_i(t')$, and $L_{tt'} = N^{-1}\Sigma_i \hat{q}_i(t)\hat{q}_i(t')$ and their conjugates. For times which are small compared with N and for simple initial conditions of the form $p_0(\boldsymbol{q}) = \Pi_i p_0(q_i)$ one thus finds

$$\overline{Z[\boldsymbol{\psi}]} = \int [DC D\hat{C}][DK D\hat{K}][DL D\hat{L}] \exp\{N[\Psi + \Phi + \Omega] + \mathcal{O}(N^0)\}. \tag{17}$$

The $\mathcal{O}(N^0)$ term in the exponent is independent of the fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$. The three relevant exponents in Eq. (17) are given by the following expressions:

$$\Psi = i \sum_{tt'} [\hat{C}_{tt'} C_{tt'} + \hat{K}_{tt'} K_{tt'} + \hat{L}_{tt'} L_{tt'}], \tag{18}$$

$$\Phi = \alpha \ln\left[\int Dw\, D\hat{w}\, Dx\, D\hat{x} \exp\left\{ i \sum_t [\hat{w}_t w_t + \hat{x}_t x_t + w_t x_t] \right\} \times \exp\left\{ -\frac{1}{2} \sum_{tt'} [w_t w_{t'} + \hat{w}_t L_{tt'} \hat{w}_{t'} + 2\hat{x}_t K_{tt'} \hat{w}_{t'} + \hat{x}_t C_{tt'} \hat{x}_{t'}] \right\} \right]. \tag{19}$$

$$\Omega = \frac{1}{N} \sum_i \ln\left[\int Dq\, D\hat{q}\; p_0[q(0)] \times \exp\left\{ i \sum_t \{\hat{q}(t)[q(t+1) - q(t) - \theta_i(t)] + \psi_i(t) q(t)\} - i \sum_{tt'} \hat{q}(t) \hat{L}_{tt'} \hat{q}(t') \right\} \times \left\langle \exp\left\{ -i \sum_{tt'} [s_i(t)\hat{C}_{tt'} s_i(t') + s_i(t)\hat{K}_{tt'}\hat{q}(t')] \right\} \right\rangle_{\boldsymbol{z}} \right]. \tag{20}$$

Here $s_i(t) = \sigma[q(t), z_t | T_i]$ and the average $\langle\ \rangle_{\boldsymbol{z}}$ has now been reduced to a single site (but many time) one $\langle g[z_1, z_2, \ldots]\rangle_{\boldsymbol{z}} = \int \Pi_t [dz_t P(z_t)] g[z_1, z_2, \ldots]$. Following [15] we have also introduced the short hands $Dq = \Pi_t[dq(t)/\sqrt{2\pi}]$, $Dw = \Pi_t[dw_t/\sqrt{2\pi}]$, and $Dx = \Pi_t[dx_t/\sqrt{2\pi}]$ (with similar definitions for $D\hat{q}$, $D\hat{w}$, and $D\hat{x}$). Note that all the quantities appearing in Eq. (17) are macroscopic; all the microscopic variables have been integrated out.

IV. THE SADDLE-POINT EQUATIONS

We can now evaluate Eq. (17) by saddle-point integration, in the limit $N \to \infty$. We define $G_{tt'} = -iK_{tt'}$. Taking derivatives with respect to the generating fields and using the normalization $\overline{Z[\boldsymbol{0}]} = 1$ then gives (at the physical saddle point) the usual identifications

$$C_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle s_i(t) s_i(t')\rangle}, \tag{21}$$

$$G_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial}{\partial \theta_i(t')} \overline{\langle s_i(t)\rangle}, \tag{22}$$

and also

$$L_{tt'} = -\lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial^2}{\partial \theta_i(t) \partial \theta_i(t')} 1 = 0. \tag{23}$$

Putting $\psi_i(t) = 0$ (they are no longer needed) and $\theta_i(t) = \theta(t)$ then simplifies Eq. (20) to

$$\Omega=\int_0^\infty dT W(T)\ln\Bigg[\int Dq\,D\hat{q}\;\; p_0(q(0))\times\exp\Bigg\{i\sum_t \hat{q}(t)[q(t+1)-q(t)-\theta(t)]-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')\Bigg\}\times\Bigg\langle\exp\Bigg\{-i\sum_{tt'}[s(t)\hat{C}_{tt'}s(t')+s(t)\hat{K}_{tt'}\hat{q}(t')]\Bigg\}\Bigg\rangle_{\mathbf{z}}\Bigg], \tag{24}$$

in which now $s(t)=\sigma[q(t),z_t|T]$, and where $W(T)$ denotes the distribution of local noise strengths:

$$W(T)=\lim_{N\to\infty}\frac{1}{N}\sum_i \delta(T-T_i). \tag{25}$$

Extremization of the extensive exponent $N[\Psi+\Phi+\Omega]$ of Eq. (17) with respect to $\{C,\hat{C},K,\hat{K},L,\hat{L}\}$ gives the saddle-point equations

$$C_{tt'}=\langle s(t)s(t')\rangle_\star \quad G_{tt'}=\frac{\partial\langle s(t)\rangle_\star}{\partial\theta(t')}, \tag{26}$$

$$\hat{C}_{tt'}=\frac{i\partial\Phi}{\partial C_{tt'}},\quad \hat{K}_{tt'}=\frac{i\partial\Phi}{\partial K_{tt'}},\quad \hat{L}_{tt'}=\frac{i\partial\Phi}{\partial L_{tt'}}. \tag{27}$$

The effective single-trader averages $\langle\ \dots\rangle_\star$, generated by taking derivatives of Eq. (20), are defined as

$$\langle f[\{q,s\}]\rangle_\star=\int_0^\infty dT W(T)\times\left\{\frac{\int Dq\;\langle M[\{q,s\}]f[\{q,s\}]\rangle_{\mathbf{z}}}{\int Dq\;\langle M[\{q,s\}]\rangle_{\mathbf{z}}}\right\}, \tag{28}$$

$$M[\{q,s\}]=p_0(q(0))\exp\Bigg[-i\sum_{tt'}s(t)\hat{C}_{tt'}s(t')\Bigg]\times\int D\hat{q}\exp\Bigg[-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')\Bigg]\times\exp\Bigg\{i\sum_t\hat{q}(t)\Bigg[q(t+1)-q(t)-\theta(t)-\sum_{t'}\hat{K}^T_{tt'}s(t')\Bigg]\Bigg\}. \tag{29}$$

Upon elimination of the trio $\{\hat{C},\hat{K},\hat{L}\}$ via Eq. (27) we obtain exact closed equations for the disorder-averaged correlation and response functions in the $N\to\infty$ limit, Eq. (26), with the effective single-trader measure (29). One recovers the theory of [15] upon putting $W(T)=\delta(T)$.

Since the introduction of decision noise into the dynamics has only affected the term Ω (24), compared to the analysis in [15], the simplifications of the term Φ (reflecting the statistical properties of the trading strategies) derived in [15] apply unaltered, so that at the physical saddle point we again find

$$\hat{L}=-\frac{1}{2}i\alpha(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}, \tag{30}$$

$$\hat{K}^T=-\alpha(\mathbb{1}+G)^{-1}, \tag{31}$$

$$\hat{C}=0, \tag{32}$$

where A^T denotes the transpose of the matrix A, and the entries of the matrix D are given by $D_{tt'}=1+C_{tt'}$. We now find our effective single-trader measure $M[\{q,s\}]$ of Eq. (29) reducing further to

$$M[\{q,s\}]=p_0(q(0))\int D\hat{q}\exp\Bigg\{-\frac{1}{2}\alpha\sum_{tt'}\hat{q}(t)\times[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}]_{tt'}\hat{q}(t')\Bigg\}\times\exp\Bigg\{i\sum_t\hat{q}(t)\Bigg[q(t+1)-q(t)-\theta(t)+\alpha\sum_{t'}(\mathbb{1}+G)^{-1}_{tt'}s(t')\Bigg]\Bigg\}. \tag{33}$$

For a given value of T, this describes a stochastic single-agent process of the form

$$q(t+1)=q(t)-\alpha\sum_{t'\leq t}(\mathbb{1}+G)^{-1}_{tt'}\sigma[q(t'),z_{t'}|T]+\theta(t)+\sqrt{\alpha}\,\eta(t). \tag{34}$$

Causality ensures that $(\mathbb{1}+G)^{-1}_{tt'}=0$ for $t'>t$. The variable z_t represents the original single-trader decision noise, with $\langle z_t\rangle=0$ and $\langle z_t z_{t'}\rangle=\delta_{tt'}$, and $\eta(t)$ is a disorder-generated Gaussian noise with zero mean and with temporal correlations given by $\langle\eta(t)\eta(t')\rangle=\Sigma_{tt'}$:

$$\Sigma=(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^T)^{-1}. \tag{35}$$

