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Diagonalization of replicated transfer matrices for disordered Ising spin systems

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Abstract

We present an alternative procedure for solving the eigenvalue problem of replicated transfer matrices describing disordered spin systems with (random) 1D nearest neighbour bonds and/or random fields, possibly in combination with (random) long range bonds. Our method is based on transforming the original eigenvalue problem for a $2^n \times 2^n$ matrix (where $n \to 0$) into an eigenvalue problem for integral operators. We first develop our formalism for the Ising chain with random bonds and fields, where we recover known results. We then apply our methods to models of spins which interact simultaneously via a one-dimensional ring and via more complex long-range connectivity structures, e.g., $(1+\infty)$-dimensional neural networks and 'small-world' magnets. Numerical simulations confirm our predictions satisfactorily.

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1. Introduction

The replica formalism, see e.g., [1], has proved to be a very powerful tool in the study of both statics and dynamics of disordered systems. In statics the presence of frozen disorder in the Hamiltonian makes a direct equilibrium statistical mechanical analysis very difficult. Instead, one starts from the key assumption (supported by numerical, experimental and sometimes analytical evidence) that in the thermodynamic limit the free energy per degree of freedom in such systems is self averaging, i.e., identical to its disorder average for any given realization of the disorder, with probability 1. This property allows one to focus on the evaluation of the disorder-averaged free energy per degree of freedom. To carry out this calculation the so-called replica trick, which introduces n replicas or copies of the original system, is invoked. The disorder average then converts the original problem of n independent but disordered systems into a new problem for n coupled but disorder-free ones. In the thermodynamic limit this new non-disordered problem can often be solved with conventional methods, e.g., saddle-point integration. The limit $n \to 0$ has to be taken in the final result. This procedure has over the years been applied with great success to many families of mostly range-free or mean-field models.

Application of the replica formalism to finite-dimensional spin models with disordered bonds and/or fields leads to the notion of replicated transfer matrices [2, 3]. For disordered one-dimensional Ising spin chains, for instance, the replica method effectively replaces an expression for the free energy in terms of products of 2×2 random transfer matrices (see, e.g., [4] for transfer matrix methods) by an expression for an n-replicated chain without disorder but with a more complicated $2^n \times 2^n$ transfer matrix which couples the n replicas at each site of the chain. In the thermodynamic limit one first has to find the largest eigenvalue of this replicated transfer matrix, and subsequently find its analytic continuation for $n \to 0$. It was shown in [2] that for the one-dimensional Ising model with random bonds and fields this procedure yields the results obtained earlier by other techniques, see e.g., [5–7]. Moreover, it was found that the smaller eigenvalues of the replicated transfer matrix contain information about disorder-averaged two-spin connected correlation functions.

In this paper, we show how the replicated transfer matrix of disordered Ising models can be diagonalized by using a particular form for the eigenvectors which transforms the original eigenvalue problem into an eigenvalue problem for integral operators. We believe our method to have a number of possible advantages. It appears more direct and explicit than existing approaches, it can be generalized in a straightforward manner to situations with RSB (which could for instance be induced by super-imposed long-range bonds), and it does not rely on the limit $n \to 0$ being taken (so that it can also be used for finite n replica calculations describing models where the disorder is not truly frozen but evolving on very large time scales, in the sense of [10–13]).

We first apply our ideas to Ising chains with random bonds and fields, where we can compare the results obtained with our method to those obtained earlier by others. Furthermore, mathematically one may express replicated transfer matrices of models which are not purely one-dimensional (due to super-imposed long-range bonds) in terms of those corresponding to random field chains, with the statistics of the random fields mediating the mean-field effect of the long-range bonds on a given site. We apply our equations to two examples of such models with one-dimensional and long-range bonds: the $(1+\infty)$ attractor neural networks of [15, 16] and the 'small-world' ferromagnet of [17], and show how one can use our methods to calculate various thermodynamic quantities.

2. Definitions

In this paper, we will deal with disordered Ising spin systems in thermal equilibrium at inverse temperature $\beta = 1/T$, of size N and with microscopic states written as $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \{-1, 1\}^N$. More specifically, we will analyse the following three models, by diagonalizing the replicated transfer matrices which they generate: the disordered Ising chain (DIC) as in [5, 6], the $(1+\infty)$-dimensional attractor neural network (ANN) as in [15, 16] and the 'small-world' ferromagnet (SWM) of [17], which are defined by the Hamiltonians

$$H_{\mathrm{DIC}}(\boldsymbol{\sigma}) = -\sum_i J_i \sigma_i \sigma_{i+1} - \sum_i \theta_i \sigma_i \tag{1}$$

$$H_{\mathrm{ANN}}(\boldsymbol{\sigma}) = -J_s \sum_i \sigma_i \sigma_{i+1} (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_{i+1}) - \frac{J_\ell}{N} \sum_{i<j} \sigma_i \sigma_j (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j) \tag{2}$$

$$H_{\mathrm{SWM}}(\boldsymbol{\sigma}) = -J_0 \sum_i \sigma_i \sigma_{i+1} - \frac{J}{c} \sum_{i<j} c_{ij} \sigma_i \sigma_j. \tag{3}$$

In (1) we have a 1D spin chain with independently identically distributed random bonds and fields $\{J_i, \theta_i\}$ at each site, drawn from some joint distribution $p(J, \theta)$. We will abbreviate $\int \mathrm{d}J\, \mathrm{d}\theta\, p(J, \theta) f(\theta, J) = \langle f(J, \theta) \rangle_{J,\theta}$. In (2) we have both 1D and long-range random bonds, but their values are not independent. The short- and long-range bonds take the values $J_s(\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_{i+1})$ and $J_\ell N^{-1}(\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j)$, respectively, where the binary vectors $\boldsymbol{\xi}_i = \left(\xi_i^1, \ldots, \xi_i^p\right)$ represent stored data and are drawn randomly and independently from $\{-1, 1\}^p$ (with uniform probabilities). Finally, in (3) we have uniform 1D ferromagnetic bonds of strength J_0, and the randomness is solely in the realization of the long-range bonds. The latter are also ferromagnetic, of strength J/c if present, but constitute a finitely connected Poissonian graph defined by dilution variables c_{ij} which for each pair (i, j) are drawn independently from $p(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0}$. Here c is the average number of long-range connections at each site $k_i = \sum_{i=1}^N c_{ij}$ in the limit $N \to \infty$. This number is taken to be finite, therefore on average the number of long-range connections at each site remains $\mathcal{O}(1)$ in the thermodynamic limit. In all three cases (1)–(3) the 1D short-range interactions are defined periodically.

At this stage, let us briefly recall from [2, 3] how a replicated transfer matrix emerges for the disordered Ising chain (1) in the context of the replica formalism. One starts from the disorder-averaged free energy per spin $\bar{f} = -\lim_{N\to\infty} \frac{1}{\beta N} \langle \log Z \rangle_{J,\theta}$, where $Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H_{\mathrm{DIC}}(\boldsymbol{\sigma})}$ is the partition function and the bar denotes averaging over the disorder variables (here the random bonds and fields). The average of the logarithm is calculated using the identity $\langle \log Z \rangle_{J,\theta} = \lim_{n\to 0} \frac{1}{n} \log \langle Z^n \rangle_{J,\theta}$. One then assumes that the order of the two limits ($N \to \infty$, $n \to 0$) can be reversed resulting in the following expression:

$$\bar{f} = -\lim_{n\to 0} \frac{1}{n} \lim_{N\to\infty} \frac{1}{\beta N} \log \langle Z^n \rangle_{J,\theta}. \tag{4}$$

Then the disorder average of the nth power of the partition function is evaluated for integer n and at the end the result is analyticaly continued to $n \to 0$. Here one finds, with $\alpha = 1, \ldots, n$ and with the short hand $\boldsymbol{\sigma}_i = \left(\sigma_i^1, \ldots, \sigma_i^n\right) \in \{-1, 1\}^n$,

$$\langle Z^n \rangle_{J,\theta} = \sum_{\boldsymbol{\sigma}_1 \ldots \boldsymbol{\sigma}_N} \prod_i \left\langle \exp\left(\beta J \sum_\alpha \sigma_{i+1}^\alpha \sigma_i^\alpha + \beta\theta \sum_\alpha \sigma_i^\alpha \right) \right\rangle_{J,\theta} = \mathrm{tr}\left(\boldsymbol{T}_n^N\right) \tag{5}$$

with a $2^n \times 2^n$ matrix $\boldsymbol{T}_n$ whose entries are given by

$$T_n(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \left\langle \exp\left(\beta J \sum_\alpha \sigma_\alpha \sigma'_\alpha + \beta\theta \sum_\alpha \sigma_\alpha \right) \right\rangle_{J,\theta}. \tag{6}$$

One can thus find the disorder-averaged free energy per spin in the usual manner, via (4), by determining the largest eigenvalue of the replicated transfer matrix $\boldsymbol{T}_n$ for integer n. The difficulty lies in the requirement to find an analytic expression for this eigenvalue for *arbitrary* integer n (in contrast to non-disordered chains, where the dimension is fixed from the start and usually small, and where direct methods can therefore be employed such as calculating the characteristic polynomial of the matrix and finding its zeros).

We will first develop our diagonalization method for the simplest case, viz the chain (1) and subsequently show that it can also serve to generate the solution of the other two models (2), (3), which involve both short- and long-range bonds, by writing the transfer matrices of the latter two models again in the form (6), but with suitably defined distributions $p(J, \theta)$ of local bonds and fields.

3. Construction and properties of eigenvectors

3.1. A detour: the Ising chain without disorder

Let us first turn to the simplest possible case: the 1D Ising chain with bonds J and uniform fields θ (without disorder), where we just have the familiar transfer matrix

$$T(\sigma,\sigma';\theta,J)=\exp(\beta J\sigma\sigma'+\beta\theta\sigma). \tag{7}$$

Diagonalizing (7) is of course trivial [4]. Here, however, we seek a method which does not require knowledge of the characteristic polynomial of the matrix, so that it can be generalized to replicated transfer matrices with arbitrary n. To this end we introduce the two vectors $\boldsymbol{u}_0[x]$, $\boldsymbol{u}_1[x,\mu]\in\mathbb{R}^2$, parametrized by $x,\mu\in\mathbb{R}$, and with components

$$u_0(\sigma;x)=\mathrm{e}^{\beta x\sigma}\qquad u_1(\sigma;x,\mu)=\mathrm{e}^{\beta x\sigma}(\sigma-\mu). \tag{8}$$

Inserting the candidates (8) into the eigenvalue equation $\sum_{\sigma'}T(\sigma,\sigma';\theta,J)u(\sigma')=\lambda u(\sigma)$, and using the general identity $f(\sigma)=\mathrm{e}^{\beta[B+A\sigma]}$, where $A=\frac{1}{2\beta}\log[f(1)/f(-1)]$ and $B=\frac{1}{2\beta}\log[f(1)f(-1)]$, for $\sigma\in\{-1,1\}$, leads to the following eigenvalue equations:

$$\exp(\beta B(J,x)+\beta[\theta+A(J,x)]\sigma)=\lambda_0\exp(\beta x\sigma) \tag{9}$$

$$\exp(\beta B(J,x)+\beta[\theta+A(J,x)]\sigma)A'(J,x)\left(\sigma-\frac{\mu-B'(J,x)}{A'(J,x)}\right)=\lambda_1\exp(\beta x\sigma)(\sigma-\mu) \tag{10}$$

where

$$A(J,x)=\frac{1}{\beta}\operatorname{arctanh}[\tanh(\beta J)\,\tanh(\beta x)] \tag{11}$$

$$B(J,x)=\frac{1}{2\beta}\log[4\cosh(\beta(J+x))\cosh(\beta(J-x))] \tag{12}$$

with partial derivatives $A'(J,x)=\partial_x A(J,x)=\frac{1}{2}[\tanh(\beta J+\beta x)+\tanh(\beta J-\beta x)]$ and $B'(J,x)=\partial_x B(J,x)=\frac{1}{2}[\tanh(\beta J+\beta x)-\tanh(\beta J-\beta x)]$, respectively. We conclude from (9) that if x^* is the solution of the algebraic equation $x=\theta+A(J,x)$, then $\boldsymbol{u}_0[x^*]$ is an eigenvector with eigenvalue $\lambda_0=\mathrm{e}^{\beta B(J,x^*)}$. This (unique) solution, which can be viewed as the stable fixed point of the iterative map $x_{i+1}=\theta+A(J,x_i)$, is given by

$$x^*=\frac{1}{2}(J+\theta)+\frac{1}{2\beta}\log\left[\mathrm{e}^{\beta J}\sinh(\beta\theta)+\sqrt{\mathrm{e}^{2\beta J}\sinh^2(\beta\theta)+\mathrm{e}^{-2\beta J}}\right]. \tag{13}$$

Inserting (13) into our expression for λ_0 then reproduces the familiar result for the largest eigenvalue of the transfer matrix of the Ising chain with uniform fields and bonds

$$\lambda_0=\mathrm{e}^{\beta B(J,x^*)}=\mathrm{e}^{\beta J}\cosh(\beta\theta)+\sqrt{\mathrm{e}^{2\beta J}\sinh^2(\beta\theta)+\mathrm{e}^{-2\beta J}}. \tag{14}$$

