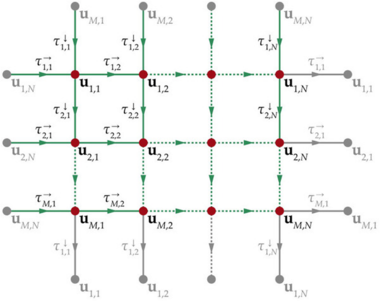
Statistical physics of tailored random graphs: entropies, processes, and generation

Lecture IV. Coupled oscillator models on graphs

ACC Coolen, King's College London



Definitions

- N phases $\theta_i \in [-\pi, \pi]$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$

$$H(\boldsymbol{\theta}) = -\sum_{i<j} c_{ij} J_{ij} \cos(\theta_i - \theta_j)$$

$$\textit{energies}: \quad J_{ij} \in \mathbb{R}, \quad \textit{drawn randomly from} \quad P(J)$$

$$\textit{topology}: \quad p(\mathbf{c}) = Z^{-1} \delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}, \quad k_i(\mathbf{c}) = \sum_j c_{ij}$$

- disorder-averaged free energy density,
  use $\overline{\log \Sigma} = \lim_{n\to 0} n^{-1} \log \overline{\Sigma^n}$:

$$\begin{aligned}
\bar{f} &= -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\left(\log \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}\ \mathrm{e}^{-\beta H(\boldsymbol{\theta})}\right)} \\
&= -\lim_{N\to\infty} \lim_{n\to 0} \frac{1}{\beta n N} \log \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}^1 \ldots \boldsymbol{\theta}^n\ \overline{\mathrm{e}^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\theta}^\alpha)}} \\
&= -\lim_{N\to\infty} \lim_{n\to 0} \frac{1}{\beta n N} \log \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}^1 \ldots \boldsymbol{\theta}^n\ \overline{\mathrm{e}^{\beta \sum_{i<j} c_{ij} J_{ij} \sum_{\alpha=1}^n \cos(\theta_i^\alpha - \theta_j^\alpha)}}
\end{aligned}$$

compared to Ising system:

integrals instead of sums, $\quad \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \to \sum_\alpha \cos(\theta_i^\alpha - \theta_j^\alpha)$

average over graphs

just substitute into Ising formulae:

$$\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \to \sum_\alpha \cos(\theta_i^\alpha-\theta_j^\alpha),\qquad 2=\sum_{\sigma=\pm 1} \;\to\; 2\pi=\int_{-\pi}^{\pi}\mathrm{d}\theta$$

$$-\beta\overline{f} \;=\; \log(2\pi)+\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{n}\Big[\Phi(\beta)-\Phi(0)\Big]$$

$$\Phi(\beta)=\frac{1}{N}\log\Big\langle\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}+\frac{\langle k\rangle}{2N}\sum_{ij}\mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)}\int\mathrm{d}J\;P(J)\mathrm{e}^{\beta J\sum_{\alpha=1}^{n}\cos(\theta_i^\alpha-\theta_j^\alpha)}}\Big\rangle_{\{\boldsymbol{\theta}_i\}}$$

order parameters:

$$\mathcal{P}(\boldsymbol{\theta},\omega|\{\boldsymbol{\theta}_i,\omega_i\})=\frac{1}{N}\sum_i\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\omega-\omega_i)$$

so that

$$\Phi(\beta)=\frac{1}{N}\log\Big\langle\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\omega}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}\cdot\mathbf{k}}\exp\Big\{\frac{1}{2}\langle k\rangle N\int\mathrm{d}\omega\mathrm{d}\omega'\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}'\mathcal{P}(\boldsymbol{\theta},\omega|\ldots)\mathcal{P}(\boldsymbol{\theta}',\omega'|\ldots)$$
$$\times\,\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\int\mathrm{d}J\;P(J)\mathrm{e}^{\beta J\sum_\alpha\cos(\theta_\alpha-\theta'_\alpha)}\Big\}\Big\rangle_{\{\boldsymbol{\theta}_i\}}$$

insert for each $(\boldsymbol{\theta},\omega)$:

$$1 = \int \{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} \mathrm{e}^{\mathrm{i}N\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\omega~\hat{\mathcal{P}}(\boldsymbol{\theta},\omega)\mathcal{P}(\boldsymbol{\theta},\omega)-\mathrm{i}\sum_i \hat{\mathcal{P}}(\boldsymbol{\theta}_i,\omega_i)}$$

with short hand (path integral measure):
$\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\} = \lim_{\Delta\omega,\Delta\theta\to 0}\prod_{\omega,\boldsymbol{\theta}}[\mathcal{P}(\boldsymbol{\theta},\omega)\mathrm{d}\hat{\mathcal{P}}(\boldsymbol{\theta},\omega)N\Delta\omega\Delta^n\theta/2\pi]$

result:
factorisation over sites!