The correlation and response functions (21) and (22) are the dynamic order parameters of the problem, and must be solved self-consistently from the closed equations

$$C_{tt'}=\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle_\star, \tag{36}$$

$$G_{tt'}=\frac{\partial}{\partial\theta(t')}\langle\sigma[q(t),z_t|T]\rangle_\star, \tag{37}$$

which, following Eq. (28), also now involve averaging over the distribution of the noise strengths T. Note that $M[\{q,s\}]$ as given by Eq. (33) is normalized, i.e., $\int DqM[\{q,s\}]=1$, so the associated averages reduce to

$$\langle f[\{q,s\}]\rangle_\star=\int_0^\infty dTW(T)\int Dq\ \ \langle M[\{q,s\}]f[\{q,s\}]\rangle_{\mathbf{z}}. \tag{38}$$

The calculation in [15] of the disorder-averaged average bid and the volatility matrix (including the single-time volatility $\sigma_t^2=\Xi_{tt}$) still hold, and hence

$$\lim_{N\to\infty}\overline{\langle A\rangle_t}=0,\qquad \lim_{N\to\infty}\overline{\Xi}_{tt'}=\frac{1}{2}\Sigma_{tt'}. \tag{39}$$

V. THE FIRST TIME STEPS

For the first few time steps one can calculate quite easily the order parameters (correlation and response functions) and the volatility, from Eq. (33), using the simplifications that follow from causality, such as

$$[G^n]_{tt'}=0 \quad \text{for} \ \ t'>t-n. \tag{40}$$

At $t=0$ this immediately allows us to conclude that $\Sigma_{00}=D_{00}=2$. We now obtain from Eq. (33) the joint statistics at times $t=1$, given a value for T:

$$p(q(1)|q(0))=\int dz_0P(z_0)\frac{\exp\{-\{q(1)-q(0)-\theta(0)+\alpha\sigma[q(0),z_0|T]\}^2/4\alpha\}}{2\sqrt{\alpha\pi}}. \tag{41}$$

Equation (41) allows us to calculate C_{10} and G_{10}, although the presence of the decision noise induces expressions which are significantly more difficult to work out explicitly than those of the noise-free case in [15], and which will depend on the choice made for $\sigma[q,z|T]$:

$$\begin{aligned}C_{10}=&\int_0^\infty dTW(T)\int dz_0\,dz_1P(z_0)P(z_1)\int dq(0)p_0(q(0))\\&\times\int\frac{dq(1)}{2\sqrt{\alpha\pi}}\exp\{-\{q(1)-q(0)-\theta(0)\\&+\alpha\sigma[\{q(0),z_0|T]\}^2/4\alpha\}\\&\times\sigma[q(0),z_0|T]\ \ \sigma[q(1),z_1|T],\end{aligned} \tag{42}$$

$$\begin{aligned}G_{10}=&\int_0^\infty dTW(T)\int dz_0\,dz_1P(z_0)P(z_1)\int dq(0)p_0(q(0))\\&\times\int\frac{dq(1)}{2\sqrt{\alpha\pi}}\exp\{-\{q(1)-q(0)-\theta(0)\\&+\alpha\sigma[q(0),z_0|T]\}^2/4\alpha\}\\&\times\frac{\partial}{\partial q(1)}\sigma[q(1),z_1|T].\end{aligned} \tag{43}$$

We can now move to the next time step, again using Eq. (40), where we need the noise covariances Σ_{11} and Σ_{10}:

$$\Sigma_{10}=1+C_{10}-2G_{10}, \tag{44}$$

$$\Sigma_{11}=2-2G_{10}[1+C_{01}]+2[G_{10}]^2. \tag{45}$$

This procedure can, in principle, be repeated for an arbitrary number of time steps.

We now specialize to the case where the game is initialized in a *tabula rasa* manner, i.e., $p(q(0))=\delta[q_0]$, and where we have no perturbation fields, i.e., $\theta(t)=0$. Now, also upon using the symmetry of $P(z)$, we can reduce the above results to

$$\begin{aligned}C_{10}=&\int_0^\infty dTW(T)\int dzP(z)\int\frac{dq}{4\sqrt{\alpha\pi}}e^{-[q+\alpha]^2/4\alpha}\\&\times\{\sigma[q,z|T]-\sigma[-q,-z|T]\},\end{aligned} \tag{46}$$

$$\begin{aligned}G_{10}=&\int_0^\infty dTW(T)\int dzP(z)\int\frac{dq}{4\sqrt{\alpha\pi}}e^{-[q+\alpha]^2/4\alpha}\\&\times\frac{\partial}{\partial q}\{\sigma[q,z|T]-\sigma[-q,-z|T]\}.\end{aligned} \tag{47}$$

Inspection of these expressions for large and small α, and for the specific choices (4) and (5) reveals the following. For $\alpha\to\infty$ one finds

$$\lim_{\alpha\to\infty}G_{10}=0,\qquad \lim_{\alpha\to\infty}\Sigma_{11}=2, \tag{48}$$

for both noise types. The order parameters C_{10} and Σ_{10}, in contrast, are sensitive to the type of noise chosen. For additive noise of the form (4) one has

$$\lim_{\alpha\to\infty}C_{10}=-1,\qquad \lim_{\alpha\to\infty}\Sigma_{10}=0, \tag{49}$$

whereas for multiplicative noise (5) one has

$$\lim_{\alpha\to\infty}C_{10}=-\int_0^\infty dTW(T)\lambda(T), \tag{50}$$

$$\lim_{\alpha\to\infty}\Sigma_{10}=1-\int_0^\infty dT W(T)\lambda(T). \tag{51}$$

In both cases the negativity of C_{10} shows that the *tabula-rasa* initialized system immediately enters an oscillation, with the $q_i(1)$ on average having opposite sign to the corresponding $q_i(0)$. Initially, additive noise is found not to play a role, and the effective disorder-generated noise components $\eta(t)$ decorrelate, compared with the deterministic case of [15]. Multiplicative noise, on the other hand, is seen to retain an impact, even for short times and large α, and to cause a reduction of the oscillation amplitude.

Now we turn to small α, where we make the choice $P(z)=(2\pi)^{-1/2}e^{-z^2/2}$ in order to work out integrals explicitly. For additive noise (4) we find

$$C_{10}=-\frac{\alpha\sqrt{2}}{\sqrt{\pi}}\int_0^\infty dT W(T)T^{-1}+\mathcal{O}(\alpha^{3/2}), \tag{52}$$

$$G_{10}=\frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty dT W(T)T^{-1}+\mathcal{O}(\alpha^{1/2}) \tag{53}$$

[provided the above integrals over T exist; if they do not, we revert back to the leading orders of the $T=0$ case [15], i.e., $C_{10}=\mathcal{O}(\sqrt{\alpha})$ and $G_{10}=\mathcal{O}(1/\sqrt{\alpha})$]. Combination with the expressions (44) and (45) shows that in leading order

$$\eta(1)=\left(\frac{1}{2}-G_{10}\right)\eta(0)+w+\cdots \tag{54}$$

in which w is a zero-average Gaussian variable, independent of $\eta(0)$, with variance $\langle w^2\rangle=3/2$. Hence we find from the effective single spin equation (34)

$$q(1)=\sqrt{\alpha}\quad\eta(0)+\mathcal{O}(\alpha), \tag{55}$$

$$q(2)=\sqrt{\alpha}\left[\left(\frac{3}{2}-G_{10}\right)\eta(0)+w\right]+\mathcal{O}(\alpha). \tag{56}$$

We observe, as in [15], that for small α and additive decision noise, the first two time steps are driven predominantly by the disorder-generated noise component in Eq. (34). However, whether this noise component starts oscillating in sign is, in the case of decision noise, crucially dependent on the distribution of temperatures; only when $\int dT W(T)\ \ T^{-1}$ is sufficiently large should we expect the system to enter the high-volatility state. For multiplicative noise, on the other hand, we arrive for small α at the leading orders

$$C_{10}=-\frac{\sqrt{\alpha}}{\sqrt{\pi}}\int_0^\infty dT W(T)\lambda(T)+\mathcal{O}(\alpha^{3/2}), \tag{57}$$

$$G_{10}=\frac{1}{\sqrt{\alpha\pi}}\int_0^\infty dT W(T)\lambda(T)+\mathcal{O}(\sqrt{\alpha})+\cdots \tag{58}$$

Here the oscillation is much stronger (provided we do not scale the temperatures with α). Combination with the expressions (44) and (45) shows that in leading order the disorder-generated noise not only drives the oscillation, but is also being amplified by a factor of the order of $\alpha^{-1/2}$:

$$\eta(1)=-G_{10}\eta(0)+\mathcal{O}(\alpha^0). \tag{59}$$

The effective single-trader equation subsequently gives

$$q(1)=\sqrt{\alpha}\,\eta(0)+\mathcal{O}(\alpha), \tag{60}$$

$$q(2)=-\frac{\eta(0)}{\sqrt{\pi}}\int_0^\infty dT W(T)\lambda(T)+\mathcal{O}(\sqrt{\alpha}). \tag{61}$$

Thus, for small α and *tabula rasa* initialization[2] additive decision noise has the most drastic effect on the dynamics, changing the leading order of the relevant observables by a factor of $\sqrt{\alpha}$ (in contrast to multiplicative noise).