Similarly we see that if $\mu^*=\frac{B'(J,x^*)}{1-A'(J,x^*)}$, with x^* as defined before, then also $\boldsymbol{u}_1[x^*,\mu^*]$ is an eigenvector with eigenvalue $\lambda_1=\mathrm{e}^{\beta B(J,x^*)}A'(J,x^*)$. Insertion of (13) leads to the familiar expression for the second eigenvalue of (7):

$$\lambda_1=\mathrm{e}^{\beta J}\cosh(\beta\theta)-\sqrt{\mathrm{e}^{2\beta J}\sinh^2(\beta\theta)+\mathrm{e}^{-2\beta J}}. \tag{15}$$

It turns out that μ^* gives the average magnetization at each site:

$$\mu^*=\frac{\tanh(\beta x^*)[1+\tanh(\beta J)]}{1+\tanh(\beta J)\tanh^2(\beta x^*)}=\frac{\sinh(\beta\theta)}{\sqrt{\sinh^2(\beta\theta)+\mathrm{e}^{-4\beta J}}}=\langle\sigma_i\rangle. \tag{16}$$

Note that our expression for (7) is not symmetric (although one could easily write the partition sum in terms of a symmetric transfer matrix), hence we have to distinguish between left and right eigenvectors; so far only right eigenvectors have been calculated. We can find the left eigenvectors $\boldsymbol{v}$ via similar ansatz to (8):

$$v_0(\sigma; y) = \mathrm{e}^{\beta y\sigma} \qquad v_1(\sigma; y, \nu) = \mathrm{e}^{\beta y\sigma}(\sigma - \nu). \tag{17}$$

Insertion into the left eigenvalue equation $\sum_{\sigma'} v(\sigma')T(\sigma', \sigma; \theta, J) = \lambda v(\sigma)$ then reveals that the two vectors $\boldsymbol{v}_0[y^*]$ and $\boldsymbol{v}_1[y^*, \nu^*]$ are left eigenvectors, where y^* is the solution of $y^* = A(J, y^* + \theta)$ and $\nu^* = \frac{B'(J,y^*+\theta)}{1-A'(J,y^*+\theta)}$. The associated eigenvalues are $\lambda_0 = \mathrm{e}^{\beta B(J,y^*+\theta)}$ and $\lambda_1 = \mathrm{e}^{\beta B(J,y^*+\theta)}A'(J, y^* + \theta)$. The fixed point y^* of the map $y_{i+1} = A(J, y_i + \theta)$ is again unique, and is given by

$$y^* = \frac{1}{2}(J - \theta) + \frac{1}{2\beta} \log\left[\mathrm{e}^{\beta J}\sinh(\beta\theta) + \sqrt{\mathrm{e}^{2\beta J}\sinh^2(\beta\theta) + \mathrm{e}^{-2\beta J}}\right]. \tag{18}$$

Obviously $x^* = y^* + \theta$, so left and right eigenvalues are identical and $\nu^* = \mu^* = \langle\sigma_i\rangle$. Furthermore, upon using the simple relation $\tanh(\beta x^* + \beta y^*) = \langle\sigma_i\rangle = \mu^* = \nu^*$ it is clear that left and right eigenvectors corresponding to different eigenvalues are orthogonal:

$$\sum_\sigma v_0(\sigma; y^*)u_1(\sigma; x^*, \mu^*) = 2\cosh(\beta x^* + \beta y^*)[\tanh(\beta x^* + \beta y^*) - \mu^*] = 0$$

$$\sum_\sigma v_1(\sigma; y^*, \mu^*)u_0(\sigma; x^*) = 2\cosh(\beta x^* + \beta y^*)[\tanh(\beta x^* + \beta y^*) - \nu^*] = 0.$$

Finally, to normalize our eigenvectors we require the constants

$$D_0(x^*, y^*) = \sum_\sigma v_0(\sigma; y^*)u_0(\sigma; x^*) = 2\cosh(\beta x^* + \beta y^*) \tag{19}$$

$$D_1(x^*, y^*) = \sum_\sigma v_1(\sigma; y^*, \mu^*)u_1(\sigma; x^*, \mu^*) = 2\cosh(\beta x^* + \beta y^*)[1 - (\mu^*)^2]. \tag{20}$$

3.2. Uncoupled replicated chains

For the ordinary Ising chain the above method would obviously not be the most efficient route towards a solution. However, in contrast to the conventional approach based on explicit diagonalization via a calculation of the zeros of the characteristic polynomial, we will show that it can be applied also to the diagonalization of replicated transfer matrices, where the dimension of the problem is no longer fixed and calculation of the characteristic polynomial is therefore not a realistic option.

As an intermediate step from the diagonalization of (7) for the simple Ising chain to diagonalization of (6) for disordered chains, let us now inspect replicated transfer matrices with uncoupled replicas, viz (6) but with δ-distributed bonds and fields:

$$T_n(\boldsymbol{\sigma}, \boldsymbol{\sigma}'; \theta, J) = \exp\left(\beta J\sum_\alpha \sigma_\alpha\sigma'_\alpha + \beta\theta\sum_\alpha \sigma_\alpha\right) \tag{21}$$

without an average over $\{\theta, J\}$. This matrix is just the n-fold Kronecker product of (7), so its left and right eigenvectors are simply (Kronecker) products of (17) and (8), respectively. Each eigenvector is characterized by an index set $\{\rho\} \subseteq \{1, \ldots, n\}$ of size $\rho \in \{0, \ldots, n\}$, indicating those indices α for which we select $\boldsymbol{u}_1[x^*]$ as opposed to $\boldsymbol{u}_0[x^*]$ (and similarly for left eigenvectors), and with $\{0\} = \emptyset$. The left and right eigenvectors of (21) can thus be written as

$$v_{\{\rho\}}(\boldsymbol{\sigma}; y^*, \mu^*) = \prod_{\alpha\in\{\rho\}} v_1(\sigma_\alpha; y^*, \mu^*)\prod_{\alpha\notin\{\rho\}} v_0(\sigma_\alpha; y^*) \tag{22}$$

$$u_{\{\rho\}}(\boldsymbol{\sigma}; x^*, \mu^*) = \prod_{\alpha\in\{\rho\}} u_1(\sigma_\alpha; x^*, \mu^*) \prod_{\alpha\notin\{\rho\}} u_0(\sigma_\alpha; x^*). \tag{23}$$

For each $\rho \in \{0, \ldots, n\}$ there are $\binom{n}{\rho}$ different index subsets giving us the required total number of 2^n eigenvectors. The associated eigenvalues follow easily, since here all spin summations factorize over replicas:

$$\begin{aligned}\sum_{\boldsymbol{\sigma}'} & T_n(\boldsymbol{\sigma}, \boldsymbol{\sigma}'; \theta, J) u_{\{\rho\}}(\boldsymbol{\sigma}'; x^*, \mu^*) \\ &= \prod_{\alpha\in\{\rho\}} \sum_{\sigma'_\alpha} T(\sigma_\alpha, \sigma'_\alpha; \theta, J) u_1(\sigma'_\alpha; x^*, \mu^*) \prod_{\alpha\notin\{\rho\}} \sum_{\sigma'_\alpha} T(\sigma_\alpha, \sigma'_\alpha; \theta, J) u_0(\sigma'_\alpha; x^*) \\ &= \lambda_1^\rho \lambda_0^{n-\rho} \prod_{a\in\{\rho\}} u_1(\sigma'_a; x^*, \mu^*) \prod_{\alpha\notin\{\rho\}} u_0(\sigma'_\alpha; x^*) = \lambda_1^\rho \lambda_0^{n-\rho} u_{\{\rho\}}(\boldsymbol{\sigma}; x^*, \mu^*).\end{aligned}$$

Hence (21) has $(n + 1)$ different eigenvalues $\lambda_\rho(n) = \lambda_1^\rho \lambda_0^{n-\rho}$, each with multiplicity $\binom{n}{\rho}$. Since $\lambda_0 > \lambda_1$, we also have the ordering relation $\lambda_0(n) > \lambda_1(n) > \cdots > \lambda_n(n)$. We can furthermore see that right and left eigenvectors satisfy the orthogonality relations

$$\boldsymbol{v}_{\{\rho'\}}[y^*, \mu^*] \cdot \boldsymbol{u}_{\{\rho\}}[x^*, \mu^*] = D_\rho(x^*, y^*) \delta_{\rho\rho'} \prod_{k=1}^{\rho} \delta_{\alpha_k \alpha'_k} \tag{24}$$

$$D_\rho(x^*, y^*) = 2\cosh^n(\beta x^* + \beta y^*)[1 - (\mu^*)^2]^\rho \tag{25}$$

where $\{\rho\} = \{\alpha_1, \ldots, \alpha_\rho\}$ and $\{\rho'\} = \{\alpha'_1, \ldots, \alpha'_{\rho'}\}$, and where the factor $\prod_{k=1}^{\rho} \delta_{\alpha_k\alpha'_k}$ in (24) is defined as unity for $\rho = 0$.

Although the above case of uncoupled replicas is still trivial, it reveals explicit suggestions for the general structure of the eigenvectors for the general class of $2^n \times 2^n$ replicated transfer matrices. This structure will serve as an efficient ansatz and will lead us below to exact solutions also for the non-trivial case of coupled replicas.

3.3. Diagonalization for the disordered Ising chain

We now turn the real problem: the diagonalization of (6), which can also be written as $\boldsymbol{T}_n = \langle \boldsymbol{T}_n[\theta, J]\rangle_{\theta,J}$. Clearly $\boldsymbol{T}_n$ shares many properties with the transfer matrix $\boldsymbol{T}_n[\theta, J]$ of the chain with uncoupled replicas, e.g., invariance under all permutations π of the permutation group S_n acting on the indices $\{1, \ldots, n\}$:

$$T_n(\pi(\boldsymbol{\sigma}), \pi(\boldsymbol{\sigma}')) = T_n(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \qquad \text{for every} \quad \pi \in S_n. \tag{26}$$

If we denote by $\boldsymbol{D}_\pi$ a $2^n \times 2^n$ matrix representation of π, i.e., $D_\pi(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \delta_{\pi(\boldsymbol{\sigma}),\boldsymbol{\sigma}'}$, it then follows that $\boldsymbol{T}_n$ and $\boldsymbol{D}_\pi$ commute: $\boldsymbol{T}_n\boldsymbol{D}_\pi = \boldsymbol{D}_\pi\boldsymbol{T}_n$. This in turn implies that if $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{T}_n$ with eigenvalue λ, then so is $\boldsymbol{D}_\pi\boldsymbol{u}$ for any $\pi \in S_n$. Furthermore in the uncoupled case (21), one observes that due to the Kronecker product form of the eigenvectors $\boldsymbol{D}_\pi \boldsymbol{u}_{\{a_1,\ldots,a_\rho\}} = \boldsymbol{u}_{\{\pi(a_1),\ldots,\pi(a_\rho)\}}$ for every $\pi \in S_n$. We will make the ansatz that the latter property also holds for the eigenvectors of (6). In order to arrive at explicit expressions for 'candidate' eigenvectors of (6) with the aforementioned properties, we retain the same spin dependence as in the eigenvectors of the uncoupled replicated transfer matrix and introduce appropriate functions P_ρ, Q_ρ which take into account the coupling between the replicas in the disordered case. We are thus being led to the following general ansatz for the right and left eigenvectors of $\boldsymbol{T}_n$:

$$u_{\{\rho\}}(\boldsymbol{\sigma}; P_\rho) = \int \mathrm{d}x\, \mathrm{d}\mu\, P_\rho(x, \mu|n)\, \exp\left(\beta x \sum_{\alpha=1}^{n} \sigma_\alpha\right) \prod_{\alpha\in\{\rho\}} (\sigma_\alpha - \mu) \tag{27}$$

$$v_{\{\rho\}}(\boldsymbol{\sigma};\, Q_\rho) = \int \mathrm{d}y\, \mathrm{d}\nu\, Q_\rho(y, \nu|n) \exp\left(\beta y \sum_{\alpha=1}^{n} \sigma_\alpha\right) \prod_{\alpha\in\{\rho\}} (\sigma_\alpha - \nu). \tag{28}$$

This structure results in a spectrum of $(n+1)$ different eigenvalues $\lambda_\rho(n)$ with $\rho = 0, 1, \ldots, n$, with multiplicity $\binom{n}{\rho}$ each, in agreement with the results in [2], which were derived using the irreducible representations of the replica permutation group.