$$\begin{aligned}\Phi(\beta) &= \frac{1}{N}\log\int\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\}\mathrm{e}^{\mathrm{i}N\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\omega~\hat{\mathcal{P}}(\boldsymbol{\theta},\omega)\mathcal{P}(\boldsymbol{\theta},\omega)} \\ &\quad\times \mathrm{e}^{\frac{1}{2}\langle k\rangle N\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}'\mathrm{d}\omega\mathrm{d}\omega'~\mathcal{P}(\boldsymbol{\theta},\omega)\mathcal{P}(\boldsymbol{\theta}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\int \mathrm{d}J~P(J)\mathrm{e}^{\beta J\sum_\alpha\cos(\theta_\alpha-\theta'_\alpha)}} \\ &\quad\times \prod_i\Big\langle\int_{-\pi}^{\pi}\mathrm{d}\omega_i~\mathrm{e}^{\mathrm{i}\omega_i k_i-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\theta}_i,\omega_i)}\Big\rangle_{\boldsymbol{\theta}_i}\end{aligned}$$

Saddle-point equations

$$\lim_{N\to\infty} \Phi(\beta) = \mathrm{extr}_{\{\mathcal{P},\hat{\mathcal{P}}\}} \Psi_\beta[\mathcal{P}.\hat{\mathcal{P}}]$$

$$\Psi_\beta[\mathcal{P},\hat{\mathcal{P}}] = \mathrm{i}\int \mathrm{d}\boldsymbol{\theta}\mathrm{d}\omega\; \hat{\mathcal{P}}(\boldsymbol{\theta},\omega)\mathcal{P}(\boldsymbol{\theta},\omega) + \sum_k p(k)\log\Big\langle \int_{-\pi}^{\pi}\mathrm{d}\omega\; \mathrm{e}^{\mathrm{i}\omega k-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\theta},\omega)}\Big\rangle_{\boldsymbol{\theta}}$$
$$+\frac{1}{2}\langle k\rangle\int \mathrm{d}\boldsymbol{\sigma}\mathrm{d}\boldsymbol{\theta}'\mathrm{d}\omega\mathrm{d}\omega'\; \mathcal{P}(\boldsymbol{\theta},\omega)\mathcal{P}(\boldsymbol{\theta}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')}\int \mathrm{d}J\; P(J)\mathrm{e}^{\beta J\sum_\alpha \cos(\theta_\alpha-\theta'_\alpha)}$$

extrema:

$$\frac{\delta\Psi}{\delta\mathcal{P}}=0: \qquad \mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\theta},\omega) = -\langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\int \mathrm{d}\boldsymbol{\theta}'\mathrm{d}\omega'\; \mathcal{P}(\boldsymbol{\theta}',\omega')\mathrm{e}^{-\mathrm{i}\omega'}\int \mathrm{d}J\; P(J)\mathrm{e}^{\beta J\sum_\alpha \cos(\theta_\alpha-\theta'_\alpha)}$$

$$\frac{\delta\Psi}{\delta\hat{\mathcal{P}}}=0: \qquad \mathcal{P}(\boldsymbol{\theta},\omega) = \sum_k p(k)\frac{\mathrm{e}^{\mathrm{i}\omega k-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\theta},\omega)}}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\theta}'\int_{-\pi}^{\pi}\mathrm{d}\omega'\; \mathrm{e}^{\mathrm{i}\omega' k-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\theta}',\omega')}}$$

- new quantities:

$$\mathcal{D}(\boldsymbol{\theta}) = \int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathcal{P}(\boldsymbol{\theta},\omega)\mathrm{e}^{-\mathrm{i}\omega}, \qquad \hat{\mathcal{P}}(\boldsymbol{\theta},\phi) = \mathrm{i}\langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\gamma(\boldsymbol{\theta}),$$

new eqns:

$$\gamma(\boldsymbol{\theta}) = \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}'\ \mathcal{D}(\boldsymbol{\theta}')\langle \mathrm{e}^{\beta J \sum_\alpha \cos(\theta_\alpha - \theta'_\alpha)}\rangle_J$$