VI. STATIONARY STATE FOR $\alpha>\alpha_C[W(T)]$

If the game has reached a time-translation invariant stationary state without long-term memory, then $G_{tt'}=G(t-t')$, $C_{tt'}=C(t-t')$, and $\Sigma_{tt'}=\Sigma(t-t')$. In this section we assume that the stationary state is one without anomalous response, i.e., $\lim_{\tau\to\infty}\sum_{t\leq\tau}G(t)=k$ exists. The lower limit of such behavior in α defines $\alpha_c(W(T))$.

In a stationary state one generally finds agents who change strategy frequently, but also agents who consistently use the same strategy. For the latter frozen agents, the values of q_i will grow linearly in time. We follow [15] and separate the two groups by introducing $\tilde{q}_i(t)=q_i(t)/t$; frozen agents will be those for whom $\lim_{t\to\infty}\tilde{q}_i(t)\neq 0$, and the quantity $\phi=\lim_{\epsilon\to 0}\lim_{t\to\infty}\langle\theta[|\tilde{q}(t)|-\epsilon]\rangle_\star$ gives the fraction of frozen agents in the original N-agent system, for $N\to\infty$. Transformation of the process (34) gives, for a given T

$$\tilde{q}_T(t)=\frac{1}{t}\tilde{q}_T(1)+\frac{\sqrt{\alpha}}{t}\sum_{t'<t}\eta(t')-\frac{\alpha}{t}\sum_{t'<t}\sum_{t''\leq t'}(\mathbb{1}+G)^{-1}_{t't''}\ \sigma[\tilde{q}_T(t''),z_{t''}|T]. \tag{62}$$

We now define $\tilde{q}_T=\lim_{t\to\infty}\tilde{q}_T(t)$ (assuming this limit exists) and take the limit $t\to\infty$ in Eq. (62), giving

$$\tilde{q}_T=-\frac{\alpha}{1+k}m_T+\sqrt{\alpha}\,\eta, \tag{63}$$

with the time averages $m_T=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}\sigma[q_t,z_t|T]$ and $\eta=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leq\tau}\eta(t)$. The variance of η follows from Eq. (35):

[2]Note that the small α expansions in this section are made for fixed $W(T)$; the observed behavior is likely to be different when $W(T)$ is allowed to scale with α.

$$\langle\eta^2\rangle=(1+k)^{-2}\left[1+\lim_{\tau,\tau'\to\infty}\frac{1}{\tau\tau'}\sum_{t\leqslant\tau}\sum_{t'\leqslant\tau'}C_{tt'}\right]$$
$$=[1+\langle m_T^2\rangle_\star]/(1+k)^2. \tag{64}$$

Note that $\langle m_T^2\rangle_\star=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leqslant\tau}C(t)=c$.

The integrated response (or static susceptibility) $k=\lim_{\tau\to\infty}\Sigma_{t\leqslant\tau}G(t)$ is also calculated along the lines of [15]. One writes the response function as $G_{tt'}=\alpha^{-1/2}\langle\partial\sigma[q(t),z_t|T]/\partial\eta(t')\rangle_\star$. Integration by parts in this expression generates

$$\langle\partial\sigma[q(t),z_t|T]/\partial\eta(t')\rangle_\star=\sum_{t''}\Sigma^{-1}_{t't''}\langle\sigma[q(t),z_t|T]\eta(t'')\rangle_\star, \tag{65}$$

and hence

$$\sqrt{\alpha}\sum_{t''}\langle\eta(t')\eta(t'')\rangle G^T_{t''t}=\langle\sigma[q(t),z_t|T]\eta(t')\rangle_\star. \tag{66}$$

Averaging over the two times t and t' now gives, in a stationary state without anomalous response, the following:

$$\langle m_T\eta\rangle_\star=k\sqrt{\alpha}\langle\eta^2\rangle. \tag{67}$$

Inserting the variance $\langle\eta^2\rangle$, as given in Eq. (64), then gives the general relation

$$\langle\eta m_T\rangle_\star=\frac{k\sqrt{\alpha}(1+c)}{(1+k)^2}. \tag{68}$$

A. Additive decision noise

In the case of additive decision noise (4) we have $\sigma[q,z|T]=\text{sgn}[q+zT]$. The effective agent is frozen if $\tilde{q}\neq 0$, in which case $m_T=\text{sgn}[\tilde{q}_T]$. This solves Eq. (63), if and only if $|\eta|>\sqrt{\alpha}/(1+k)$. If $|\eta|<\sqrt{\alpha}/(1+k)$, on the other hand, the agent is not frozen; now $\tilde{q}_T=0$ and $m_T=(1+k)\eta/\sqrt{\alpha}$. As a result, we can calculate $c=\langle m_T^2\rangle_\star$ and the fraction $\phi=\langle\theta[|\eta|-\sqrt{\alpha}/(1+k)]\rangle=1-\text{erf}[\sqrt{\alpha/2(1+c)}]$ of frozen agents exactly as in in the case [15] without decision noise, giving the deterministic [i.e., $W(T)=\delta(T)$] result

$$c=1-\left(1-\frac{1+c}{\alpha}\right)\text{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right]-2\sqrt{\frac{1+c}{2\pi\alpha}}e^{-\alpha/2(1+c)}. \tag{69}$$

We use Eq. (68) and calculate the covariance $\langle\eta m_T\rangle_\star$ exactly as in [15]. The final result is

$$\frac{1}{k}=\frac{\alpha}{\text{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right]}-1, \tag{70}$$

with the value of c to be determined by solving Eq. (69). We find exactly the same transition point $\alpha_c\approx 0.337\,40$, signaling the divergence of the integrated response k, as was found in the noise-free case, in accord with earlier numerical observations [20,12–14] and theoretical predictions [18].

Numerical simulations of the (batch) dynamics of the present model (which we will not present here, for brevity) confirm quite convincingly that, upon measuring objects such as c or ϕ, in the case of additive decision noise, one indeed exactly recovers the graphs of [15], without any dependence on the noise parameters. This, however, will turn out to be quite different in the case of multiplicative noise.

B. Homogeneous multiplicative decision noise

Next we turn to the case of multiplicative noise (5), at first with the simplest distribution $W(T)=\delta(T-\overline{T})$, where $\sigma[q,z|T]=\text{sgn}[q]\text{sgn}[1+\overline{T}z]$, and where $m_T=\lim_{\tau\to\infty}\tau^{-1}\Sigma_{t\leqslant\tau}\,\text{sgn}[q_T(t)]\text{sgn}[1+\overline{T}z_t]$. Since there is now only one noise strength in the system, $\overline{T}$, we may drop the subscripts T for variables such as $q(t)$ or m, without danger of confusion. For a frozen agent one now finds

$$m=\lambda(\overline{T})\text{sgn}[\tilde{q}]. \tag{71}$$

This solves Eq. (63) when $|\eta|>\sqrt{\alpha}\lambda(\overline{T})/(1+k)$. If $|\eta|<\sqrt{\alpha}\lambda(\overline{T})/(1+k)$, on the other hand, the agent is not frozen; now $\tilde{q}_T=0$ and $m=(1+k)\eta/\sqrt{\alpha}$. We can again calculate $c=\langle m^2\rangle_\star$ self-consistently, upon distinguishing between the two possibilities:

$$c=\lambda^2(\overline{T})\left\langle\theta\left[|\eta|-\frac{\sqrt{\alpha}\lambda(\overline{T})}{1+k}\right]\right\rangle+\frac{(1+k)^2}{\alpha}\left\langle\theta\left[\frac{\sqrt{\alpha}\lambda(\overline{T})}{1+k}-|\eta|\right]\eta^2\right\rangle. \tag{72}$$

Working out the Gaussian integrals describing the statics of η, with variance as given by (64), subsequently gives

$$c=\lambda^2(\overline{T})-\left[\lambda^2(\overline{T})-\frac{1+c}{\alpha}\right]\text{erf}\left[\sqrt{\frac{\alpha\lambda^2(\overline{T})}{2(1+c)}}\right]-2\lambda(\overline{T})\sqrt{\frac{1+c}{2\pi\alpha}}e^{-\alpha\lambda^2(\overline{T})/2(1+c)}. \tag{73}$$

From this equation the value of c is solved numerically. The fraction ϕ of frozen agents is given by

$$\phi=\left\langle\theta\left[|\eta|-\frac{\sqrt{\alpha}\lambda(\overline{T})}{1+k}\right]\right\rangle=1-\text{erf}\left[\sqrt{\frac{\alpha\lambda^2(\overline{T})}{2(1+c)}}\right]. \tag{74}$$