The unknown functions P_ρ and Q_ρ are determined by inserting (27) into the right eigenvalue equation $\boldsymbol{T}_n \boldsymbol{u}_{\{\rho\}}[P_\rho] = \lambda_\rho(n)\boldsymbol{u}_{\{\rho\}}[P_\rho]$ and (28) into the left eigenvalue equation $\boldsymbol{v}_{\{\rho\}}[Q_\rho]\boldsymbol{T}_n = \lambda_\rho(n)\boldsymbol{v}_{\{\rho\}}[Q_\rho]$, respectively. Working out the first equation gives, with the definitions (11), (12):

$$\begin{aligned}\sum_{\boldsymbol{\sigma}'} T_n(\boldsymbol{\sigma}, \boldsymbol{\sigma}')u_{\{\rho\}}(\boldsymbol{\sigma}';\, P_\rho) = \int \mathrm{d}x'\, \mathrm{d}\mu'\, P_\rho(x', \mu'|n)\Bigg\langle \prod_{\alpha\notin\{\rho\}} \exp(\beta B(J, x') + \beta[\theta + A(J, x')]\sigma_\alpha)\\ \times \prod_{\alpha\in\{\rho\}} \exp(\beta B(J, x') + \beta[\theta + A(J, x')]\sigma_\alpha) A'(J, x')\left(\sigma_\alpha - \frac{\mu' - B'(J, x')}{A'(J, x')}\right)\Bigg\rangle_{J,\theta}.\end{aligned}$$

Upon inserting suitable integrals over δ-functions, viz $1 = \int \mathrm{d}x\, \delta[x - \theta - A(J, x')]$ and $1 = \int \mathrm{d}\mu\, \delta\big[\mu - \frac{\mu' - B'(J,x')}{A'(J,x')}\big]$, we then find our right eigenvalue equation taking the form

$$\begin{aligned}\int \mathrm{d}x\, \mathrm{d}\mu\Bigg[\int \mathrm{d}x'\, \mathrm{d}\mu'\, P_\rho(x', \mu'|n)\Bigg\langle \exp(n\beta B(J, x'))[A'(J, x')]^\rho \delta[x - \theta - A(J, x')]\\ \times \delta\left[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}\right]\Bigg\rangle_{J,\theta}\Bigg]\left[\exp\left(\beta x \sum_{a=1}^{n} \sigma_a\right) \prod_{a\in\{\rho\}} (\sigma_a - \mu)\right]\\ = \lambda_\rho(n) \int \mathrm{d}x\, \mathrm{d}\mu\, P_\rho(x, \mu|n)\left[\exp\left(\beta x \sum_{a=1}^{n} \sigma_a\right) \prod_{a\in\{\rho\}} (\sigma_a - \mu)\right].\end{aligned}$$

We conclude from this that, in order for (27) to be an eigenvector of our replicated transfer matrix, the function P_ρ must satisfy the following eigenvalue equation:

$$\int \mathrm{d}x'\, \mathrm{d}\mu'\, \Lambda^{(P)}_\rho(x, \mu, x', \mu'|n) P_\rho(x', \mu'|n) = \lambda_\rho(n) P_\rho(x, \mu|n) \tag{29}$$

with the kernel

$$\Lambda^{(P)}_\rho(x, \mu, x', \mu'|n) = \left\langle \mathrm{e}^{n\beta B(J,x')}[A'(J, x')]^\rho \delta[x - \theta - A(J, x')]\delta\left[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}\right]\right\rangle_{J,\theta}. \tag{30}$$

Upon repeating the above procedure also for the left eigenvectors (28) we find a similar eigenvalue problem for the functions Q_ρ, but now with a different kernel $\Lambda^{(Q)}_\rho$:

$$\int \mathrm{d}y'\, \mathrm{d}\nu'\, \Lambda^{(Q)}_\rho(y, \nu, y', \nu'|n) Q_\rho(y', \nu'|n) = \lambda_\rho(n) Q_\rho(y, \nu|n) \tag{31}$$

$$\begin{aligned}\Lambda^{(Q)}_\rho(y, \nu, y', \nu'|n) = \Bigg\langle \exp(n\beta B(J, y'+\theta))[A'(J, y'+\theta)]^\rho\\ \times \delta[y - A(J, y'+\theta)]\delta\left[\nu - \frac{\nu' - B'(J, y'+\theta)}{A'(J, y'+\theta)}\right]\Bigg\rangle_{J,\theta}.\end{aligned} \tag{32}$$

We have now transformed the problem of diagonalizing the $2^n \times 2^n$ replicated transfer matrix (6) into a problem involving integral operators (30), (32), where the limit $n \to 0$ can be taken.

The variable n no longer controls the dimension of the operator to be diagonalized, but has become simply a parameter of a continuous kernel.

We note that, at least for the purpose of finding the eigenvalues $\lambda_\rho(n)$, the two eigenvalue problems (29), (31) can be integrated over μ and ν, respectively, and replaced by a simpler eigenvalue problem for the two single-argument functions $\Phi_\rho(x|n) = \int \mathrm{d}\mu\, P_\rho(x,\mu|n)$ and $\Psi_\rho(y|n) = \int \mathrm{d}\nu\, Q_\rho(y,\nu|n)$:

$$\int \mathrm{d}x'\, \Lambda_\rho^{(P)}(x,x'|n)\Phi_\rho(x'|n) = \lambda_\rho(n)\Phi_\rho(x|n) \tag{33}$$

$$\int \mathrm{d}y'\, \Lambda_\rho^{(Q)}(y,y'|n)\Psi_\rho(y'|n) = \lambda_\rho(n)\Psi_\rho(y|n) \tag{34}$$

with

$$\Lambda_\rho^{(P)}(x,x'|n) = \langle \exp(n\beta B(J,x'))[A'(J,x')]^\rho \delta[x-\theta-A(J,x')]\rangle_{J,\theta} \tag{35}$$

$$\Lambda_\rho^{(Q)}(y,y'|n) = \langle \exp(n\beta B(J,y'+\theta))[A'(J,y'+\theta)]^\rho \delta[y-A(J,y'+\theta)]\rangle_{J,\theta}. \tag{36}$$

Once we know the functions P_ρ and Q_ρ, the form of the kernels (35), (36) enables us to integrate (29) and (31) over x and y and to obtain relatively expressions for the corresponding eigenvalues:

$$\lambda_\rho(n) = \frac{\int \mathrm{d}x\, \Phi_\rho(x|n)\langle \exp(n\beta B(J,x))[A'(J,x)]^\rho\rangle_J}{\int \mathrm{d}x\, \Phi_\rho(x|n)} \tag{37}$$

$$\lambda_\rho(n) = \frac{\int \mathrm{d}y\, \Psi_\rho(y|n)\langle \exp(n\beta B(J,y+\theta))[A'(J,y+\theta)]^\rho\rangle_{J,\theta}}{\int \mathrm{d}y\, \Psi_\rho(y|n)}. \tag{38}$$

Let us quickly inspect special cases. We see that for δ-distributed fields and bonds the integral equations (29), (31) admit the expected solutions $P_\rho(x,\mu|n) = \delta(x-x^*)\delta(\mu-\mu^*)$ and $Q_\rho(y,\nu|n) = \delta(y-y^*)\delta(\nu-\mu^*)$, the eigenvectors (27), (28) reduce to the eigenvectors of (21), and the eigenvalues become $\lambda_\rho(n) = \lambda_1^\rho\lambda_0^{n-\rho}$, as they should. Also the special case of a chain without external fields, i.e., $p(J,\theta) = p(J)\delta(\theta)$, can easily be solved analytically. Here $A(J,0) = B'(J,0) = 0$ and $A'(J,0) = \tanh(\beta J)$ for every J, which enables us to verify that (29), (31) have the trivial solutions $P_\rho(x,\mu|n) = \delta(x)\delta(\mu)$ and $Q_\rho(y,\nu|n) = \delta(y)\delta(\nu)$. Hence, the eigenvectors become

$$u_{\{\rho\}}(\boldsymbol{\sigma}) = v_{\{\rho\}}(\boldsymbol{\sigma}) = \prod_{\alpha\in\{\rho\}} \sigma_\alpha.$$

They satisfy $\boldsymbol{v}_\rho[Q]\cdot\boldsymbol{u}_{\rho'}[P] = 2^n\delta_{\rho\rho'}\prod_{k=1}^{\rho}\delta_{\alpha_k\alpha_k'}$. These eigenvectors are in fact common to all matrices of the form $T(\boldsymbol{\sigma},\boldsymbol{\sigma}') = T(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}')$ [14], and our replicated transfer matrix falls in this category when the external fields are zero. The eigenvalues are given by $\lambda_\rho(n) = \langle[2\cosh(\beta J)]^n \tanh^\rho(\beta J)\rangle_J$, and it is clear that the largest eigenvalue corresponds to $\rho = 0$.

3.4. Properties of the kernel eigenvalue problems for $n \to 0$

Let us consider in more detail the $n \to 0$ limits of the eigenvalue problems (30) and (32). We first turn to $\rho = 0$. The eigenvectors corresponding to eigenvalue $\lambda_0(0)$ do not depend on $\{\nu,\mu\}$, so upon writing simply $\Phi_0(x|0) = \Phi(x)$ and $\Psi_0(y|0) = \Psi(y)$ we obtain for $\rho = 0$:

$$\int \mathrm{d}x'\, \Phi(x')\langle\delta[x-\theta-A(J,x')]\rangle_{J,\theta} = \lambda_0(0)\Phi(x) \tag{39}$$

$$\int \mathrm{d}y'\, \Psi(y')\langle\delta[y - A(J, y' + \theta)]\rangle_{J,\theta} = \lambda_0(0)\Psi(y). \tag{40}$$

If we assume that $\int \mathrm{d}x\, \Phi(x) \neq 0$ and $\int \mathrm{d}y\, \Psi(y) \neq 0$, then integration of (39), (40) over x and y, respectively, gives us in both equations $\lambda_0(0) = 1$. This, in turn, implies that $\Phi(x)$ and $\Psi(y)$ are the stationary distributions of the two random maps

$$x_{i+1} = \theta_i + A(J_i, x_i) \qquad y_{i+1} = A(J_i, y_i + \theta_i).$$

These maps describe the propagation of the fields x and y along the chain. The two distributions are connected via the following equations:

$$\Phi(x) = \int \mathrm{d}y\, \Psi(y)\langle\delta[x - \theta - y]\rangle_\theta \qquad \Psi(y) = \int \mathrm{d}x\, \Phi(x)\langle\delta[y - A(J, x)]\rangle_J \tag{41}$$

which can be verified upon substituting into (39), (40), using $\lambda_0(0) = 1$.

The case $\rho > 0$ is more complicated. Here, we find the $n \to 0$ eigenvalue problems

$$\begin{aligned}\int \mathrm{d}x'\, \mathrm{d}\mu' \bigg\langle [A'(J, x')]^\rho \delta[x - \theta - A(J, x')]\delta\bigg[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}\bigg]\bigg\rangle_{J,\theta} P_\rho(x', \mu'|0)\\ = \lambda_\rho(0) P_\rho(x, \mu|0)\end{aligned} \tag{42}$$

$$\begin{aligned}\int \mathrm{d}y'\, \mathrm{d}\nu' \bigg\langle [A'(J, y'+\theta)]^\rho \delta[y - A(J, y'+\theta)]\delta\bigg[\mu - \frac{\mu' - B'(J, y'+\theta)}{A'(J, y'+\theta)}\bigg]\bigg\rangle_{J,\theta} Q_\rho(y', \nu'|0)\\ = \lambda_\rho(0) Q_\rho(y, \nu|0).\end{aligned} \tag{43}$$

As for $\rho = 0$ we can show that these equations admit solutions P_ρ and Q_ρ which can be interpreted as probability densities. The difference with $\rho = 0$, where these distributions are the stationary measures of the random maps of the propagated fields $\{x, y\}$, is that here the quantities which are propagated are the distributions themselves, via deterministic but nonlinear functional maps:

$$P_{\rho,i+1} = \mathcal{A}_{P,\rho}(P_{\rho,i}) \qquad Q_{\rho,i+1} = \mathcal{A}_{Q,\rho}(Q_{\rho,i})$$

where

$$\begin{aligned}[\mathcal{A}_{P,\rho}(P)](x, \mu) = \int \mathrm{d}x'\, \mathrm{d}\mu' \bigg\langle &\frac{P(x', \mu')[A'(J, x')]^\rho}{\int \mathrm{d}x''\, \mathrm{d}\mu''\, P(x'', \mu'')\langle[A'(J'', x'')]^\rho\rangle_{J''}}\\ &\times \delta[x - \theta - A(J, x')]\delta\bigg[\mu - \frac{\mu' - B'(J, x')}{A'(J, x')}\bigg]\bigg\rangle_{J,\theta}\end{aligned} \tag{44}$$

$$\begin{aligned}[\mathcal{A}_{Q,\rho}(Q)](y, \nu) = \int \mathrm{d}y'\, \mathrm{d}\nu' \bigg\langle &\frac{Q(y', \nu')[A'(J, y'+\theta)]^\rho}{\int \mathrm{d}y''\mathrm{d}\nu''\, Q(y'', \nu'')\langle[A'(J'', y''+\theta'')]^\rho\rangle_{J'',\theta''}}\\ &\times \delta[y - A(J, y' + \theta)]\delta\bigg[\nu - \frac{\nu' - B'(J, y'+\theta)}{A'(J, y'+\theta)}\bigg]\bigg\rangle_{J,\theta}.\end{aligned} \tag{45}$$