$$\mathcal{D}(\boldsymbol{\theta}) = \sum_k p(k) \frac{\int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathrm{e}^{\mathrm{i}\omega(k-1)+\langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\gamma(\boldsymbol{\theta})}}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\theta}' \int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathrm{e}^{\mathrm{i}\omega k+\langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\gamma(\boldsymbol{\theta}')}}$$

after ω-integrals:

$$\mathcal{D}(\boldsymbol{\theta}) = \sum_{k>0} p(k) \frac{k}{\langle k\rangle} \frac{\gamma^{k-1}(\boldsymbol{\theta})}{\int_{-\pi}^{\pi}\mathrm{d}\boldsymbol{\theta}'\ \gamma^k(\boldsymbol{\theta}')}$$

$$\gamma(\boldsymbol{\theta}) = \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}'\ \mathcal{D}(\boldsymbol{\theta}')\langle \mathrm{e}^{\beta J \sum_\alpha \cos(\theta_\alpha - \theta'_\alpha)}\rangle_J$$

use saddle point eqns:

$$\lim_{N\to\infty} \Phi(\beta) = \mathrm{extr}_{\{\mathcal{P},\hat{\mathcal{P}}\}} \Psi_\beta[\mathcal{P}.\hat{\mathcal{P}}]$$
$$= \log(2\pi) - \frac{1}{2}\langle k\rangle + \sum_k p(k)\log\langle \gamma^k(\boldsymbol{\theta})\rangle_{\boldsymbol{\theta}} + \sum_k p(k)\log[\langle k\rangle^k/k!]$$

$\beta = 0$: $\gamma(\boldsymbol{\theta}) = 1$

free energy:

$$-\beta\bar{f} = \log(2\pi) + \lim_{N\to\infty}\lim_{n\to 0} n^{-1}\left[\Phi(\beta) - \Phi(0)\right]$$
$$= \lim_{n\to 0}\frac{1}{n}\sum_k p(k)\log\langle \gamma^k(\boldsymbol{\theta})\rangle_{\boldsymbol{\theta}}$$

Replica symmetric solutions

$\mathcal{D}(\boldsymbol{\theta})$ invariant under all permutations of $\{1, \ldots, n\}$

$$\mathcal{D}(\boldsymbol{\theta}) = \int \{\mathrm{d}\pi\}\, \mathcal{D}[\{\pi\}] \prod_{\alpha=1}^{n} \pi(\theta_\alpha)$$

$\mathcal{D}[\{\pi\}]$: measure on the set of normalised distributions $\pi : [-\pi, \pi] \to \mathbb{R}$

- work out $\gamma(\boldsymbol{\theta})$:

$$\gamma(\boldsymbol{\theta}) \quad = \quad \int \{\mathrm{d}\pi\}\, \mathcal{D}[\{\pi\}] \int \mathrm{d}J\, P(J) \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}' \prod_{\alpha} \pi(\theta'_\alpha) \mathrm{e}^{\beta J \cos(\theta_\alpha - \theta'_\alpha)}$$

$$C_{n,k} = \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}\, \gamma^k(\boldsymbol{\theta})$$

$$\lim_{n\to 0} C_{n,k} = \lim_{n\to 0} \int \{\mathrm{d}\pi\} \mathrm{d}J\, \mathcal{D}[\{\pi\}] P(J) \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta} \mathrm{d}\boldsymbol{\theta}' \prod_{\alpha} \pi(\theta'_\alpha) \mathrm{e}^{\beta J \cos(\theta_\alpha - \theta'_\alpha)}$$

$$= \lim_{n\to 0} \int \{\mathrm{d}\pi\} \mathrm{d}J\, \mathcal{D}[\{\pi\}] P(J) \Big[\int_{-\pi}^{\pi} \mathrm{d}\theta \mathrm{d}\theta' \pi(\theta') \mathrm{e}^{\beta J \cos(\theta - \theta')} \Big]^n \; = \; 1$$

eqn for $\mathcal{D}[\{\pi\}]$:

$$\int\{\mathrm{d}\pi\}\ \mathcal{D}[\{\pi\}]\prod_{\alpha=1}^{n}\pi(\theta_\alpha) = \sum_{k>0}p(k)\frac{k}{\langle k\rangle C_{n,k}}\gamma^{k-1}(\boldsymbol{\sigma})$$

$$= \sum_{k>0}\frac{p(k)k}{\langle k\rangle C_{n,k}}\int\Big[\prod_{\ell<k}\{\mathrm{d}\pi_\ell\}\mathcal{D}[\{\pi_\ell\}]\Big]\Big\langle\prod_{\alpha}\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta_\alpha-\theta_\ell)}\Big\rangle_{J_1\ldots J_{k-1}}$$