We calculate the remaining object $\langle\eta m\rangle_\star$ in Eq. (68) by again distinguishing between frozen and nonfrozen agents and by using the two identities $m=\lambda(T)\text{sgn}[\eta]$ (for frozen agents) and $m=\eta(1+k)/\sqrt{\alpha}$ (for fickle ones), both of which follow from Eq. (63), giving

$$\langle \eta m\rangle_\star = \lambda(\overline{T})\left\langle \theta\left[|\eta| - \frac{\sqrt{\alpha}\lambda(\overline{T})}{1+k}\right]|\eta|\right\rangle + \frac{1+k}{\sqrt{\alpha}}\left\langle \theta\left[\frac{\sqrt{\alpha}\lambda(\overline{T})}{1+k} - |\eta|\right]\eta^2\right\rangle = \frac{1+c}{(1+k)\sqrt{\alpha}}\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2(\overline{T})}{2(1+c)}}\right].$$

Insertion into Eq. (68), together with Eq. (64), then gives the desired expression for the integrated response:

$$\frac{1}{k} = \frac{\alpha}{\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2(T)}{2(1+c)}}\right]} - 1, \tag{75}$$

with the value of c to be determined by solving Eq. (74). Equivalently, using Eq. (74) we find, as in the $T=0$ case [15]

$$k = \frac{1-\phi}{\alpha - 1 + \phi}. \tag{76}$$

The integrated response k is positive and finite, and our solution exact, for $\alpha > \alpha_c(W(T))$. At $\alpha_c(W(T))$ one finds that k diverges; this transition is, as for $T=0$, found to happen when the fraction of fickle agents equals α [10]. Finally, according to Eqs. (73) and (75) we can write $\alpha_c(W(T))$ as $\alpha_c(W(T)) = \mathrm{erf}[x]$, where x is the solution of the transcendental equation

$$\lambda^2(\overline{T})\left\{\mathrm{erf}[x] - 1 + \frac{1}{x\sqrt{\pi}}e^{-x^2}\right\} = 1. \tag{77}$$

Equivalently, we can write our transition line explicitly in terms of the inverse error function as

$$\lambda(\overline{T}_c) = \left\{\alpha_c + \frac{e^{-[\mathrm{erf}^{\mathrm{inv}}[\alpha_c]]^2}}{\mathrm{erf}^{\mathrm{inv}}[\alpha_c]\sqrt{\pi}} - 1\right\}^{-1/2}, \tag{78}$$

where $\lambda(\overline{T}) \in [0,1]$, see Eq. (6).

In Figs. 1 and 2 we show the solution of Eq. (73) and the corresponding fraction ϕ of frozen agents as functions of α, together with the values for c and ϕ as obtained by carrying out numerical simulations of the batch minority game (8) with homogeneous multiplicative decision noise. The two figures for c and ϕ both show excellent agreement between theory and experiment above $\alpha_c(W(T))$. One observes that, in addition to a reduction in the persistent correlation, another effect of the introduction of multiplicative decision noise is an overall increase in the fraction of frozen agents. This is consistent with our solution of the first few iteration steps, where introducing decision noise had the effect of dampening the oscillations. In Fig. 3 we show the system's phase diagrams for $W(T) = \delta(T-\overline{T})$, defined by the transition line, where $k=\infty$. This line is given by the solution of Eq. (78) in the case of multiplicative noise, and by

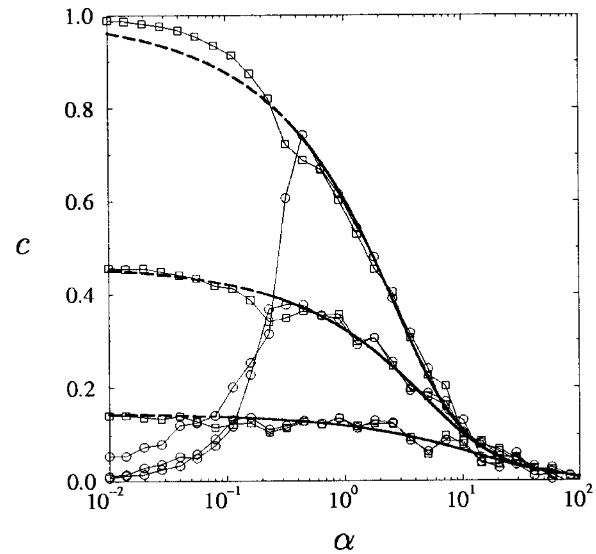

FIG. 1. The persistent correlation c as a function of $\alpha = p/N$, for multiplicative noise with $W(T) = \delta(T-\overline{T})$ and different choices of the noise strength ($\overline{T}=0$, 1, 2 from top to bottom). Connected markers: individual simulation runs, with $pN = \alpha N^2 = 10^6$ and homogeneous initial conditions where $q_i(0) = q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Thick solid curves for $\alpha > \alpha_c(W(T))$: analytical predictions for homogeneous multiplicative decision noise. For $\alpha < \alpha_c(W(T))$, where they should no longer be correct, they have been continued as thick-dashed line. For additive decision noise our theory predicts independence of $\overline{T}$ for $\alpha > \alpha_c(W(T))$, i.e., c as given by the $\overline{T}=0$ curve of multiplicative noise.

$\alpha_c(W(T)) \approx 0.337\,40$ [i.e., the value corresponding to $\lambda(0) = 1$] for additive noise. Below $\alpha_c(W(T))$ our simulations show, as has been observed and reported earlier for the deterministic case, that in the anomalous response region the stationary state reached by the system depends critically on the initial conditions. For small values of the $|q_i(0)|$ (i.e., weak initial strategy preferences) the system enters a high-volatility state with low c and ϕ, whereas for large values of the $|q_i(0)|$ (i.e., strong initial strategy preferences) the system enters a low-volatility state with large c and ϕ.

C. Inhomogeneous multiplicative decision noise

Finally we turn to the more complicated situation of multiplicative noise (5) with arbitrary distributions. For a frozen agent and for a given value of T one has

$$m_T = \lambda(T)\mathrm{sgn}[\tilde{q}]. \tag{79}$$

As before, this solves Eq. (63) if $|\eta| > \sqrt{\alpha}\lambda(T)/(1+k)$, whereas for $|\eta| < \sqrt{\alpha}\lambda(T)/(1+k)$ the agent is fickle, i.e., $\tilde{q}_T = 0$ and $m_T = (1+k)\eta/\sqrt{\alpha}$. According to Eqs. (36) and (37), the calculation of persistent order parameters will now also involve averaging over the noise distribution. Since the macroscopic dynamics turns out to depend on T only via $\lambda(T)$, it will be advantageous to define $w(\lambda) = \int_0^\infty dT W(T)\delta(\lambda - \lambda(T))$, or

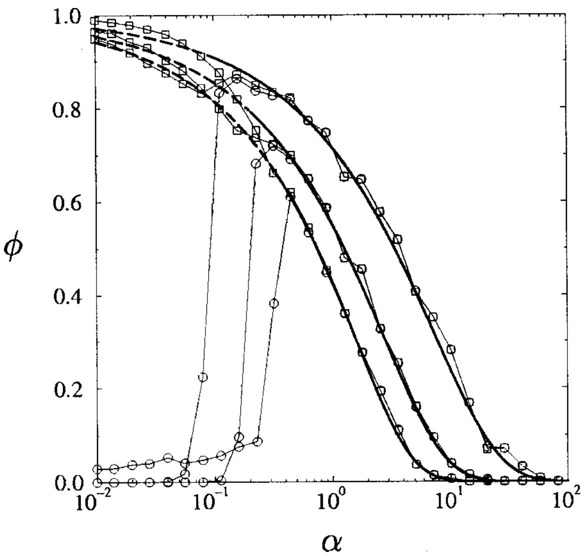

FIG. 2. The asymptotic fraction of frozen agents ϕ as a function of $\alpha=p/N$, for multiplicative noise with $W(T)=\delta(T-\overline{T})$ and different choices of the noise strength ($\overline{T}=0$, 1, 2 from top to bottom). Markers: individual simulation runs, with $pN=\alpha N^2=10^6$ and homogeneous initial conditions, where $q_i(0)=q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Thick-solid curves for $\alpha>\alpha_c(W(T))$: analytical predictions for homogeneous multiplicative decision noise. For $\alpha<\alpha_c(W(T))$, where they should no longer be correct, they have been continued as thick-dashed lines. For additive decision noise our theory predicts independence of $\overline{T}$, i.e., ϕ as given by the $\overline{T}=0$ curve of multiplicative noise.