We see that the defining properties of a probability density, viz non-negativity and normalization, are preserved by both functional maps. Hence, we may indeed view the eigenvalue problems (42), (43) as the fixed point equations of the functional maps (44), (45). The corresponding eigenvalues are

$$\lambda_\rho(0) = \int \mathrm{d}x\, \Phi_\rho(x|0)\langle[A'(J, x)]^\rho\rangle_J = \int \mathrm{d}y\, \Psi_\rho(y|0)\langle[A'(J, y + \theta)]^\rho\rangle_{\theta,J} \tag{46}$$

where Φ_ρ and Ψ_ρ are as before the marginals of P_ρ and Q_ρ, i.e., $\Phi_\rho(x|0) = \int \mathrm{d}\mu\, P_\rho(x, \mu|0)$ and $\Psi_\rho(y) = \int \mathrm{d}\nu\, Q_\rho(y, \nu|0)$. Moreover, using the property $A'(J, x) < 1$ for every J, x we obtain the ordering relation $\lambda_\rho(0) < \lambda_0(0) = 1$ for every $\rho > 1$. We may also generalize equations (41) which give the relation between the solutions of the two $\rho = 0$ eigenvalue problems. It is straightforward to check by substitution into (42), (43) that for $\rho > 1$ we have

$$P_\rho(x,\mu|0) = \int \mathrm{d}y\,\mathrm{d}\nu\, Q_\rho(y,\nu|0)\langle\delta(x-\theta-y)\rangle_\theta\delta(\mu-\nu) \tag{47}$$

$$Q_\rho(y,\nu|0) = \frac{\int \mathrm{d}x\,\mathrm{d}\mu\, P_\rho(x,\mu|0)\big\langle[A'(J,x)]^\rho\delta[y-A(J,x)]\delta\big[\nu-\frac{\mu-B'(J,x)}{A'(J,x)}\big]\big\rangle_J}{\int \mathrm{d}x\,\Phi_\rho(x|0)\langle[A'(J,x)]^\rho\rangle_J}. \tag{48}$$

3.5. Spectral decompositions

Standard linear algebra guarantees that left and right eigenvectors corresponding to different eigenvectors are orthogonal. Thus, given that our eigenvalues $\lambda_\rho(n)$ depend only on the size ρ of the index sets we know that

$$\rho \neq \rho' : \sum_{\boldsymbol{\sigma}} u_{\{\rho\}}(\boldsymbol{\sigma}; P_\rho)v_{\{\rho'\}}(\boldsymbol{\sigma}; Q_{\rho'}) = 0. \tag{49}$$

It follows that we may always use the decomposition

$$T_n(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \sum_{\rho=0}^{n}\lambda_\rho(n)U_n^{(\rho)}(\boldsymbol{\sigma},\boldsymbol{\sigma}') \tag{50}$$

in which the matrices $\boldsymbol{U}_n^{(\rho)}$ are projection matrices, each formed of linear combinations of $\lambda_\rho(n)$ eigenvectors and each acting only in one of the orthogonal eigenspaces. We note that also $\boldsymbol{T}_n^k = \sum_{\rho=0}^n \lambda_\rho^k(n)\boldsymbol{U}_n^{(\rho)}$ for any integer $k > 0$, and that the trace of a projection operator reduces to the dimension of the space which it projects, i.e., $\mathrm{tr}\big(\boldsymbol{U}_n^{(\rho)}\big) = \binom{n}{\rho}$. Since the dimensions of both the $\lambda_0(n)$ and the $\lambda_n(n)$ eigenspaces are 1, the corresponding eigenvectors are pairwise orthogonal and orthogonal to all other eigenvectors, and therefore

$$U_n^{(0)}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \frac{u_{\{0\}}(\boldsymbol{\sigma})v_{\{0\}}(\boldsymbol{\sigma}')}{D_0(n)} \qquad U_n^{(n)}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \frac{u_{\{n\}}(\boldsymbol{\sigma})v_{\{n\}}(\boldsymbol{\sigma}')}{D_n(n)} \tag{51}$$

with

$$\begin{aligned} D_\rho(n) &= \sum_{\boldsymbol{\sigma}} v_{\{\rho\}}(\boldsymbol{\sigma})u_{\{\rho\}}(\boldsymbol{\sigma}) \\ &= \int \mathrm{d}x\,\mathrm{d}\mu\, P_\rho(x,\mu|n)\int \mathrm{d}y\,\mathrm{d}\nu\, Q_\rho(y,\nu|n) \\ &\quad\times [2\cosh(\beta x+\beta y)]^n[1+\mu\nu-\tanh(\beta x+\beta y)[\mu+\nu]]^\rho. \end{aligned} \tag{52}$$

We note that $\lim_{n\to 0} D_0(n) = 1$. Expression (50) will prove useful in calculating observables such as magnetizations and correlation functions. If also within each eigenspace, characterized by an index set size $1 \leqslant \rho \leqslant (n-1)$, the eigenvectors would be orthogonal (as in chains without disorder, or as in the random bond chain without external fields), then we would have the identity $U_n^{(\rho)}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \sum_{\{\rho\}} u_{\{\rho\}}(\boldsymbol{\sigma})v_{\{\rho\}}(\boldsymbol{\sigma}')/D_\rho(n)$ for *all* ρ, and hence

$$T_n(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \sum_{\rho=0}^{n}\lambda_\rho(n)\sum_{\{\rho\}}\frac{u_{\{\rho\}}(\boldsymbol{\sigma})v_{\{\rho\}}(\boldsymbol{\sigma}')}{D_\rho(n)} \tag{53}$$

where the summation $\sum_{\{\rho\}}$ is over all different replica indices sets with ρ elements.

4. Applications of the theory: the random field Ising model

As a benchmark test, let us first calculate the free energy and various observables for the random field Ising chain (1) with nearest neighbour bonds of strength J_0.

4.1. The free energy

We recall that the free energy is given by $\bar{f} = -\lim_{n\to 0}\frac{1}{n}\lim_{N\to\infty}\frac{1}{\beta N}\log \mathrm{tr}\left(\boldsymbol{T}_n^N\right)$, where $\boldsymbol{T}_n$ is the replicated transfer matrix (6). Assuming that the largest eigenvalue is $\lambda_0(n)$, we may write the trace as

$$\mathrm{tr}\left(\boldsymbol{T}_n^N\right) = \sum_{\rho=0}^{n}[\lambda_\rho(n)]^N\,\mathrm{tr}\left(\boldsymbol{U}_n^{(\rho)}\right) = \lambda_0^N(n)\left[1+\sum_{\rho=1}^{n}\binom{n}{\rho}\left(\frac{\lambda_\rho(n)}{\lambda_0(n)}\right)^N\right]. \quad (54)$$

Since $\lim_{N\to\infty}(\lambda_{\rho<0}(n)/\lambda_0(n))^N = 0$, only the contribution of the largest eigenvalue survives, so that, upon writing $\lambda_0(n) = 1+\lambda n+\mathcal{O}(n^2)$ (for we had already established that $\lambda_0(0)=1$):

$$\bar{f} = -\frac{1}{\beta}\lim_{n\to 0}\frac{1}{n}\log\lambda_0(n) = -\frac{1}{\beta}\lim_{n\to 0}\frac{1}{n}\log[1+n\lambda+\mathcal{O}(n^2)] = -\frac{\lambda}{\beta}. \quad (55)$$

The $\mathcal{O}(n)$ contribution λ to $\lambda_0(n)$ can be found upon expanding (37) for small n, and is found to be $\lambda = \beta\int \mathrm{d}x\,\Phi(x)B(J_0,x)$. Insertion of this result into (55) gives us

$$\bar{f} = -\frac{1}{2\beta}\int \mathrm{d}x\,\Phi(x)\log 4\cosh(\beta(J_0+x))\cosh(\beta(J_0-x)). \quad (56)$$

This expression can be converted into a form more familiar from the one-dimensional random systems literature [2, 5, 6]. If we define a new random variable $\tilde{x}$ and an associated density $\tilde{\Phi}(\tilde{x})$ via $\tilde{\Phi}(\tilde{x}) = \int \mathrm{d}x\,\Phi(x)\delta[\tilde{x}-\mathrm{e}^{2\beta x}]$, we find after some straightforward manipulations that

$$\bar{f} = \langle\theta\rangle_\theta - \frac{1}{\beta}\int \mathrm{d}\tilde{x}\,\tilde{\Phi}(\tilde{x})\log[\mathrm{e}^{\beta J_0}+\tilde{x}\mathrm{e}^{-\beta J_0}] \quad (57)$$

where

$$\tilde{\Phi}(\tilde{x}) = \int \mathrm{d}\tilde{x}'\,\tilde{\Phi}(\tilde{x}')\left\langle\delta\left[\tilde{x}-\mathrm{e}^{2\beta\theta}\frac{\mathrm{e}^{-\beta J_0}+\tilde{x}'\,\mathrm{e}^{\beta J_0}}{\mathrm{e}^{\beta J_0}+\tilde{x}'\,\mathrm{e}^{-\beta J_0}}\right]\right\rangle_\theta. \quad (58)$$

The resulting (correct) expression (57) for the free energy[1] justifies *a posteriori* our assumption that $\lambda_0(n)$ is generally the largest eigenvalue, and confirms that our ansatz for the associated right and left eigenvectors, which are seen themselves to be replica symmetric (i.e., $u_0(\pi(\boldsymbol{\sigma})) = u_0(\boldsymbol{\sigma})$ and $v_0(\pi(\boldsymbol{\sigma})) = v_0(\boldsymbol{\sigma})$ for every permutation $\pi\in S_n$), was correct.

4.2. Single-site expectation values and their powers

Let us next show how single-site observables of the form $\overline{\langle\sigma_i\rangle^\rho}$ (integer ρ), with brackets denoting a thermal average over the Boltzman measure and $\overline{\cdots}$ denoting averaging over the disorder, can also be calculated. We use the following replica identity which will enable us to express single-site expectation values in terms of the replicated transfer matrix

$$\begin{aligned}\overline{\langle\sigma_i\rangle^\rho} &= \lim_{n\to 0}\overline{\left[\sum_{\boldsymbol{\sigma}}\sigma_i\,\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^\rho\left[\sum_{\boldsymbol{\sigma}}\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^{n-\rho}}\\ &= \lim_{n\to 0}\sum_{\{\boldsymbol{\sigma}\}}\sigma_i^{\alpha_1}\cdots\sigma_i^{\alpha_\rho}\overline{\prod_{\alpha=1}^{n}\mathrm{e}^{-\beta H(\boldsymbol{\sigma}^\alpha)}}\\ &= \lim_{n\to 0}\sum_{\{\boldsymbol{\sigma}\}}\sigma_i^{\alpha_1}\cdots\sigma_i^{\alpha_\rho}\prod_i T_n(\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_{i+1})\end{aligned} \quad (59)$$

[1] In the traditional method to solve one-dimensional random bond or random field spin chains [2, 5, 6], based on constructing iterative relations for the partition function as the size of the chain is increased by one spin, the variable $\tilde{x}$ in (57) represents the ratio of conditioned partition functions.

and define the diagonal $2^n \times 2^n$ matrix $\boldsymbol{S}_{\{\rho\}}$ with entries

$$S_{\{\rho\}}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \prod_{\alpha\in\{\rho\}} \sigma_\alpha. \tag{60}$$

Upon using the replicated transfer matrix (6) to evaluate (59), and upon dividing (59) by $1 = \lim_{n\to 0} \overline{Z^n} = \lim_{n\to 0} \operatorname{tr}(\boldsymbol{T}^N)$, expression (59) can be written in the form

$$\overline{\langle\sigma_i\rangle^\rho} = \lim_{n\to 0} \frac{\operatorname{tr}(\boldsymbol{S}_{\{\rho\}}\boldsymbol{T}^N)}{\operatorname{tr}(\boldsymbol{T}^N)}.$$

For large N our spectral decomposition (50) now gives us

$$\begin{aligned}
\overline{\langle\sigma_i\rangle^\rho} &= \lim_{n\to 0} \lim_{N\to\infty} \frac{\operatorname{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(0)}\right) + \sum_{\rho'=1}^n [\lambda_{\rho'}(n)/\lambda_0(n)]^N \operatorname{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(\rho')}\right)}{1 + \sum_{\rho'=1}^n [\lambda_{\rho'}(n)/\lambda_0(n)]^N} \\
&= \lim_{n\to 0} \operatorname{tr}(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(0)}) \\
&= \lim_{n\to 0} D_0^{-1}(n) \sum_{\boldsymbol{\sigma}} v_{\{0\}}(\boldsymbol{\sigma}) u_{\{0\}}(\boldsymbol{\sigma}) \prod_{\alpha\in\{\rho\}} \sigma_\alpha \\
&= \lim_{n\to 0} \int \mathrm{d}x\, \mathrm{d}y\, \Phi_0(x|n)\Psi_0(y|n)\, [2\cosh(\beta x + \beta y)]^n \tanh^\rho(\beta x + \beta y).
\end{aligned} \tag{61}$$

We note that the dependence on the particular realization of the index set $\{\rho\}$ has disappeared, as it should, leaving only a dependence on the size ρ of this set. We may now take the limit $n \to 0$, and find our transparent and appealing final result

$$\overline{\langle\sigma_i\rangle^\rho} = \int \mathrm{d}x\, \mathrm{d}y\, \Phi(x)\Psi(y) \tanh^\rho(\beta x + \beta y). \tag{62}$$

Equation (62) also shows that $\Phi(x)$ and $\Psi(y)$ represent distributions of effective fields.