$$= \sum_{k>0}\frac{p(k)k}{\langle k\rangle C_{n,k}}\int\Big[\prod_{\ell<k}\{\mathrm{d}\pi_\ell\}\mathcal{D}[\{\pi_\ell\}]\Big]\Big\langle\prod_{\alpha}\frac{\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta_\alpha-\theta_\ell)}}{\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}}$$

$$\times\Big(\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}\Big)^{n}\Big\rangle_{J_1\ldots J_{k-1}}$$

$$= \int\{\mathrm{d}\pi\}\prod_{\alpha=1}^{n}\pi(\theta_\alpha)\sum_{k>0}\frac{p(k)k}{\langle k\rangle C_{n,k}}\int\Big[\prod_{\ell<k}\{\mathrm{d}\pi_\ell\}\mathcal{D}[\{\pi_\ell\}]\Big]$$

$$\times\Big\langle\prod_{\theta}\delta\Big[\pi(\theta)-\frac{\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta-\theta_\ell)}}{\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}}\Big]$$

$$\times\Big(\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}\Big)^{n}\Big\rangle_{J_1\ldots J_{k-1}}$$

after $n \to 0$:

$$\mathcal{D}[\{\pi\}] \;=\; \sum_{k>0} \frac{p(k)k}{\langle k\rangle} \int \Big[\prod_{\ell<k} \{\mathrm{d}\pi_\ell\} \mathcal{D}[\{\pi_\ell\}]\Big] \times \Big\langle \prod_\theta \delta\Big[\pi(\theta) - \frac{\int_{-\pi}^{\pi} \prod_{\ell<k} \mathrm{d}\theta_\ell \; \pi_\ell(\theta_\ell)\, \mathrm{e}^{\beta J_\ell \cos(\theta-\theta_\ell)}}{\int_{-\pi}^{\pi} \mathrm{d}\theta' \int_{-\pi}^{\pi} \prod_{\ell<k} \mathrm{d}\theta_\ell \; \pi_\ell(\theta_\ell)\, \mathrm{e}^{\beta J_\ell \cos(\theta'-\theta_\ell)}}\Big]\Big\rangle_{J_1 \ldots J_{k-1}}$$

interpretation?

- return to $\mathcal{P}(\boldsymbol{\theta}, \omega)$:

$$\mathcal{P}(\boldsymbol{\theta}) = \int_{-\pi}^{\pi} \mathrm{d}\omega \; \mathcal{P}(\boldsymbol{\theta}, \omega) = \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{N} \sum_i \lim_{n\to 0} \overline{\langle \prod_{\alpha\le n} \delta[\theta_\alpha - \theta_i^\alpha] \rangle}$$

hence

$$\lim_{N\to\infty} \frac{1}{N} \sum_i \begin{pmatrix} \overline{\langle \cos(\theta_i)\rangle} \\ \overline{\langle \sin(\theta_i)\rangle} \end{pmatrix} = \lim_{n\to 0} \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta} \; \mathcal{P}(\boldsymbol{\theta}) \frac{1}{n} \sum_{\alpha=1}^{n} \begin{pmatrix} \cos(\theta_\alpha) \\ \sin(\theta_\alpha) \end{pmatrix}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_i \begin{pmatrix} \overline{\langle \cos^2(\theta_i)\rangle} \\ \overline{\langle \cos(\theta_i)\sin(\theta_i)\rangle} \\ \overline{\langle \sin^2(\theta_i)\rangle} \end{pmatrix} = \lim_{n\to 0} \int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta} \; \mathcal{P}(\boldsymbol{\theta}) \frac{1}{n(n-1)} \sum_{\alpha\neq\beta=1}^{n} \begin{pmatrix} \cos^2(\theta_\alpha) \\ \cos(\theta_\alpha)\sin(\theta_\alpha) \\ \sin^2(\theta_\alpha) \end{pmatrix}$$