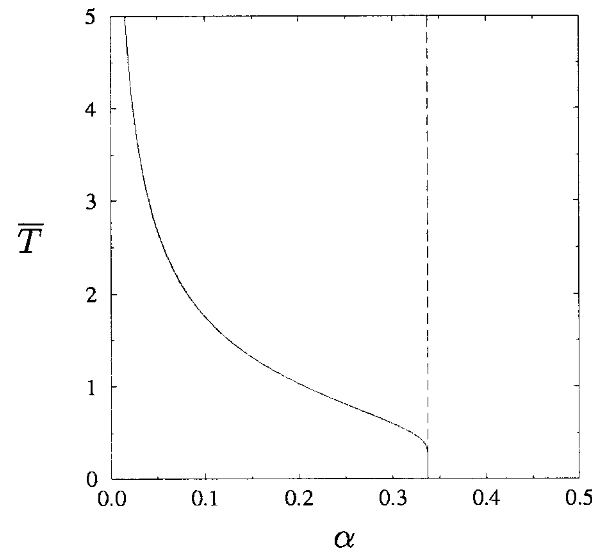

FIG. 3. Phase diagram in the $(\alpha,1-\lambda(\overline{T}))$ plane for homogeneous multiplicative noise, i.e., $W(T)=\delta(T-\overline{T})$. The solid line separates a nonergodic phase with anomalous response (left) from an ergodic one without anomalous response (right). For additive noise our theory predicts the $\overline{T}$-independent transition given by the dashed line.

$$w(\lambda)=\int_0^\infty dT\, W(T)\,\delta\left(\lambda-\int dz\, P(z)\mathrm{sgn}[1+Tz]\right). \tag{80}$$

Here $\lambda\in[0,1]$, with $\lambda=0$ reflecting $T\to\infty$ contributions and $\lambda=1$ reflecting $T\to 0$ ones. Now we may write

$$\begin{aligned}c&=\int_0^1 d\lambda\, w(\lambda)\left\{\lambda^2\left\langle \theta\left[|\eta|-\frac{\sqrt{\alpha}\lambda}{1+k}\right]\right\rangle\right.\\&\quad\left.+\frac{(1+k)^2}{\alpha}\left\langle \theta\left[\frac{\sqrt{\alpha}\lambda}{1+k}-|\eta|\right]\eta^2\right\rangle\right\}\\&=\int_0^1 d\lambda\, w(\lambda)\left\{\lambda^2-2\lambda\sqrt{\frac{1+c}{2\pi\alpha}}\,e^{-\alpha\lambda^2/2(1+c)}\right.\\&\quad\left.-\left[\lambda^2-\frac{1+c}{\alpha}\right]\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2}{2(1+c)}}\right]\right\}. \end{aligned}\tag{81}$$

From this equation the value of c is solved numerically. The fraction ϕ of frozen agents is given by

$$\phi=1-\int_0^1 d\lambda\, w(\lambda)\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2}{2(1+c)}}\right]. \tag{82}$$

We calculate the remaining object $\langle \eta m_T\rangle_\star$ in Eq. (68) by again distinguishing between frozen and nonfrozen agents and by using the two identities $m_T=\lambda(T)\mathrm{sgn}[\eta]$ (for frozen agents) and $m_T=\eta(1+k)/\sqrt{\alpha}$ (for the nonfrozen ones), both of which follow from Eq. (63), giving

$$\langle \eta m_T\rangle_\star=\frac{1+c}{(1+k)\sqrt{\alpha}}\int_0^1 d\lambda\, w(\lambda)\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2}{2(1+c)}}\right].$$

Insertion into Eq. (68), together with Eq. (64), then gives the desired expression for the integrated response:

$$\frac{1}{k}=\frac{\alpha}{\int_0^1 d\lambda\, w(\lambda)\mathrm{erf}\left[\sqrt{\frac{\alpha\lambda^2}{2(1+c)}}\right]}-1, \tag{83}$$

with the value of c to be determined by solving Eq. (81). Using Eq. (82) this can again be written in the familiar form (77), which suggests that the $k=\infty$ transition is of a geometrical nature.

Unless we revert back to uniform noise levels, a transformation like $\alpha_c(W(T))=\mathrm{erf}[x]$ will now no longer be helpful; to find the location of the phase transition one has to solve Eq. (81), together with the condition $k=\infty$. Upon putting $y^2=\alpha/2(1+c)$ one can, however, compactify these two coupled equations to

$$1=\int_0^1 d\lambda\, w(\lambda)\lambda^2\left\{\mathrm{erf}[y\lambda]-1+\frac{e^{-y^2\lambda^2}}{y\lambda\sqrt{\pi}}\right\}, \tag{84}$$

$$\alpha=\int_0^1 d\lambda\, w(\lambda)\mathrm{erf}[y\lambda]. \tag{85}$$

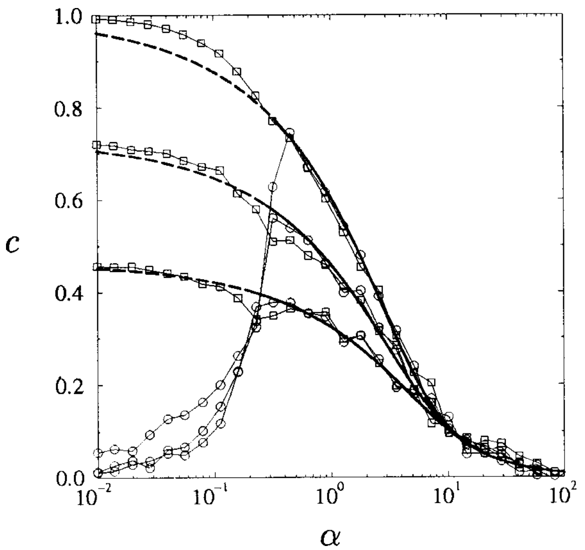

FIG. 4. The persistent correlation c as a function of $\alpha=p/N$, for multiplicative noise with $W(T')=\epsilon\delta(T'-T)+(1-\epsilon)\delta(T')$, for $T=1$ and different choices of the width ($\epsilon=0$, 0.5, 1 from top to bottom). Markers: individual simulation runs, with $pN=\alpha N^2=10^6$ and homogeneous initial conditions where $q_i(0)=q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Solid curves for $\alpha>\alpha_c(W(T))$: analytical predictions. For $\alpha<\alpha_c(W(T))$, where they should no longer be correct, they have been continued as dashed lines.

We will finally work out our equations describing the system with inhomogeneous multiplicative decision noise explicitly for the following simple bimodal distribution

$$W(T')=\epsilon\delta(T'-T)+(1-\epsilon)\delta(T'), \tag{86}$$

with $\epsilon\in[0,1]$. For $\epsilon=1$ we revert back to the homogeneous case studied earlier in this section; for $\epsilon=0$ we return to the model of [15]. Here we have

$$w(\lambda)=\epsilon\delta(\lambda-\lambda(T))+(1-\epsilon)\delta(\lambda-1), \tag{87}$$

with the function $\lambda(T)$ as defined in Eq. (6). The general Eqs. (81) and (82) from which to solve c and ϕ, reduce to

$$\begin{aligned} c=\epsilon\Bigg\{&\lambda^2(T)-2\lambda(T)\sqrt{\frac{1+c}{2\pi\alpha}}e^{-\alpha\lambda^2(T)/2(1+c)}\\ &-\left[\lambda^2(T)-\frac{1+c}{\alpha}\right]\operatorname{erf}\left[\sqrt{\frac{\alpha\lambda^2(T)}{2(1+c)}}\right]\Bigg\}\\ &+(1-\epsilon)\Bigg\{1-2\sqrt{\frac{1+c}{2\pi\alpha}}e^{-\alpha/2(1+c)}\\ &-\left[1-\frac{1+c}{\alpha}\right]\operatorname{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right]\Bigg\}, \end{aligned} \tag{88}$$

$$\phi=1-\epsilon\operatorname{erf}\left[\sqrt{\frac{\alpha\lambda^2(T)}{2(1+c)}}\right]-(1-\epsilon)\operatorname{erf}\left[\sqrt{\frac{\alpha}{2(1+c)}}\right]. \tag{89}$$

Similarly, the two coupled Eqs. (84) and (85) which define the phase transition, reduce to

$$\begin{aligned} 1=\epsilon\lambda^2(T)&\left\{\operatorname{erf}[y\lambda(T)]-1+\frac{e^{-y^2\lambda^2(T)}}{y\lambda(T)\sqrt{\pi}}\right\}\\ &+(1-\epsilon)\left\{\operatorname{erf}[y]-1+\frac{e^{-y^2}}{y\sqrt{\pi}}\right\}, \end{aligned} \tag{90}$$

$$\alpha=\epsilon\operatorname{erf}[y\lambda(T)]+(1-\epsilon)\operatorname{erf}[y]. \tag{91}$$

Note that for $T\to 0$ our transition line equations reduce once more to those of the noise-free case, as derived in [15], giving $\alpha_c\approx 0.33\,740$. For $T\to\infty$, in contrast, we find a strong dependence on ϵ (the fraction of traders who experience decision noise). In particular, there is a qualitative difference between $\epsilon<1$ and $\epsilon=1$ (where one of the two noise levels in the system becomes zero).