4.3. Multiple-site observables

Finally, we apply our methods to the evaluation of disorder-averaged powers of two-spin correlations of the form $\overline{\langle\sigma_i\sigma_j\rangle^\rho}$ with integer ρ. As in the previous section, we express the observables in terms of the replicated transfer matrix and exploit the associated spectral decomposition to take the limit $N \to \infty$ after which only contributions from the largest eigenvalue remain. The resulting expressions can then be analytically continued to real values of n. However, the calculations are obviously more involved than those of single-site quantities, and will at some point require further ansätze. We choose $j > i$ and start from the identity

$$\begin{aligned}
\overline{\langle\sigma_i\sigma_j\rangle^\rho} &= \lim_{n\to 0} \overline{\left[\sum_{\boldsymbol{\sigma}} \sigma_i\sigma_j\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^\rho \left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^{n-\rho}} \\
&= \lim_{n\to 0} \sum_{\{\boldsymbol{\sigma}\}} \sigma_i^{\alpha_1}\sigma_j^{\alpha_1} \cdots \sigma_i^{\alpha_\rho}\sigma_j^{\alpha_\rho} \overline{\prod_{\alpha=1}^n \mathrm{e}^{-\beta H(\boldsymbol{\sigma}^\alpha)}} \\
&= \lim_{n\to 0} \frac{\operatorname{tr}(\boldsymbol{S}_{\{\rho\}}\boldsymbol{T}^{j-i}\boldsymbol{S}_{\{\rho\}}\boldsymbol{T}^{N-j+i})}{\operatorname{tr}(\boldsymbol{T}^N)}.
\end{aligned} \tag{63}$$

Our spectral decomposition (50), together with $\lambda_0(n) = 1$, enables us to write for $N \to \infty$:

$$\begin{aligned}
\overline{\langle\sigma_i\sigma_j\rangle^\rho} &= \lim_{n\to 0} \lim_{N\to\infty} \frac{\sum_{\rho'\rho''=0}^n \left[\frac{\lambda_{\rho'}(n)}{\lambda_0(n)}\right]^{j-i} \left[\frac{\lambda_{\rho''}(n)}{\lambda_0(n)}\right]^{N-j+i} \operatorname{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(\rho')}\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(\rho'')}\right)}{1 + \sum_{\rho'=1}^n \left[\frac{\lambda_{\rho'}(n)}{\lambda_0(n)}\right]^N} \\
&= \lim_{n\to 0} \sum_{\rho'=0}^n \lambda_{\rho'}(0)^{j-i} \operatorname{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(\rho')}\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(0)}\right).
\end{aligned} \tag{64}$$

To work out the trace in (64) we write the entries of our projection matrices as follows:

$$U_n^{(\rho')}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \sum_{\substack{\{\varsigma\},\{\varsigma'\}\\ |\{\varsigma\}|=|\{\varsigma'\}|=\rho'}} V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}} u_{\{\varsigma\}}(\boldsymbol{\sigma}) v_{\{\varsigma'\}}(\boldsymbol{\sigma}') \tag{65}$$

where the summation is over all different replica index sets $\{\varsigma\}, \{\varsigma'\}$ with fixed number of elements ρ'. We may now write

$$\begin{aligned}\mathrm{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(\rho')}\boldsymbol{S}_{\{\rho\}}\boldsymbol{U}_n^{(0)}\right) &= \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\Big[v_{\{0\}}(\boldsymbol{\sigma})\prod_{\alpha\in\{\rho\}}\sigma_\alpha\Big]U_n^{(\rho')}(\boldsymbol{\sigma},\boldsymbol{\sigma}')\Big[u_{\{0\}}(\boldsymbol{\sigma}')\prod_{\alpha\in\{\rho\}}\sigma'_\alpha\Big]\\ &= \sum_{\substack{\{\varsigma\},\{\varsigma'\}\\ |\{\varsigma\}|=|\{\varsigma'\}|=\rho'}} V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}} A^{(\{0\},\{\varsigma\})}_{\{\rho\}} A^{(\{\varsigma'\},\{0\})}_{\{\rho\}}\end{aligned} \tag{66}$$

with

$$A^{(\{0\},\{\varsigma\})}_{\{\rho\}} = \sum_{\boldsymbol{\sigma}} v_{\{0\}}(\boldsymbol{\sigma}) u_{\{\varsigma\}}(\boldsymbol{\sigma}) \prod_{\alpha\in\{\rho\}}\sigma_\alpha \tag{67}$$

$$A^{(\{\varsigma\},\{0\})}_{\{\rho\}} = \sum_{\boldsymbol{\sigma}} u_{\{0\}}(\boldsymbol{\sigma}) v_{\{\varsigma\}}(\boldsymbol{\sigma}) \prod_{\alpha\in\{\rho\}}\sigma_\alpha. \tag{68}$$

Our correlations (64) can apparently be written in the simplified form

$$\overline{\langle\sigma_i\sigma_j\rangle^\rho} = \lim_{n\to 0}\sum_{\rho'=0}^{n}\lambda_{\rho'}(0)^{j-i}\sum_{\substack{\{\varsigma\},\{\varsigma'\}\\ |\{\varsigma\}|=|\{\varsigma'\}|=\rho'}} V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}} A^{(\{0\},\{\varsigma\})}_{\{\rho\}} A^{(\{\varsigma'\},\{0\})}_{\{\rho\}}. \tag{69}$$

Inserting the eigenvectors (27), (28) into expressions (67), (68) for the coefficients $A^{(\{0\},\{\varsigma\})}_{\{\rho\}}(n)$ and $A^{(\{\varsigma\},\{0\})}_{\{\rho\}}(n)$, followed by summation over the spin variables, gives

$$\begin{aligned}A^{(\{0\},\{\varsigma\})}_{\{\rho\}}(n) = &\int \mathrm{d}x\,\mathrm{d}\mu\, P_{\rho'}(x,\mu|n)\int \mathrm{d}y\;\Psi_0(y|n)\\ &\times[2\cosh(\beta x+\beta y)]^n[1-\mu\tanh(\beta x+\beta y)]^{|\{\rho\}\cap\{\varsigma\}|}\\ &\times[\tanh(\beta x+\beta y)]^{|\{\rho\}\cap\overline{\{\varsigma\}}|}[\tanh(\beta x+\beta y)-\mu]^{|\overline{\{\rho\}}\cap\{\varsigma\}|}\end{aligned} \tag{70}$$

$$\begin{aligned}A^{(\{\varsigma\},\{0\})}_{\{\rho\}}(n) = &\int \mathrm{d}x\;\Phi_0(x|n)\int \mathrm{d}y\,\mathrm{d}\nu\, Q_{\rho'}(y,\nu|n)\\ &\times[2\cosh(\beta x+\beta y)]^n[1-\nu\tanh(\beta x+\beta y)]^{|\{\rho\}\cap\{\varsigma\}|}\\ &\times[\tanh(\beta x+\beta y)]^{|\{\rho\}\cap\overline{\{\varsigma\}}|}[\tanh(\beta x+\beta y)-\nu]^{|\overline{\{\rho\}}\cap\{\varsigma\}|}.\end{aligned} \tag{71}$$

These quantities no longer depend on the detailed realizations of the index sets, but only on the sizes of these sets and of their intersections. Let us denote the number of elements in the intersection of $\{\rho\}$ and $\{\varsigma\}$ by $k=|\{\rho\}\cap\{\varsigma\}|$, $k=0,\ldots,\min\{\rho,\rho'\}$ (since $|\{\varsigma\}|=\rho'$):

$$|\{\rho\}\cap\{\varsigma\}|=k,\qquad |\{\rho\}\cap\overline{\{\varsigma\}}|=\rho-k,\qquad |\overline{\{\rho\}}\cap\{\varsigma\}|=\rho'-k \tag{72}$$

with similar definitions in the case of $\{\varsigma'\}$, defining the variable k'. We may now write (69) as

$$\begin{aligned}\overline{\langle\sigma_i\sigma_j\rangle^\rho} = \lim_{n\to 0}&\sum_{\rho'=0}^{n}\lambda_{\rho'}(0)^{j-i}\sum_{k,k'=0}^{\min\{\rho,\rho'\}} A^{(0,\rho')}_{\rho,k}A^{(\rho',0)}_{\rho,k'}\\ &\times\sum_{\substack{\{\varsigma\},\,|\{\varsigma\}|=\rho'\\ |\{\rho\}\cap\{\varsigma\}|=k}}\;\sum_{\substack{\{\varsigma'\},\,|\{\varsigma'\}|=\rho'\\ |\{\rho\}\cap\{\varsigma'\}|=k'}} V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}}\end{aligned} \tag{73}$$

in which $A^{(0,\rho')}_{\rho,k}$ and $A^{(\rho',0)}_{\rho,k}$ denote the $n\to 0$ limits of (70) and (71), respectively (with the conventions as laid down in (72)):

$$A^{(0,\rho')}_{\rho,k} = \int \mathrm{d}x\,\mathrm{d}\mu\, P_{\rho'}(x,\mu|0) \int \mathrm{d}y\, \Psi_0(y|0)[\tanh(\beta x+\beta y)]^{\rho-k} \times [1-\mu\tanh(\beta x+\beta y)]^k[\tanh(\beta x+\beta y)-\mu]^{\rho'-k} \tag{74}$$

$$A^{(\rho',0)}_{\rho,k} = \int \mathrm{d}x\, \Phi_0(x|0) \int \mathrm{d}y\,\mathrm{d}\nu\, Q_{\rho'}(y,\nu|0)[\tanh(\beta x+\beta y)]^{\rho-k} \times [1-\nu\tanh(\beta x+\beta y)]^k[\tanh(\beta x+\beta y)-\nu]^{\rho'-k}. \tag{75}$$

The rigorous evaluation of the last line in (73) for arbitrary models requires the explicit calculation of the expansion factors $V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}}$. Although one can easily write formal expressions for these quantities in terms of the inverse of the matrix of inner products of the eigenvectors within a given eigenspace ρ', this leads as yet only to expressions in which it is not clear how the limit $n\to 0$ can be taken.

We can at present only push the evaluation of (73) to its conclusion for those cases where the eigenvectors within each eigenspace are either explicitly orthogonal for any n (as in chains without disorder, or in the random bond chain without external fields), or become effectively orthogonal in the $n\to 0$ limit. The latter is very hard to verify or disprove *a priori*, but can serve as an efficient ansatz, to be verified later using numerical simulations. In these cases, we are allowed to write simply $V^{(\rho')}_{\{\varsigma\},\{\varsigma'\}} = D^{-1}_{\rho'}(n)\delta_{\{\varsigma\},\{\varsigma'\}}$ and find (73) reducing to

$$\overline{\langle\sigma_i\sigma_j\rangle^{\rho}} = \lim_{n\to 0}\left\{\sum_{\varsigma=0}^{\rho}\frac{\lambda_\varsigma(0)^{j-i}}{D_\varsigma(0)}\sum_{k=0}^{\varsigma}\binom{\rho}{k}\binom{n-\rho}{\varsigma-k}A^{(0,\varsigma)}_{\rho,k}A^{(\varsigma,0)}_{\rho,k} + \sum_{\varsigma>\rho}\frac{\lambda_\varsigma(0)^{j-i}}{D_\varsigma(0)}\sum_{k=0}^{\rho}\binom{\rho}{k}\binom{n-\rho}{\varsigma-k}A^{(0,\varsigma)}_{\rho,k}A^{(\varsigma,0)}_{\rho,k}\right\}. \tag{76}$$

It turns out that in (76) only the terms with $k=\varsigma$ will survive the limit $n\to 0$. In the special case of non-disordered models, where $P_\rho(x,\mu|n)=\delta(x-x^*)\delta(\mu-\mu^*)$ and $Q_\rho(y,\nu|n)=\delta(y-y^*)\delta(\nu-\mu^*)$, with $\mu^*=\tanh(\beta(x^*+y^*))$, we see that $A^{(0,\varsigma)}_{\rho,k}$ and $A^{(\varsigma,0)}_{\rho,k}$ vanish unless $k=\varsigma$. More generally we show in the appendix that for integer ρ and ℓ:

$$\rho\geqslant 1,\ \ell\geqslant 0: \qquad \lim_{n\to 0}\binom{n-\rho}{\ell} = \delta_{\ell,0} \tag{77}$$

It follows that the second line of (76) must vanish entirely since there one always has the ordering $k\leqslant\rho<\varsigma$, whereas in the first line we retain only the terms with $k=\varsigma$. Thus, together with (52) we arrive at

$$\overline{\langle\sigma_i\sigma_j\rangle^{\rho}} = \sum_{\varsigma=0}^{\rho} D^{-1}_\varsigma\binom{\rho}{\varsigma}A^{(0,\varsigma)}_{\rho}A^{(\varsigma,0)}_{\rho}\lambda_\varsigma(0)^{j-i} \tag{78}$$

$$A^{(0,\varsigma)}_{\rho} = \int \mathrm{d}x\,\mathrm{d}\mu\, P_\varsigma(x,\mu|0)\int \mathrm{d}y\,\Psi_0(y|0)[1-\mu\tanh(\beta x+\beta y)]^{\varsigma}[\tanh(\beta x+\beta y)]^{\rho-\varsigma} \tag{79}$$

$$A^{(\varsigma,0)}_{\rho} = \int \mathrm{d}x\,\Phi_0(x|0)\int \mathrm{d}y\,\mathrm{d}\nu\, Q_\varsigma(y,\nu|0)[1-\nu\tanh(\beta x+\beta y)]^{\varsigma}[\tanh(\beta x+\beta y)]^{\rho-\varsigma} \tag{80}$$

$$D_\rho = \int \mathrm{d}x\,\mathrm{d}\mu\, P_\rho(x,\mu|0)\int \mathrm{d}y\,\mathrm{d}\nu\, Q_\rho(y,\nu|0)\Big[1+\mu\nu-\tanh(\beta x+\beta y)[\mu+\nu]\Big]^{\rho}. \tag{81}$$

This concludes our calculations for the random field Ising chain. The limit $n \to 0$ has been taken, and we are left with an explicit theory with which to calculate not only the free energy per spin but also the relevant observables and correlation functions.