$\mathcal{P}(\boldsymbol{\theta})$ in terms of $\mathcal{D}[\{\pi\}]$:

$$\mathcal{P}(\boldsymbol{\theta}) = \sum_k p(k) \frac{\int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathrm{e}^{\mathrm{i}\omega k + \langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\gamma(\boldsymbol{\theta})}}{\int_{-\pi}^{\pi} \mathrm{d}\boldsymbol{\theta}' \int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathrm{e}^{\mathrm{i}\omega k + \langle k\rangle \mathrm{e}^{-\mathrm{i}\omega}\gamma(\boldsymbol{\sigma}')}} = \sum_k \frac{p(k)}{C_{n,k}} \gamma^k(\boldsymbol{\theta})$$

$$= \sum_k p(k) \int \Big[\prod_{\ell\leq k} \{\mathrm{d}\pi_\ell\} \mathcal{D}[\{\pi_\ell\}]\Big]$$

$$\times \Big\langle \prod_\alpha \frac{\int_{-\pi}^{\pi} \prod_{\ell<k} \mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell \cos(\theta_\alpha - \theta_\ell)}}{\int_{-\pi}^{\pi} \mathrm{d}\theta' \int_{-\pi}^{\pi} \prod_{\ell<k} \mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell \cos(\theta' - \theta_\ell)}} \Big[1 + \mathcal{O}(n)\Big] \Big\rangle_{J_1 \ldots J_k}$$

for $n \to 0$:

$$\mathcal{P}(\boldsymbol{\theta}) = \int \{\mathrm{d}\pi\}\ D[\{\pi\}] \prod_{\alpha=1}^{n} \pi(\theta_\alpha)$$

$$D[\{\pi\}] = \sum_{k>0} \frac{p(k)k}{\langle k\rangle} \int \Big[\prod_{\ell\leq k} \{\mathrm{d}\pi_\ell\} \mathcal{D}[\{\pi_\ell\}]\Big] \Big\langle \delta\big[\pi - F\pi\big] \Big\rangle_{J_1 \ldots J_k}$$

$$\mathcal{D}[\{\pi\}] = \sum_{k>0} \frac{p(k)k}{\langle k\rangle} \int \Big[\prod_{\ell<k} \{\mathrm{d}\pi_\ell\} \mathcal{D}[\{\pi_\ell\}]\Big] \Big\langle \delta\big[\pi - F\pi\big] \Big\rangle_{J_1 \ldots J_{k-1}}$$

$$(F\pi)(\theta) = \frac{\int_{-\pi}^{\pi} \prod_{\ell\leq k} \mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell \cos(\theta - \theta_\ell)}}{\int_{-\pi}^{\pi} \mathrm{d}\theta' \int_{-\pi}^{\pi} \prod_{\ell\leq k} \mathrm{d}\theta_\ell\ \pi_\ell(\theta_\ell)\ \mathrm{e}^{\beta J_\ell \cos(\theta' - \theta_\ell)}}$$

Continuous phase transitions

$$\mathcal{D}[\{\pi\}] = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\int\Big[\prod_{\ell<k}\{\mathrm{d}\pi_\ell\}\mathcal{D}[\{\pi_\ell\}]\Big]\Big\langle\delta\big[\pi - F[\pi_1,\ldots,\pi_{k-1}]\big]\Big\rangle_{J_1\ldots J_{k-1}}$$

$$F[\theta|\pi_1,\ldots,\pi_{k-1}] = \frac{\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\;\pi_\ell(\theta_\ell)\;\mathrm{e}^{\beta J_\ell\cos(\theta-\theta_\ell)}}{\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\prod_{\ell<k}\mathrm{d}\theta_\ell\;\pi_\ell(\theta_\ell)\;\mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}}$$

- at $T=\beta^{-1}=\infty$: $\mathcal{D}[\{\pi\}] = \delta[\pi-\pi_0]$, $\pi_0(\theta) = (2\pi)^{-1}$

  gives: $D[\{\pi\}] = \delta[\pi-\pi_0]$

  paramagnetic state,
  is a soln at any β

- bifurcations away from $\delta[\pi-\pi_0]$:
  expand eqns in deviation from π_0 $\quad \pi(\theta) = \frac{1}{2\pi} + \epsilon\,\xi(\theta), \quad \int_{-\pi}^{\pi}\mathrm{d}\theta\,\xi(\theta) = 0$

1st order bifurc : $\quad \delta[\pi-\pi_0] \to \mathcal{D}[\{\pi\}]$ *with* $\int\{\mathrm{d}\pi\}\,\mathcal{D}[\{\pi\}]\pi(\theta) \neq \frac{1}{2\pi}$

2nd order bifurc : $\quad \delta[\pi-\pi_0] \to \mathcal{D}[\{\pi\}]$ *with* $\int\{\mathrm{d}\pi\}\,\mathcal{D}[\{\pi\}]\pi(\theta) = \frac{1}{2\pi}$