For $\epsilon=1$ we return to the case of uniform decision noise, and Eqs. (90) and (91) dictate that the transition line obeys $\alpha\to 0$ as $T\to\infty$. For $\epsilon<1$ (i.e., a nonzero fraction of the traders take decisions deterministically) on the other hand, we find for $T\to\infty$ the Eqs. (90) and (91) (which will now have a solution with finite y) reducing to

$$1=(1-\epsilon)\left\{\operatorname{erf}[y]-1+\frac{e^{-y^2}}{y\sqrt{\pi}}\right\}, \tag{92}$$

$$\alpha=(1-\epsilon)\operatorname{erf}[y]. \tag{93}$$

Equivalently,

$$\sqrt{\pi}\left[\frac{2-\epsilon-\alpha}{1-\epsilon}\right]\operatorname{erf}^{\text{inv}}\left[\frac{\alpha}{1-\epsilon}\right]=e^{-[\operatorname{erf}^{\text{inv}}[(\alpha/1-\epsilon)]]^2}. \tag{94}$$

The solution of this equation defines the point $\alpha_c(\epsilon,T=\infty)$, which obeys $\alpha_c(\epsilon<1,T=\infty)>0$ and $\alpha_c(1,T=\infty)=0$.

In Figs. 4 and 5 we show the (numerical) solution of Eq. (89) for the persistent correlation c, and the corresponding value for the fraction ϕ of frozen agents, as given by Eq. (90), as functions of α and for different choices of the parameters $\{T,\epsilon\}$, together with the corresponding values for c and ϕ, as obtained by carrying out numerical simulations. Here we have chosen Gaussian distributed $z_j(l)$, i.e., $\lambda(T)=\operatorname{erf}[1/T\sqrt{2}]$. As before, one observes excellent agreement between theory and experiment above α_c, and a strong dependence on initial conditions below α_c. Finally, in Fig. 6 we show, in the $(\alpha,\overline{T})$ plane, the system's phase diagram as defined by the $k=\infty$ transition line, obtained by solving numerically the coupled Eqs. (90) and (91), for different values of ϵ.

VII. STATIONARY VOLATILITY FOR $\alpha>\alpha_c(W(T))$

As in the noise-free case [15], one finds that the volatility matrix (12), which is to be calculated from expressions (35) and which in a stationary state is time-translation-invariant $\Xi_{tt'}=\Xi(t-t')$, generally involves both long-term and

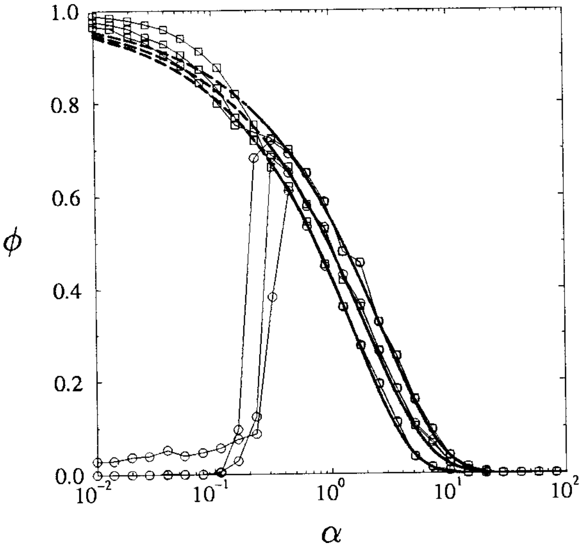

FIG. 5. The asymptotic fraction of frozen agents ϕ as a function of $\alpha=p/N$, for multiplicative noise with $W(T')=\epsilon\delta(T'-T)+(1-\epsilon)\delta(T')$, for $T=1$ and different choices of the width ($\epsilon=0$, 0.5, 1 from bottom to top). Markers: individual simulation runs, with $pN=\alpha N^2=10^6$ and homogeneous initial conditions where $q_i(0)=q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Solid curves for $\alpha>\alpha_c(W(T))$: analytical predictions. For $\alpha<\alpha_c(W(T))$, where they should no longer be correct, they have been continued as dashed lines.

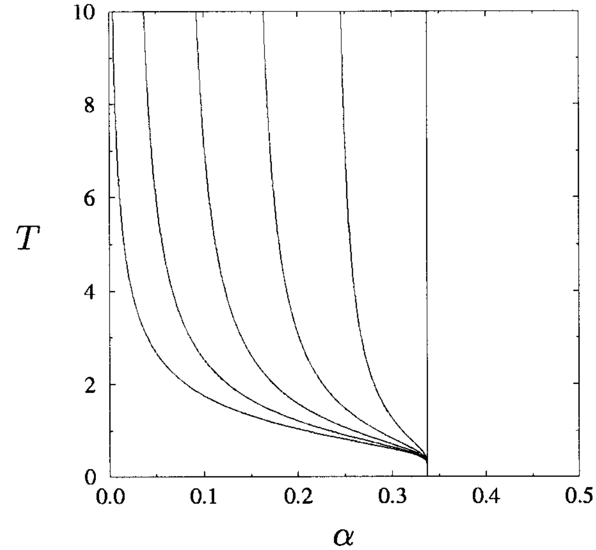

FIG. 6. The phase diagram for multiplicative noise with $W(T')=\epsilon\delta(T'-T)+(1-\epsilon)\delta(T')$ and Gaussian distributed z, shown in the (α,T) plane for different values of ϵ ($\epsilon\in\{0,0.2,0.4,0.6,0.8,1\}$, from right to left). For each value of ϵ, the solid line separates a nonergodic phase with anomalous response (left) from an ergodic one without anomalous response (right). For additive noise our theory predicts the T-independent transition given by the vertical line (i.e., the $\epsilon=0$ curve).

short-term fluctuations. Hence even the ordinary single-time stationary volatility $\sigma^2=\Xi(0)$ cannot be expressed in terms of the persistent order parameter c (or its relatives k and ϕ). Upon separating in the functions C and G the persistent from the nonpersistent terms, i.e., $C(t)=c+\tilde{C}(t)$ and $G(t)=\tilde{G}(t)$ (there is no persistent response for $\alpha>\alpha_c$) we find, as in [15]:

$$\sigma^2=\frac{1+c}{2(1+k)^2}+\lim_{\tau\to\infty}\frac{1}{2\tau}\sum_{u\leqslant\tau}\sum_{t't''}(\mathbb{1}+\tilde{G})^{-1}_{ut'}\tilde{C}_{t't''}(\mathbb{1}+\tilde{G}^T)^{-1}_{t''u}. \tag{95}$$

Obtaining an exact expression for σ^2 would require solving our coupled saddle-point Eqs. (36) and (37) for $C_{tt'}$ and $G_{tt'}$ for large times but finite temporal separations $t-t'$, hence in practice one has to resort to approximations. The approximation chosen in [10,11], for instance, is in our language equivalent to substituting

$$\langle\sigma[q_i(t),z_i(t)|T]\sigma[q_j(t),z_j(t)|T]\rangle \to \delta_{ij}+(1-\delta_{ij})\langle\sigma[q_i(t),z_i(t)|T]\rangle \times\langle\sigma[q_j(t),z_j(t)|T]\rangle$$

.

Here we will generalize to the case of decision noise (at least for the batch MG) the slightly more accurate approximation proposed in [15]. We will abbreviate the double averages $\int dT W(T)\langle\cdots\rangle$ as $\langle\langle\cdots\rangle\rangle$. In order to find the volatility we separate the correlations at stationarity in a frozen and a fickle contribution:

$$C(t-t')=\phi\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\rm fr} \tag{96}$$

$$+(1-\phi)\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\rm fi}, \tag{97}$$

which gives, using $\tilde{C}(t-t')=C(t-t')-c$, and upon rewriting the fickle contribution to the volatility

$$\sigma^2=\frac{1}{2(1+k)^2}+\lim_{\tau\to\infty}\frac{1-\phi}{2\tau}\times\sum_{u\leqslant\tau}\left\langle\left\langle\left[\sum_t(\mathbb{1}+\tilde{G})^{-1}_{ut}\sigma[q(t),z_t|T]\right]^2\right\rangle\right\rangle_{\rm fi}+\lim_{\tau\to\infty}\frac{\phi}{2\tau}\sum_{u\leqslant\tau}\sum_{tt'}(\mathbb{1}+\tilde{G})^{-1}_{ut}(\mathbb{1}+\tilde{G}^T)^{-1}_{t'u}\times\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\rm fr}. \tag{98}$$

The approximation of [15] consists of retaining in the contribution from fickle agents only the instantaneous $u=t$ terms, the rationale being that the $u\neq t$ ones represent, in the original single-trader equation, a retarded self-interaction, which is assumed to be significant only for frozen agents. Hence we obtain