4.4. Comparison with simulations

We have tested the final predictions (62), (78) for the observables and correlations functions in the random field Ising chain with $p(\theta) = p\delta(\theta - \tilde{\theta}) + (1-p)\delta(\theta + \tilde{\theta})$. Objects such as $\langle\sigma_i\rangle^2$ or $\langle\sigma_i\sigma_j\rangle^2$ were measured by simulating two copies of the system, each with identical disorder realizations but each evolving independently according to standard Glauber dynamics towards equilibrium following a randomly chosen microscopic initial state. The results are shown in figure 1. In all simulations the system size was $N = 20\,000$ spins. We concentrated on the following macroscopic quantities:

$$m = \frac{1}{N}\sum_i \langle\sigma_i\rangle, \qquad a_1 = \frac{1}{N}\sum_i \langle\sigma_i\sigma_{i+1}\rangle, \qquad a_2 = \frac{1}{N}\sum_i \langle\sigma_i\sigma_{i+2}\rangle \tag{82}$$

$$q = \frac{1}{N}\sum_i \langle\sigma_i\rangle^2, \qquad r = \frac{1}{N}\sum_i \langle\sigma_i\sigma_{i+1}\rangle^2. \tag{83}$$

Ideally in the thermodynamic limit the evolution of those observables are given by smooth curves. The fluctuations in the plots we present are due to the finite (although very large) system size and the fact that our data points correspond to integer values of t with the intermediate lines segments serving as guides to the eye.

The evaluation of the theoretical predictions (62), (78) involved solving the relevant functional eigenvalue equations numerically. For m and q, which both follow from (62), one just needs to solve (39) for $\lambda_0(0) = 1$, which is straightforward (either by iteration, or using a population dynamics algorithm). The function $\Psi(y)$ subsequently follows via identity (41). We see in figure 1 that for m and q the agreement between theory and experiment is excellent. Figure 2 shows the corresponding shapes of the integrated field distribution $\hat{\Phi}(x) = \int_{-\infty}^{x} \mathrm{d}z\, \Phi(z)$, and of the integrated single-site magnetization distribution

$$\hat{W}(m) = \int_{-1}^{m} \mathrm{d}n \int \mathrm{d}x\, \mathrm{d}y \Phi(x)\Psi(y)\delta[n - \tanh(\beta(x+y))]$$

which are smoother functions than the distributions themselves. We can see that they exhibit the by now familiar characteristics of random field Ising models (see, e.g., [5, 6, 8, 9]).

For those observables which require evaluation of (78), and therefore numerical solution of the eigenvalue problems (42), (43) for different values of ρ (which is feasible but extremely demanding in computing time), we have used the approximation consisting of replacing $P_\rho(\cdots)$ and $Q_\rho(\cdots)$ for $\rho > 0$ by $P_0(\cdots)$ and $Q_0(\cdots)$, respectively. This would formally be allowed only in the non-disordered case (where also the assumed orthogonality of our eigenvectors within eigenspaces is correct), but is seen to give surprisingly accurate results even for those cases where the shape of these distributions is highly non-trivial, see figures 1 and 2.

5. Applications of the theory: neural networks and 'small-world' systems

The theory in section 3 can be applied to any model which involves replicated transfer matrices. Here, we demonstrate how it may be used to analyse models which are structurally different from the random field Ising model, in having not only short-range but also long-range bonds.

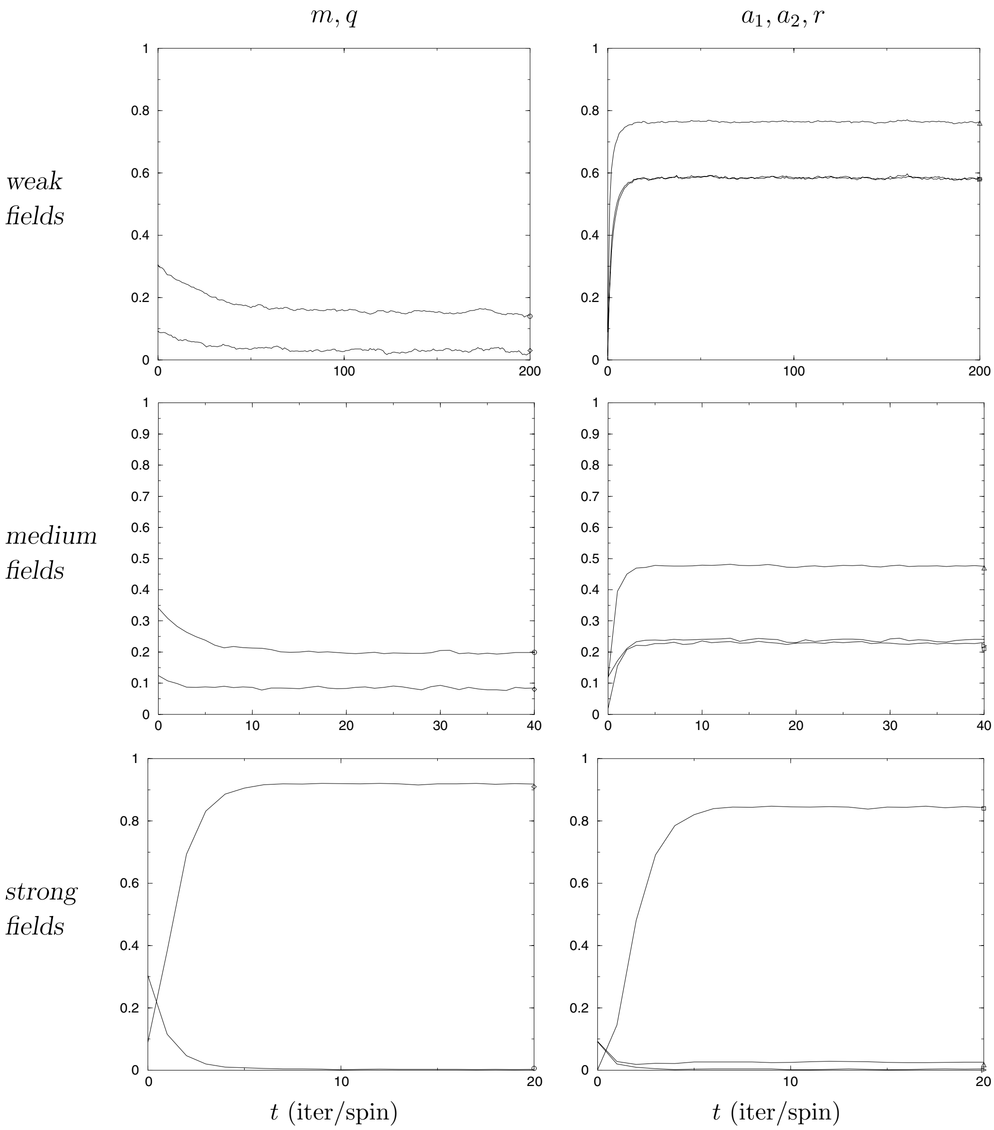

Figure 1. Relaxation of observables towards equilibrium at $T = 1$, in two random field Ising chains with identical disorder realizations, of size $N = 20\,000$ and with field distribution $p(\theta) = p\delta(\theta - \tilde{\theta}) + (1 - p)\delta(\theta + \tilde{\theta})$. Left column: evolution of the magnetization $m = N^{-1}\sum_i \sigma_i$ and the order parameter $q = N^{-1}\sum_i \sigma_i\sigma_i'$. Right column: evolution of the multiple site quantities $a_1 = N^{-1}\sum_i \sigma_i\sigma_{i+1}$, $a_2 = N^{-1}\sum_i \sigma_i\sigma_{i+2}$ and $r = N^{-1}\sum_i \sigma_i\sigma_{i+1}\sigma_i'\sigma_{i+1}'$. Different rows correspond to different control parameters. Top row: weak random fields, with $J_0 = 1$, $\tilde{\theta} = 0.05$, $p = 0.7$, where the theoretical equilibrium predictions are $m \simeq 0.14$, $q \simeq 0.03$, $a_1 \simeq 0.76$, $a_2 \simeq 0.58$, $r \simeq 0.58$. Middle row: intermediate fields, with $J_0 = 0.5$, $\tilde{\theta} = 0.2$, $p = 0.7$, where our theory predicts $m \simeq 0.20$, $q \simeq 0.08$, $a_1 \simeq 0.47$, $a_2 \simeq 0.22$, $r \simeq 0.21$. Bottom row: strong random fields, with $J_0 = 0.2$, $\tilde{\theta} = 2$, $p = 0.5$, where the theory predicts the equilibrium values $m \simeq 0.006$, $q \simeq 0.91$, $a_1 \simeq 0.018$, $a_2 \simeq 0.0003$, $r = 0.84$. In all cases, the predictions are indicated by markers at the right of the graphs.

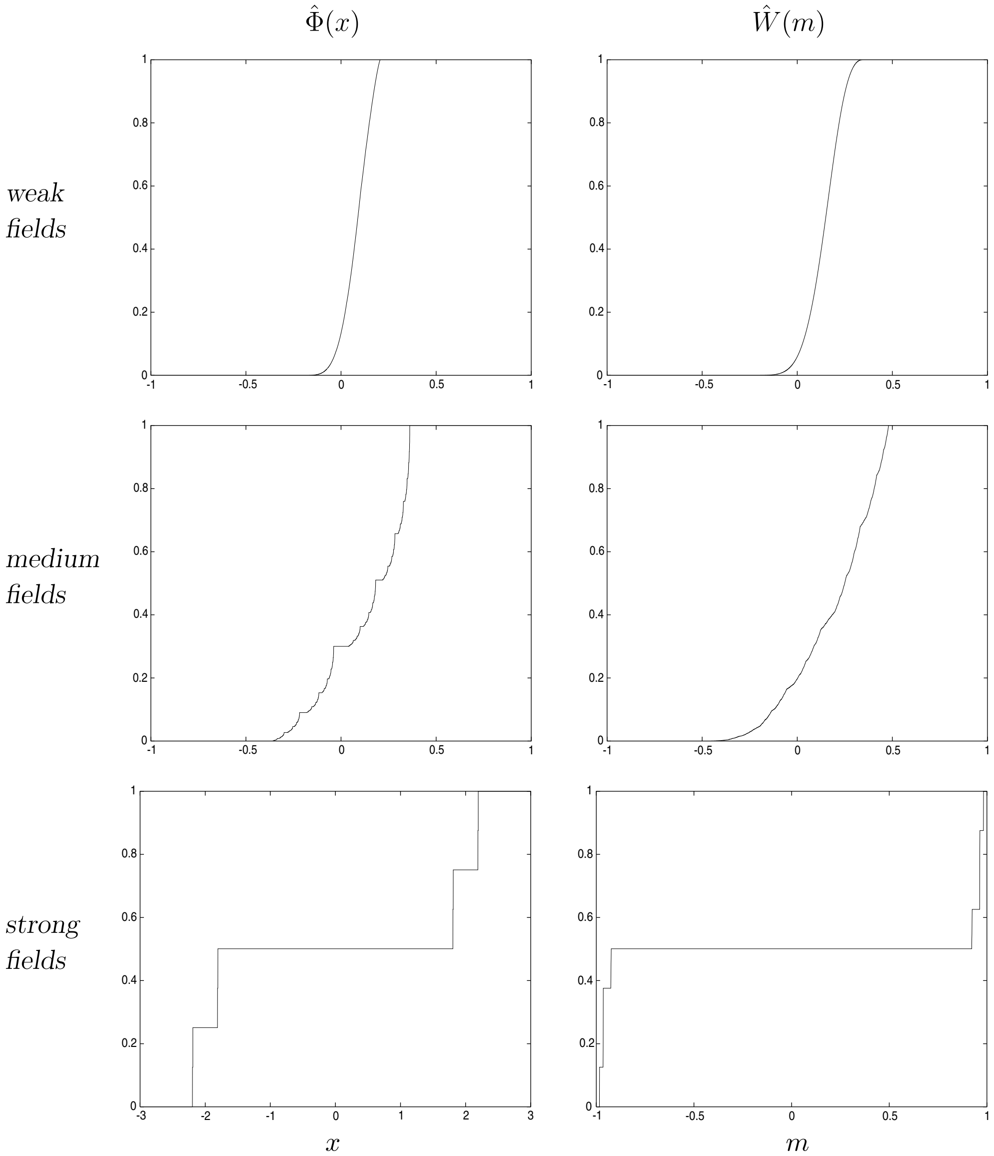

Figure 2. The integrated field and site magnetization distributions corresponding to the data of the previous figure 1, as obtained by numerical solution of our integral eigenvalue equations (39), (40) via a population dynamics algorithm. The rows correspond again to weak random fields (top row), intermediate random fields (middle row) and strong random fields (bottom row). Note that in the second case the integrated distribution $\hat{\Phi}(x)$ has the form of devil's staircase which in turn implies that the associated distribution $\Phi(x)$ is highly non-trivial.