- first order:

$$\int\{\mathrm{d}\pi\}\mathcal{D}[\{\pi\}]\pi(\theta)=\sum_{k>0}p(k)\frac{k}{\langle k\rangle}\int\Big[\prod_{\ell<k}\{\mathrm{d}\pi_\ell\}\mathcal{D}[\{\pi_\ell\}]\Big]\Big\langle F[\theta|\pi_1,\ldots,\pi_{k-1}]\Big\rangle_{J_1\ldots J_{k-1}}$$

$$\frac{1}{2\pi}+\epsilon\int\{\mathrm{d}\xi\}W[\{\xi\}]\xi(\theta)=\sum_{k>0}p(k)\frac{k}{\langle k\rangle}\int\Big[\prod_{\ell<k}\{\mathrm{d}\xi_\ell\}W[\{\xi_\ell\}]\Big]$$
$$\times\Big\langle F[\theta|\pi_0+\epsilon\xi_1,\ldots,\pi_0+\epsilon\xi_{k-1}]\Big\rangle_{J_1\ldots J_{k-1}}$$

expand F:

$$F[\theta|\pi_0+\epsilon\xi_1,\ldots,\pi_0+\epsilon\xi_{k-1}]=\frac{\prod_{\ell<k}\Big[I_0(\beta J_\ell)+\epsilon\int_{-\pi}^{\pi}\mathrm{d}\theta_\ell\;\xi_\ell(\theta_\ell)\mathrm{e}^{\beta J_\ell\cos(\theta-\theta_\ell)}\Big]}{\int_{-\pi}^{\pi}\mathrm{d}\theta'\prod_{\ell<k}\Big[I_0(\beta J_\ell)+\epsilon\int_{-\pi}^{\pi}\mathrm{d}\theta_\ell\;\xi_\ell(\theta_\ell)\mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}\Big]}$$

$$=\frac{1}{2\pi}\frac{1+\epsilon\sum_{\ell<k}\int_{-\pi}^{\pi}\mathrm{d}\theta_\ell\;\xi_\ell(\theta_\ell)\mathrm{e}^{\beta J_\ell\cos(\theta-\theta_\ell)}/I_0(\beta J_\ell)}{1+\epsilon\sum_{\ell<k}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta'}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta_\ell\;\xi_\ell(\theta_\ell)\mathrm{e}^{\beta J_\ell\cos(\theta'-\theta_\ell)}/I_0(\beta J_\ell)}+\mathcal{O}(\epsilon^2)$$

$$=\frac{1}{2\pi}+\frac{\epsilon}{I_0(\beta J_\ell)}\sum_{\ell<k}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_\ell}{2\pi}\;\xi_\ell(\theta_\ell)\mathrm{e}^{\beta J_\ell\cos(\theta-\theta_\ell)}+\mathcal{O}(\epsilon^2)$$

$\psi(\theta)=\int\{\mathrm{d}\xi\}W[\{\xi\}]\xi(\theta)$:
bifurcation if

$$\psi(\theta)=\sum_{k>0}p(k)\frac{k(k-1)}{\langle k\rangle}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta'}{2\pi}\psi(\theta')\Big\langle\frac{\mathrm{e}^{\beta J\cos(\theta-\theta')}}{I_0(\beta J)}\Big\rangle_J,\qquad\int_{-\pi}^{\pi}\mathrm{d}\theta\;\psi(\theta)=0$$

$$\psi(\theta) = \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta'}{2\pi} \psi(\theta') \Big\langle \frac{\mathrm{e}^{\beta J \cos(\theta-\theta')}}{I_0(\beta J)} \Big\rangle_J, \qquad \int_{-\pi}^{\pi} \mathrm{d}\theta \; \psi(\theta) = 0$$

modified Bessel functions:
$I_n(z) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \cos(n\theta) \mathrm{e}^{z\cos(\theta)}$

bifurcation condition has form of *convolution*,
so solns are Fourier modes:

$$\psi_q(\theta) = \mathrm{e}^{\mathrm{i}q\theta}, \; q \in \mathbb{N} : \qquad 1 = \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \Big\langle \frac{I_q(\beta J)}{I_0(\beta J)} \Big\rangle_J$$

first to occur: $q = 1$,
bifurcation of phase-ordered state at:

$$(\langle k^2 \rangle / \langle k \rangle - 1) \int \mathrm{d}J \; P(J) \frac{I_1(\beta J)}{I_0(\beta J)} = 1$$

(similarly for 2nd order bifurcation)