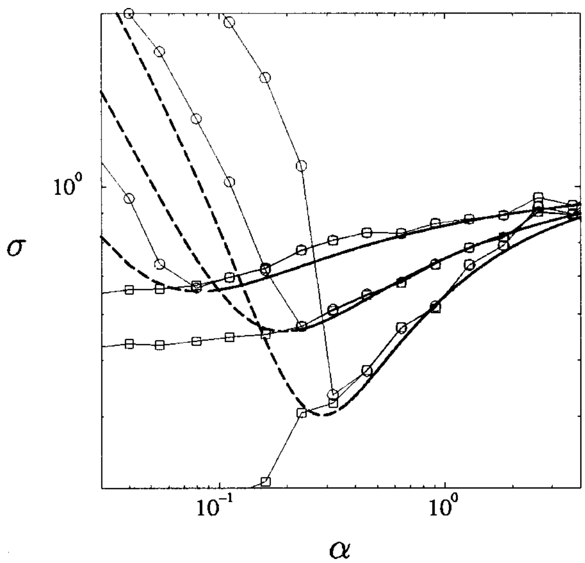

FIG. 7. The asymptotic volatility σ as a function of α, for multiplicative noise with $W(T)=\delta[T-\overline{T}]$ and different choices of the noise strength ($\overline{T}=0$, 1, 2 from bottom to top in $\alpha>\alpha_c$ regime). Markers: individual simulation runs, with $pN=\alpha N^2=10^6$ and homogeneous initial conditions where $q_i(0)=q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Thick-solid curves for $\alpha>\alpha_c(W(T))$: analytical predictions for homogeneous multiplicative decision noise. For $\alpha<\alpha_c(W(T))$, where they should no longer be correct, they have been continued as thick-dashed lines. For additive decision noise our theory predicts independence of $\overline{T}$, i.e., σ as given by the $\overline{T}=0$ curve of multiplicative noise.

$$\sigma^2=\frac{1}{2(1+k)^2}+\frac{1}{2}(1-\phi)+\lim_{\tau\to\infty}\frac{\phi}{2\tau}\sum_{u\leq\tau}\sum_{tt'}(\mathbb{1}+\tilde{G})^{-1}_{ut}(\mathbb{1}+\tilde{G}^T)^{-1}_{t'u}\times\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\text{fr}}. \tag{99}$$

Note that, according to Eqs. (82) and (83), the integrated response k can be expressed in terms of the order parameter ϕ as $k=(1-\phi)/(\alpha-1+\phi)$.

At this stage we again have to distinguish between additive noise and multiplicative noise, in order to work out the remaining averages. For additive noise one simply finds

$$\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\text{fr}}=\langle\langle\sigma[\tilde{q}(t)]\sigma[\tilde{q}(t')]\rangle\rangle_{\text{fr}}=1,$$

and hence we recover the expression describing the noise-free case in [15]:

$$\sigma^2=\frac{1+\phi}{2(1+k)^2}+\frac{1}{2}(1-\phi). \tag{100}$$

Since the order parameters ϕ and k are, for additive noise, independent of the noise distribution, the same is true for the volatility. This independence of the noise parameters, at least for $\alpha>\alpha_c$ (in line with [13,14]), again finds confirmation in numerical simulations [that is, within the limits imposed by our approximation; one does observe some weak effect, which could either be due to excessive relation times or due to the retarded self-interaction of fickle traders, which we neglected in deriving Eq. (100)].

The more interesting case, as before, is that of multiplicative noise. Here we have

$$\langle\langle\sigma[q(t),z_t|T]\sigma[q(t'),z_{t'}|T]\rangle\rangle_{\text{fr}}=\langle\langle\lambda^2(T)\rangle\rangle_{\text{fr}}+\delta_{tt'}[1-\langle\langle\lambda^2(T)\rangle\rangle_{\text{fr}}]. \tag{101}$$

Hence the approximation (99) reduces to

$$\sigma^2=\frac{1+\phi\chi}{2(1+k)^2}+\frac{1}{2}(1-\phi)+\frac{1}{2}\phi(1-\chi)[(\mathbb{1}+\tilde{G})^{-1}(\mathbb{1}+\tilde{G}^T)^{-1}](0). \tag{102}$$

Here we have used time-translation invariance of the stationary state, giving $[\dots]_{tt}\to[\dots](t-t)=[\dots](0)$ for the relevant matrix elements in Eq. (103). The conditional average $\chi=\langle\langle\lambda^2(T)\rangle\rangle_{\text{fr}}$, constrained by $|\eta|>\sqrt{\alpha}\lambda(T)/(1+k)$ (which, in the case of multiplicative noise, is the condition for an agent to be frozen) and calculated using the variance $\langle\eta^2\rangle=(1+c)/(1+k)^2$ (64) of the zero-average persistent noise term, is given by

$$\chi=\langle\langle\lambda^2(T)\rangle\rangle_{\text{fr}}=\frac{\int_0^\infty dT\,W(T)\lambda^2(T)\int Dz\,\theta\left[|z|-\frac{\sqrt{\alpha}\lambda(T)}{\sqrt{1+c}}\right]}{\int_0^\infty dT\,W(T)\int Dz\,\theta\left[|z|-\frac{\sqrt{\alpha}\lambda(T)}{\sqrt{1+c}}\right]}=\frac{\int_0^1 d\lambda\,w(\lambda)\lambda^2\left[1-\text{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)\right]}{\int_0^1 d\lambda\,w(\lambda)\left[1-\text{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)\right]}. \tag{103}$$

We note that only for $W(T)=\delta(T)$ [15], i.e., $w(\lambda)=\delta(\lambda-1)$, where $\chi=1$, will Eq. (103) involve only persistent observables. In the presence of decision noise, as in this study, one always has $\chi<1$, and additional approximations are required to also reduce the last term in Eq. (103) further to an expression in terms of persistent order parameters only. This is done in detail in Appendix A, where we show that a reasonable approximation is obtained by simply putting $[(\mathbb{1}+\tilde{G})^{-1}(\mathbb{1}+\tilde{G}^T)^{-1}](0)\to 1$. The end result is the following final approximation for the stationary-state volatility:

$$\sigma^2=\frac{1+\phi\chi}{2(1+k)^2}+\frac{1}{2}(1-\phi\chi), \tag{104}$$

with χ as given by Eq. (103).

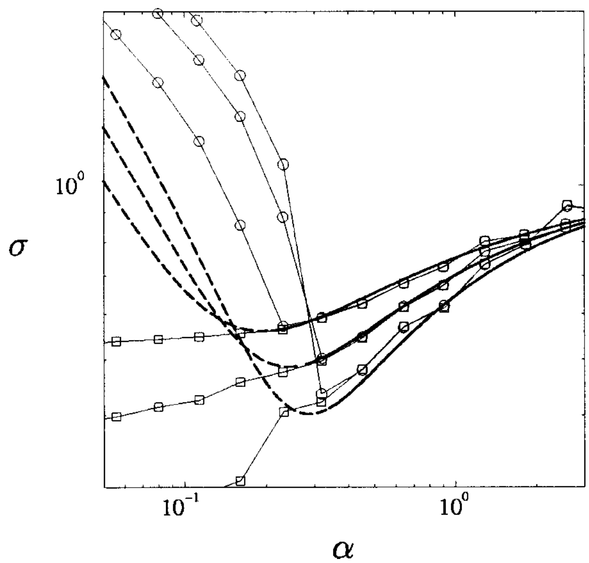

FIG. 8. The asymptotic volatility σ as a function of α, for multiplicative noise with $W(T')=\epsilon\delta(T'-T)+(1-\epsilon)\delta(T')$, for $T=1$ and different choices of the width ($\epsilon=0$, 0.5, 1 from bottom to top in the $\alpha>\alpha_c$ regime). Markers: individual simulation runs, with $pN=\alpha N^2=10^6$ and homogeneous initial conditions where $q_i(0)=q(0)$ [circles: $q(0)=0$, squares: $q(0)=10$] and in excess of 1000 iteration steps. Solid curves for $\alpha>\alpha_c(W(T))$: analytical predictions. For $\alpha<\alpha_c(W(T))$, where they should no longer be correct, they have been continued as dashed lines.

Expression (104), which reverts back to that of [15] for $T\to 0$ and which also reduces correctly to the random trading limit $\sigma=1$ for $T\to\infty$ (where $\phi=1$, $c=k=\chi=0$), turns out to be a surprisingly accurate approximation of the volatility for $\alpha>\alpha_c$ (i.e., in its regime of validity). This can be observed in Figs. 7 and 8, where we compare the approximate prediction (104) to the volatility as observed in numerical simulations, for both homogeneous multiplicative noise defined by $W(T)=\delta(T-\overline{T})$ and for inhomogeneous multiplicative noise defined by Eqs. (86) and (87), respectively. In all cases $\lambda(T)=\mathrm{erf}[1/T\sqrt{2}]$. (Note the persistent order parameters have already been calculated in the previous section.) We note that expression (104) is very similar to that obtained by different means for the on-line case in [10,11], from which it can be obtained by the replacement $c\to\phi\chi$.