5.1. $(1+\infty)$*-dimensional attractor neural networks*

We now turn to the attractor neural network described by the Hamiltonian (2), where short-range interactions compete with long-range ones. A detailed study of the model, based on the more conventional methods of [5, 6] can be found in [15, 16]; here our objective is only to demonstrate how the present replicated transfer matrix diagonalization formalism can also be put to use in the context of such models. Upon introducing the p overlap order parameters $m_\mu(\boldsymbol{\sigma}) = N^{-1}\sum_i \xi_i^\mu \sigma_i$, each of which measures the similarity between the system's microscopic configuration $\boldsymbol{\sigma}$ and a given stored pattern, one arrives after some standard manipulations at the following expression for the partition function:

$$Z = \int \mathrm{d}\boldsymbol{m}\, \exp\left(N\left[-\frac{1}{2}\beta J_\ell \boldsymbol{m}^2 + r(\boldsymbol{m})\right]\right) \tag{84}$$

where $\boldsymbol{m} = (m_1, \ldots, m_p)$, $\boldsymbol{m}^2 = \sum_\mu m_\mu^2$ and $r(\boldsymbol{m}) = \frac{1}{N}\log R(\boldsymbol{m})$ with

$$R(\boldsymbol{m}) = \sum_{\sigma_1\ldots\sigma_N} \exp\left(\beta J_s \sum_i \sigma_i\sigma_{i+1}(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_{i+1}) + \beta J_\ell \sum_i \sigma_i(\boldsymbol{m}\cdot\boldsymbol{\xi}_i)\right). \tag{85}$$

One may now calculate $r(\boldsymbol{m})$ by regarding the random patterns as disorder and use the replica approach to calculate the disorder average. In the thermodynamic limit, $r(\boldsymbol{m})$ (which is itself mathematically identical to the free energy per spin of a suitably defined chain) must be identical to its disorder average, with probability 1. Therefore, we consider

$$\overline{r(\boldsymbol{m})} = \lim_{n\to 0}\frac{1}{n}\lim_{N\to\infty}\frac{1}{N}\log\overline{R^n(\boldsymbol{m})}.$$

In particular, we have

$$\begin{aligned}\overline{R^n(\boldsymbol{m})} &= 2^{-pN}\sum_{\boldsymbol{\xi}_1\cdots\boldsymbol{\xi}_N}\sum_{\boldsymbol{\sigma}_1\cdots\boldsymbol{\sigma}_N}\prod_i \exp\left(\beta J_s(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_{i+1})(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_{i+1}) + \beta J_\ell(\boldsymbol{m}\cdot\boldsymbol{\xi}_i)\sum_{\alpha=1}^n \sigma_i^\alpha\right)\\ &= \mathrm{tr}(\boldsymbol{T}^N(\boldsymbol{m}))\end{aligned}$$

where $\boldsymbol{\sigma}_i = \left(\sigma_i^1, \ldots, \sigma_i^n\right)$, and $\boldsymbol{T}(\boldsymbol{m})$ is a $2^{np}\times 2^{np}$ transfer matrix with entries

$$T_{\boldsymbol{\xi},\boldsymbol{\xi}'}(\boldsymbol{\sigma},\boldsymbol{\sigma}';\boldsymbol{m}) = 2^{-p}\exp\left(\beta J_s(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}') + \beta J_\ell(\boldsymbol{m}\cdot\boldsymbol{\xi})\sum_{\alpha=1}^n \sigma_\alpha\right). \tag{86}$$

In order to determine the largest eigenvalue of this replicated transfer matrix we make the by now familiar type of ansatz for its left and right eigenvectors, but now applied to each value of $\boldsymbol{\xi}\in\{-1,1\}^P$ separately:

$$v_{\boldsymbol{\xi}}(\boldsymbol{\sigma}) = \int \mathrm{d}y\,\Psi_{\boldsymbol{\xi}}(y|n)\exp\left(\beta y\sum_{\alpha=1}^n\sigma_\alpha\right) \qquad u_{\boldsymbol{\xi}}(\boldsymbol{\sigma}) = \int \mathrm{d}x\,\Phi_{\boldsymbol{\xi}}(x|n)\exp\left(\beta x\sum_{\alpha=1}^n\sigma_\alpha\right). \tag{87}$$

Our motivation for this particular choice of the dependence on the pattern vectors $\boldsymbol{\xi}$ is that for $p = 1$ the dependence on the remaining pattern can be transformed away by the gauge transformation $\boldsymbol{\sigma}_i \to \xi_i\boldsymbol{\sigma}_i$. This would leave a replicated transfer matrix of an Ising chain with constant bonds, where the role of the external field is played by $J_\ell m$. Thus, for $p = 1$ the present eigenvectors must reduce to those as studied in section 3.3. Secondly, the group (87) obviously represents only a subset of all eigenvectors (to be precise: the $\rho = 0$ family, in the language of the previous section). Building the full set is straightforward, but here we restrict ourselves for brevity to the main ones, i.e., those which control the free energy and the single-site observables (the other eigenvectors only play a role in the calculation of multiple-site observables).

Having introduced our eigenvectors, we proceed as in the random field Ising model, adding $\boldsymbol{m}$ as a conditioning label wherever needed. We then find in the limit $n\to 0$ that $\lambda(0;\boldsymbol{m}) = 1$, and that our final eigenvalue problems are defined in terms of joint field-pattern distributions:

$$\Psi_{\boldsymbol{\xi}}(y|0) = 2^{-p}\sum_{\boldsymbol{\xi}'}\int \mathrm{d}y'\,\Psi_{\boldsymbol{\xi}'}(y'|0)\delta[y - A(J_s(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'), y' + J_\ell(\boldsymbol{m}\cdot\boldsymbol{\xi}'))] \quad (88)$$

$$\Phi_{\boldsymbol{\xi}}(x|0) = 2^{-p}\sum_{\boldsymbol{\xi}'}\int \mathrm{d}x'\,\Phi_{\boldsymbol{\xi}'}(x'|0)\delta[x - J_\ell(\boldsymbol{m}\cdot\boldsymbol{\xi}) - A(J_s(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'), x')]. \quad (89)$$

These distributions are normalized according to $\int \mathrm{d}x\,\Phi_{\boldsymbol{\xi}}(x|0) = \int \mathrm{d}y\,\Psi_{\boldsymbol{\xi}}(y|0) = 1$ for all $\boldsymbol{\xi}$. The actual value to be inserted for the vector $\boldsymbol{m}$ in the above expressions is to be solved from the saddle-point equations which determine the stationary point of the extensive exponent in the partition sum. This equation can simply be written as $\boldsymbol{m} = \lim_{N\to\infty} N^{-1}\sum_i \overline{\langle\sigma_i\boldsymbol{\xi}_i^\mu\rangle}$. Upon repeating the steps taken earlier in solving the random field Ising model, we get

$$\begin{aligned} m_\mu &= \lim_{n\to 0}\frac{\mathrm{tr}\left(\boldsymbol{S}^\mu_{\{1\}}\boldsymbol{T}^N(\boldsymbol{m})\right)}{\mathrm{tr}(\boldsymbol{T}^N(\boldsymbol{m}))} \\ &= 2^{-p}\sum_{\boldsymbol{\xi}}\int \mathrm{d}x\,\mathrm{d}y\,\Phi_{\boldsymbol{\xi}}(x|0)\Psi_{\boldsymbol{\xi}}(y|0)\xi_\mu\tanh(\beta x + \beta y) \end{aligned} \quad (90)$$

in which $\boldsymbol{S}^\mu_{\{1\}}$ is a diagonal $2^{np}\times 2^{np}$ matrix with elements

$$S^\mu_{\{1\},\boldsymbol{\xi}\boldsymbol{\xi}'}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \delta_{\boldsymbol{\xi},\boldsymbol{\xi}'}\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'}\xi_\mu\sigma_1.$$

The $n=0$ eigenvalue problems for $\Phi_{\boldsymbol{\xi}}$ and $\Psi_{\boldsymbol{\xi}}$ are coupled to the saddle-point equations for the 'mean-field' order parameters. This feature is typical, within the replica formalism, for all models where a one-dimensional structure is embedded in a mean-field (or range-free) architecture, as is the case here.

In order to calculate the free energy we need to know the $\mathcal{O}(n)$ contribution $\lambda(\boldsymbol{m})$ to $\lambda(n;\boldsymbol{m})$ (i.e., $\lambda(n;\boldsymbol{m}) = 1 + n\lambda(\boldsymbol{m}) + \mathcal{O}(n^2)$). The latter can be expressed in terms of the $n=0$ effective field distributions, and is found to be given by

$$\lambda(\boldsymbol{m}) = 2^{-2p}\sum_{\boldsymbol{\xi},\boldsymbol{\xi}'}\int \mathrm{d}y\,\Psi_{\boldsymbol{\xi}'}(y|0)\beta B(J_s(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'), y + J_\ell(\boldsymbol{m}\cdot\boldsymbol{\xi}')).$$

Hence,

$$\begin{aligned} \overline{r(\boldsymbol{m})} &= \lim_{n\to 0}\frac{1}{n}\lim_{N\to\infty}\frac{1}{N}\log\lambda^N(n;\boldsymbol{m}) \\ &= \lim_{n\to 0}\frac{1}{n}\log[1 + n\lambda(\boldsymbol{m}) + \mathcal{O}(n^2)] = \lambda(\boldsymbol{m}). \end{aligned} \quad (91)$$

Substitution of this result for $r(\boldsymbol{m})$ into the partition leads to our final result

$$f = \tfrac{1}{2}J_\ell\boldsymbol{m}^2 - T\lambda(\boldsymbol{m}) \quad (92)$$

in which $\boldsymbol{m}$ is given by the solution of (90). The link with the results of [15] can now be established upon defining a new random variable k, which in [15] represents the ratio of conditioned partition functions, and is subject to a random nonlinear map as one builds up the chain iteratively from $N=1$ to $N=\infty$. With the following definition the two solutions (the one in [15] and the other in this paper) become fully identical:

$$P(k,\boldsymbol{\xi}) = 2^{-p}\int \mathrm{d}y\,\Psi_{\boldsymbol{\xi}}(y)\delta[k - \mathrm{e}^{-2\beta y}]. \quad (93)$$

5.2. 'Small-world' ferromagnets

Our final application example is the so-called 'small-world' ferromagnet, defined by the Hamiltonian (3). As in the previous example this model represents a combination of one-dimensional short-range interactions and long-range ones. In contrast to the previous example the long-range bonds are not 'all-to-all', but represent a finitely connected Poissonian random graph. This model was studied in more detail in [17], where it was shown that application of the replica formalism generates the following replicated transfer matrix, with $\boldsymbol{\sigma}, \boldsymbol{\sigma}', \boldsymbol{\tau} \in \{-1, 1\}^n$:

$$T(\boldsymbol{\sigma}, \boldsymbol{\sigma}'|P) = \exp\left(\beta J_0 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' + c \sum_{\boldsymbol{\tau}} P(\boldsymbol{\tau}) \exp\left[\frac{\beta J}{c} \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\right] - c\right). \tag{94}$$

Here the mean-field order parameter is a function $P(\boldsymbol{\tau})$, which gives the fraction of sites where the replicated spin $\boldsymbol{\sigma}_i$ equals $\boldsymbol{\tau}$. The saddle-point equations are here found to take the form of an expression for $P(\boldsymbol{\tau})$ in terms of those eigenvectors of $\boldsymbol{T}$ which correspond to the largest eigenvalue:

$$P(\boldsymbol{\tau}) = \frac{v_0(\boldsymbol{\tau}) u_0(\boldsymbol{\tau})}{\sum_{\boldsymbol{\tau}} v_0(\boldsymbol{\tau}) u_0(\boldsymbol{\tau})} \tag{95}$$

(assuming this eigenspace to be non-degenerated, similar to our previous models). In this model one expects a replica symmetric solution (RS) to describe the physics correctly, which for the order parameter $P(\boldsymbol{\tau})$ implies the form

$$P(\boldsymbol{\tau}) = \int \mathrm{d}h\, W(h) \frac{\exp\left(\beta h \sum_{\alpha=1}^{n} \tau_\alpha\right)}{[2\cosh(\beta h)]^n}. \tag{96}$$

Insertion of this RS expression into (94) results in the following replicated transfer matrix:

$$T^{\mathrm{RS}}(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \int \mathrm{d}\theta\, p(\theta|n) \exp\left(\beta J_0 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' + \beta\theta \sum_{\alpha} \sigma_\alpha\right) \tag{97}$$

$$p(\theta|n) = \sum_k \frac{\mathrm{e}^{-c} c^k}{k!} \int \left\{ \prod_{r=1}^{k} \frac{\mathrm{d}h_r\, W(h_r)\, \exp(n\beta B(J/c, h_r))}{[2\cosh(\beta h_r)]^n} \right\} \delta\left[\theta - \sum_r A(J/c, h_r)\right]. \tag{98}$$

Again we observe that our replicated transfer matrix may be viewed as equivalent to that of a one-dimensional chain with suitably chosen random fields. The associated 'distribution' of these fields represents the overall effect within the system of the sparse Poissonian long-range bonds on a given site of the ring. We note that $p(\theta|n)$ is normalized only for $n = 0$.

Having identified the structure of our RS replicated transfer matrix, one may proceed to solve this model using the eigenvectors introduced in section 3.3. As in the Ising chain, this results in a transformation of the eigenvalue problem to integral equations, viz (29), (30) and (31), (32), involving now the above field distribution $p(\theta|n)$. In addition, the integral eigenvalue equations become coupled with the new distribution $W(h)$ in (96), which may be viewed as the fundamental 'mean-field' order parameter in this model. In the limit $n \to 0$ one finds that $W(h)$ is given by

$$W(h) = \int \mathrm{d}x\, \mathrm{d}y\, \Phi(x) \Psi(y) \delta(h - x - y). \tag{99}$$

In order to find also correlation functions in the present model we return to the previous derivation in section 4 and invoke the identity:

$$\overline{\langle \sigma_i \sigma_j \rangle^\rho} = \lim_{n\to 0} \sum_{\{\sigma\}} \sigma_1^{\alpha_1} \sigma_j^{\alpha_1} \cdots \sigma_i^{\alpha_\rho} \sigma_j^{\alpha_\rho} \overline{\prod_{\alpha=1}^{n} \mathrm{e}^{-\beta H(\boldsymbol{\sigma}^\alpha)}}.$$

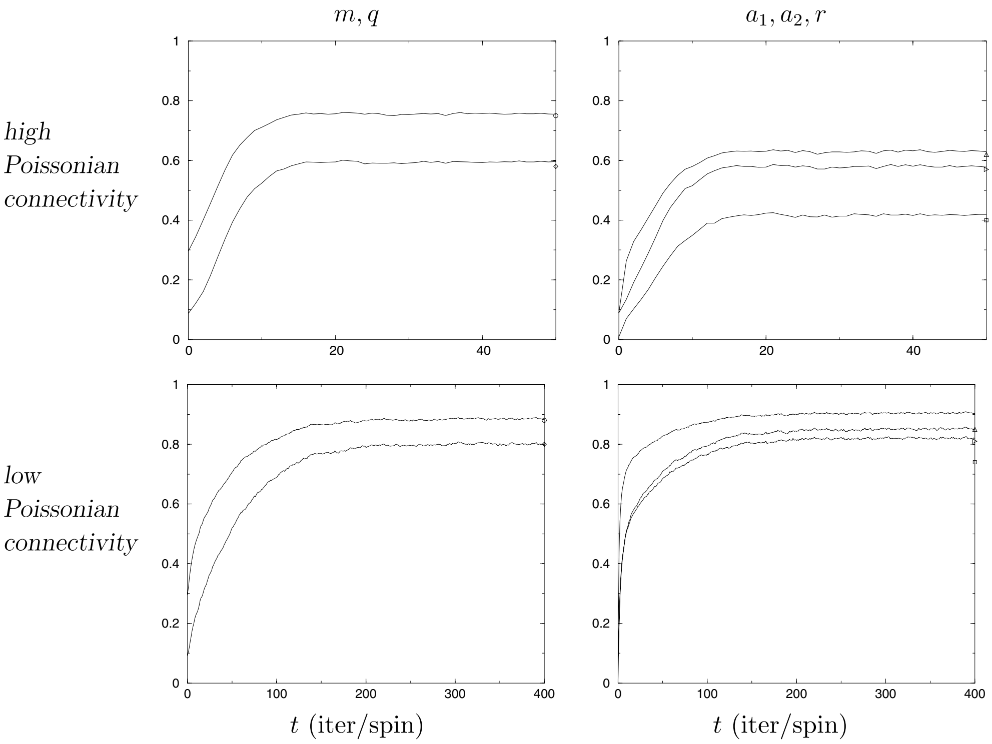

Figure 3. Relaxation of observables towards equilibrium at $T = J = 1$, in two 'small-world' ferromagnets with identical realizations of the disorder (i.e., the Poissonian graph), of size $N = 20\,000$. Left column: evolution of the magnetization $m = N^{-1}\sum_i \sigma_i$ and the order parameter $q = N^{-1}\sum_i \sigma_i\sigma_i'$. Right column: evolution of the multiple-site quantities $a_1 = N^{-1}\sum_i \sigma_i\sigma_{i+1}$, $a_2 = N^{-1}\sum_i \sigma_i\sigma_{i+2}$ and $r = N^{-1}\sum_i \sigma_i\sigma_{i+1}\sigma_i'\sigma_{i+1}'$. Different rows correspond to different control parameters. Top row: high Poissonian connectivity, viz $J_0 = 0.25$ and $c = 4$, where the predicted equilibrium values are $m \simeq 0.75$, $q \simeq 0.58$, $a_1 \simeq 0.62$, $a_2 \simeq 0.57$, $r \simeq 0.40$. Bottom row: low Poissonian connectivity, viz $J_0 = 1$ and $c = 0.5$, where the theory predicts $m \simeq 0.88$, $q \simeq 0.80$, $a_1 \simeq 0.85$, $a_2 \simeq 0.81$, $r \simeq 0.74$. In all cases the predictions are indicated by markers at the right of the graphs.

We find, after some straightforward and by now standard manipulations (viz averaging over the disorder, insertion of the relevant order parameters and use of saddle-point equations) that correlation functions can be again written in the form

$$\overline{\langle\sigma_i\sigma_j\rangle^\rho} = \lim_{n\to 0} \frac{\mathrm{tr}\left(\boldsymbol{S}_{\{\rho\}}\boldsymbol{T}^{j-i}[P]\boldsymbol{S}_{\{\rho\}}\boldsymbol{T}^{N-j+i}[P]\right)}{\mathrm{tr}(\boldsymbol{T}^N[P])}$$

where P is now given by expression (96). Since the steps which led us earlier for the random field Ising chain to (78) apply again, we may simply use (78) again to find also the correlation functions for the present model. The results of solving the relevant order parameter equations numerically (via population dynamics algorithms) are shown in figure 3, where we show the predicted equilibrium values for the scalar observables (82), (83) together with the corresponding measurements in numerical simulations, for comparison. The corresponding integrated field and site magnetization distributions are shown in figure 4. As with the random field Ising model, the order parameter functions required for the calculation of m and q have been calculated using the exact equations, whereas those required for the

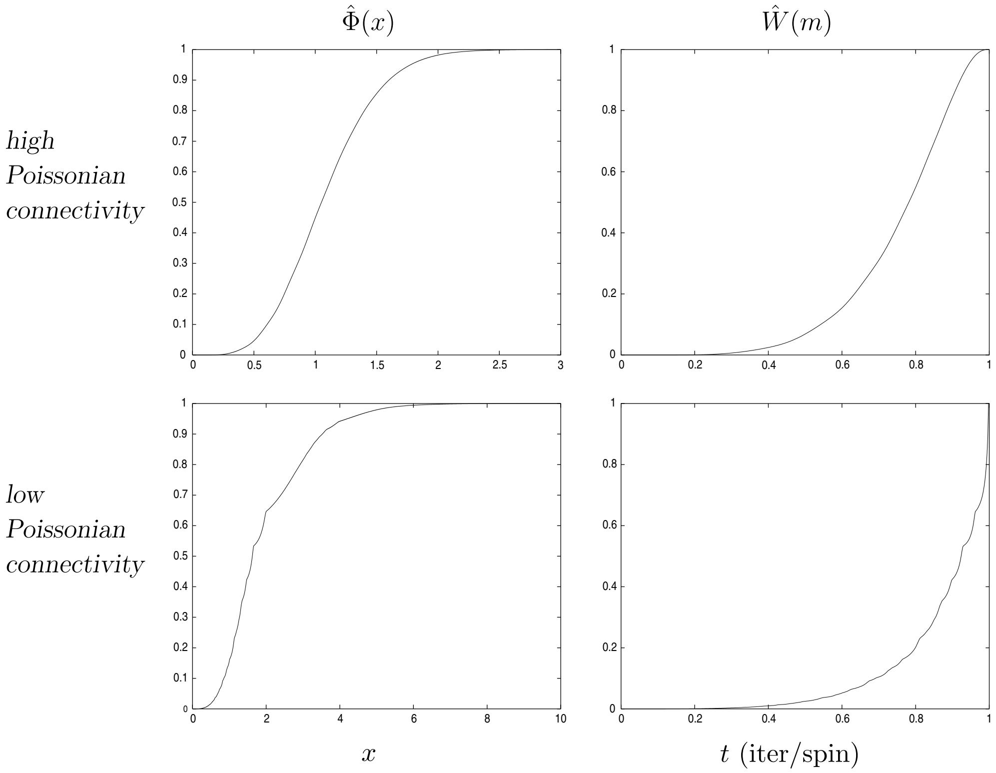

Figure 4. Integrated field and site magnetization distributions corresponding to the data of the previous figure 3, obtained by numerical solution of our integral eigenvalue equations (39), (40) via a population dynamics algorithm. The rows correspond to examples with high (top row) and low (bottom row) Poissonian connectivity. In the latter case the integrated field distribution is less smooth indicating that the assosiated $\Phi(x)$ is a complicated function.

multiple-site observables $\{a_1, a_2, r\}$ have been solved approximately. This is borne out by figure 3, which indeed shows excellent agreement between theory and simulations for m and q (left column), but deviations for the three quantities that have been calculated in approximation (right column).

6. Discussion

In this paper, we have developed new tools for the diagonalization of replicated transfer matrices, which arise upon applying the replica method to disordered models with one-dimensional short-range bonds, possibly in combination with (random) long-range ones. Our method was based on mapping the problem of diagonalizing $2^n \times 2^n$ matrices which are invariant under the replica permutation group onto the problem of diagonalizing appropriate n-dependent integral operators, in which the limit $n \to 0$ can be taken much more easily, via a suitable ansatz for the eigenvectors. The result, similar to that obtained earlier via more traditional methods, is an integral eigenvalue problem, which is exact in the relevant limits $N \to \infty$ and $n \to 0$, but which has to be solved numerically (using, e.g., population dynamics). Given our explicit expressions for the eigenvectors, the route is open to the evaluation of the free energy and several families of disorder-averaged observables, including the magnetization and the spin-glass order parameter, but also multiple-site correlation functions. It should be emphasized, however, that to evaluate the latter types of objects we had to make two simplifying assumptions, for which the only basis as yet is their validity in simpler and thereby verifiable cases.

We have developed our theory in full detail for the random field Ising chain, and we showed subsequently how the solution of other more complicated models can be obtained from this, especially those where short-range bonds are combined with long-range ones and where one effectively ends up with a random field Ising problem embedded within a mean-field calculation. In particular, we have worked out our equations and predictions for $(1+\infty)$-dimensional recurrent neural networks and for 'small-world' ferromagnets.

Possible future applications of the alternative approach presented in this paper would be to the analysis of two-dimensional disordered spin systems, or to spin models which require finite $-n$ replica calculations (e.g., those where the disorder is not truly frozen, but slowly and stochastically evolving in time according to equations which involve expectation values of the spins), or to situations where one has broken replica symmetry (RSB) in $(1+\infty)$-dimensional or 'small-world' spin systems. Especially, the latter two types of calculations would not seem to be easily carried out using the more conventional random field methods as in, e.g., [5, 6], if at all, but would appear to be quite feasible and straightforward extensions of the procedures presented here.

Acknowledgment

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Appendix. Combinatorial terms in the $n \to 0$ limit

Here we prove identity (77). We note that the natural continuation of factorials to non-integer values is via the Gamma function [18], viz $n! = \Gamma(n+1)$. For integer $\rho \geqslant 1$, integer $\ell > 0$ and real-valued $n < 1$ (so that always $\ell > n - \rho + 1$) we may therefore write

$$\begin{aligned}
\binom{n-\rho}{\ell} &= \frac{1}{\ell!}\lim_{\epsilon\downarrow 0}\frac{\int_\epsilon^\infty \mathrm{d}x\, x^{n-\rho}\,\mathrm{e}^{-x}}{\int_\epsilon^\infty \mathrm{d}x\, x^{n-\rho-\ell}\,\mathrm{e}^{-x}} \\
&= \frac{1}{\ell!}\lim_{\epsilon\downarrow 0}\frac{\int_\epsilon^1 \mathrm{d}x\, x^{n-\rho}\,\mathrm{e}^{-x} + \mathcal{O}(\epsilon^0)}{\int_\epsilon^1 \mathrm{d}x\, x^{n-\rho-\ell}\,\mathrm{e}^{-x} + \mathcal{O}(\epsilon^0)} \\
&= \frac{1}{\ell!}\frac{n-\rho-\ell+1}{n-\rho+1}\lim_{\epsilon\downarrow 0}\frac{\epsilon^{n-\rho+1} + \mathcal{O}(\epsilon^0)}{\epsilon^{n-\rho-\ell+1} + \mathcal{O}(\epsilon^0)} \\
&= \frac{1}{\ell!}\frac{n-\rho-\ell+1}{n-\rho+1}\lim_{\epsilon\downarrow 0}\frac{\epsilon^{\ell} + \mathcal{O}(\epsilon^{\ell+\rho-n-1})}{1 + \mathcal{O}(\epsilon^{\ell+\rho-n-1})} = 0.
\end{aligned} \tag{A.1}$$

We are left only with the case $\ell = 0$, for which the above factorial terms would be equal to 1. This proves (77).

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