The above results emphasize once more the qualitative difference between additive and multiplicative noise; in contrast to additive noise, the system remains sensitive to multiplicative noise even for $\alpha>\alpha_c$. The resulting dependence of the volatility on the multiplicative noise strength is very similar to that reported in [6] for additive noise (which was later understood to be caused by insufficient equilibration [13,14]).

VIII. DISCUSSION

In this paper we have generalized the thermal minority game [6] to the case of imhomogeneous agent populations (where the decision noise, which can be either additive or multiplicative, is of nonuniform strength). We have solved the dynamics of the batch version of this model by generalizing the recent application [15] to the minority game of the generating functional techniques of [16] (note that in [15] only the fully deterministic case was studied). This formalism reduces the N-agent dynamics, in the limit $N\to\infty$, to a stochastic process for a single "effective agent," with dynamic equations involving colored noise and a retarded self-interaction. It leads to exact closed (but implicit and nontrivial) equations for correlation and response functions.

Our theory enables us to (i) obtain a better understanding of previously observed but only partially explained phenomena (e.g., the suppression of the volatility by decision noise [6,12], even below random for $\alpha<\alpha_c$, due to damping of the "crowd anti-crowd" oscillations [7–9], or the increase in the volatility in the presence of multiplicative decision noise for $\alpha>\alpha_c$), (ii) derive exact phase diagrams, and (iii) calculate macroscopic observables (e.g., the fraction of frozen agents and the persistent correlations) in ergodic stationary states exactly.[3] In the case of additive decision noise we find a phase diagram identical to that of deterministic decision making in the onset of equilibrium properties of the higher α ergodic phase, with nonergodic behavior at lower α. In the case of multiplicative decision noise, in contrast, we arrive at phase diagrams with nontrivial decision noise dependencies of the phase separation line as well as the behavior of both phases. Here the control parameters are the relative number of possible values for the external information, $\alpha=p/N$, and the parameters characterizing the noise statistics. In the nonergodic regime of the model (i.e., for sufficiently small α) our closed equations in terms of correlation and response functions are still exact, and can be solved in principle iteratively for arbitrary times; however, finding the stationary states is hard (see e.g., the calculations for the simpler case [15]).[4] Here we have restricted our calculations in the nonergodic regime to the the first few time steps, finding noise dependence for both additive and multiplicative decision noise.

In the present paper we have only worked out explicitly two types of choices for the decision noise strengths statistics: a delta distribution (i.e., decision noise of uniform strength), and a parametrized class of bimodel distributions.

[3]Although the stationary-state equations, derived upon assuming ergodicity and absence of long-term memory, are no longer valid in the nonergodic regime, Figs. 1, 2, 4, and 5 show that for $\alpha<\alpha_c(W(T))$ their predictions regarding the persistent observables c and ϕ, nevertheless, give good qualitative agreement with the results of simulations from a highly biased start (for the volatility σ, which also involves nonpersistent order parameters, this is no longer the case).

[4]Note that a recently proposed procedure [22] for calculating at least the high-volatility stationary state in the nonergodic regime, based on assuming the integrated response function (which diverges exactly at the critical point) to remain infinite throughout the $\alpha<\alpha_c$ region, is not likely to work for the case of decision noise. It would, for instance, predict the simple relation $\phi=1-\alpha$ (i.e., ϕ being independent of the noise parameters), which is clearly in conflict with the simulation experiments presented in this paper.

Due to the general nature of our solution, however, there is no limit to the different types of noise statistics we could have studied. This emphasizes once more the remarkable potential and appropriateness to the minority games of the generating functional analysis methods of [16]. Two natural next steps would be (1) to develop the generating functional formalism for the original "on-line" formulation of the game, where the external information is fed to the agents sequentially,[5] or (2) to analyze our present (exact) order-parameter equations further in the nonergodic region $\alpha < \alpha_c(W(T))$.

APPENDIX: APPROXIMATION OF NONPERSISTENT TERMS IN THE STATIONARY VOLATILITY

The term $Q = [(\mathbb{1}+\tilde{G})^{-1}(\mathbb{1}+\tilde{G}^T)^{-1}](0)$ in Eq. (102), which contains contributions of nonpersistent order parameters, can be written as

$$Q = \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \frac{1}{|1+\hat{G}(\omega)|^2}, \tag{A1}$$

with the definition $\hat{G}(\omega) = \Sigma_t \tilde{G}(t) e^{-i\omega t}$. The simplest approximation for $\tilde{G}(t)$, which respects causality and also meets the requirement $\Sigma_t \tilde{G}(t) = k$, is an exponential expression of the form $\tilde{G}(t>0) \to k(1-\gamma)\gamma^{t-1}$ [with $-1<\gamma<1$ and with $\tilde{G}(t \leq 0) = 0$]. This gives $\hat{G}(\omega) = k(1-\gamma)/(e^{i\omega} - \gamma)$, and thus

$$Q = \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} \frac{|e^{i\omega}-\gamma|^2}{|e^{i\omega}-\gamma+k(1-\gamma)|^2}. \tag{A2}$$

We will obtain an estimate for γ by carrying out an approximate calculation of the one-step response function

$$\tilde{G}(1) = \frac{\partial}{\partial \theta(t)} \langle \sigma[q(t+1), z_{t+1}|T] \rangle_\star. \tag{A3}$$

We insert Eq. (34), and using the fact that the response of frozen agents will be zero, we repeat our previous ansatz that fickle agents do not experience a retarded self-interaction, and we carry out the average over the decision noise variable z_t. This is followed by carrying out the average over $\eta(t)$ [which is Gaussian, with variance $\langle \eta^2(t) \rangle = 2\sigma^2$; we assume, within the context of the present approximation, the correlations between $\eta(t)$ and the persistent noise η not to be important for fickle agents]. This gives

$$\tilde{G}(1) = \frac{1-\phi}{\sigma\sqrt{\pi\alpha}} \left\langle\left\langle \lambda(T) \exp\{-[q^2(t)/\alpha+\alpha]/4\sigma^2\} \times \left\{ \cosh\left[\frac{|q(t)|}{2\sigma^2}\right] + \lambda(T)\sinh\left[\frac{|q(t)|}{2\sigma^2}\right] \right\} \right\rangle\right\rangle_{\text{fi}}. \tag{A4}$$

In this expression we simply replace $|q(t)| \to 0$ (fickle agents being described by values of $q(t)$, which oscillate around zero) and we calculate the residual average $\langle\langle \lambda(T) \rangle\rangle_{\text{fi}}$ similar to our calculation of Eq. (103). Hence we arrive at the approximation

$$\tilde{G}(1) \approx \frac{1-\phi}{\sigma\sqrt{\pi\alpha}} e^{-\alpha/4\sigma^2} \left\{ \frac{\int_0^1 d\lambda\, w(\lambda)\lambda\, \mathrm{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)}{\int_0^1 d\lambda\, w(\lambda) \mathrm{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)} \right\}. \tag{A5}$$

On the other hand, according to our ansatz $\tilde{G}(t>0) = k(1-\gamma)\gamma^{t-1}$ we must demand $\tilde{G}(1) = k(1-\gamma)$, so that Eq. (A5) leads to the following estimate of γ:

$$\gamma \approx 1 - \frac{1-\phi}{\sigma k\sqrt{\pi\alpha}} e^{-\alpha/4\sigma^2} \left\{ \frac{\int_0^1 d\lambda\, w(\lambda)\lambda\, \mathrm{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)}{\int_0^1 d\lambda\, w(\lambda) \mathrm{erf}\left(\frac{\lambda\sqrt{\alpha}}{\sqrt{2(1+c)}}\right)} \right\}. \tag{A6}$$

Since for $\alpha \to \infty$ we must find $\sigma \to 1$ (random trading), and since $k \sim \alpha^{-1}$ (76), we conclude from Eq. (A6) that $\gamma \to 1$ for $\alpha \to \infty$. Conversely, as α is lowered, we find a divergence of k at finite α_c (where also ϕ is finite). Hence Eq. (A6) also predicts that $\gamma \to 1$ for $\alpha \to \alpha_c$. We now assume that $\gamma \to 1$ will give a sensible approximation in the whole range $\alpha > \alpha_c$, and use Eq. (A2) to arrive at the approximate result

$$[(\mathbb{1}+\tilde{G})^{-1}(\mathbb{1}+\tilde{G}^T)^{-1}](0) \approx 1. \tag{A7}$$

The above derivation is clearly far from rigorous, and not quite satisfactory; it simply appears the best one can do without actually solving the order-parameter equations for finite-time differences in the stationary state. Yet Eq. (A7) turns out to lead to a surprisingly accurate approximation for the volatility (see the main text).

[5]This is the subject of [19], where one also finds a detailed analysis of the derivation of the correct continuous time microscopic stochastic equations, of the effect of truncations in differential formalisms, and of the relation between the batch and on-line minority game.

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