Modelling of Complex Real-World Systems
Part B. Tools for Heterogeneous Systems

B1. Homogeneous systems

Ton Coolen
Department of Biophysics, Radboud University
module NWI-NM127, January 2021

Homogeneous systems

discrete

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma})=\sum_i\left[w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma})-w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\right],\qquad w_i(\boldsymbol{\sigma})=\frac{1}{2}[1-\sigma_i\tanh(\beta h_i(\boldsymbol{\sigma}))]$$

$$p_{t+1}(\boldsymbol{\sigma})=\sum_{\boldsymbol{\sigma}'}W[\boldsymbol{\sigma};\boldsymbol{\sigma}']p_t(\boldsymbol{\sigma}'),\qquad W[\boldsymbol{\sigma};\boldsymbol{\sigma}']=\prod_{i=1}^{N}\left[\frac{1}{2}+\frac{1}{2}\sigma_i\tanh(\beta h_i(\boldsymbol{\sigma}'))\right]$$

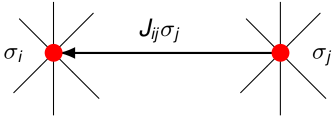
continuous

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma})=-\sum_i\frac{\partial}{\partial\sigma_i}\left[p_t(\boldsymbol{\sigma})f_i(\boldsymbol{\sigma})\right]+T\sum_i\frac{\partial^2}{\partial\sigma_i^2}p_t(\boldsymbol{\sigma})$$

- forces

$$h_i(\boldsymbol{\sigma})=\sum_j J_{ij}\sigma_j+\theta_i$$

$$f_i(\boldsymbol{\sigma})=\sum_j J_{ij}\sigma_j+\theta_i+g(\sigma_i)$$



homogeneous, with $h_i(\boldsymbol{\sigma})=\mathcal{O}(1)$ for $N\to\infty$:

$J_{ij}=J/N,\quad \theta_i=\theta$:

$$h_i(\boldsymbol{\sigma})=Jm(\boldsymbol{\sigma})+\theta$$

$$f_i(\boldsymbol{\sigma})=Jm(\boldsymbol{\sigma})+\theta+g(\sigma_i)\qquad m(\boldsymbol{\sigma})=\tfrac{1}{N}\sum_i\sigma_i$$

Sequential master equation

Dynamical solution

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma}) = \sum_{i=1}^{N}\left[w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma}) - w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\right], \quad w_i(\boldsymbol{\sigma}) = \frac{1}{2}[1-\sigma_i \tanh(\beta(Jm(\boldsymbol{\sigma})+\theta))]$$

key macroscopic object

$$m(\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \sigma_i, \qquad P_t(m) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})\delta[m-m(\boldsymbol{\sigma})]$$

- macroscopic dynamics

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}P_t(m) &= \sum_{\boldsymbol{\sigma}} \frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma})\delta[m-m(\boldsymbol{\sigma})] \\
&= \sum_i \sum_{\boldsymbol{\sigma}} \left[w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma}) - w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\right]\delta[m-m(\boldsymbol{\sigma})] \\
&= \sum_i \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})w_i(\boldsymbol{\sigma})\Big(\delta[m-m(F_i\boldsymbol{\sigma})] - \delta[m-m(\boldsymbol{\sigma})]\Big)
\end{aligned}$$

- expand, using $m(F_i\boldsymbol{\sigma}) = m(\boldsymbol{\sigma}) - \frac{2}{N}\sigma_i$

$$\delta[m - m(F_i\boldsymbol{\sigma})] \quad = \quad \delta[m - m(\boldsymbol{\sigma})] + \frac{2\sigma_i}{N}\frac{\mathrm{d}}{\mathrm{d}m}\delta[m - m(\boldsymbol{\sigma})] + \mathcal{O}(N^{-2})$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}P_t(m) &= \sum_i \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) w_i(\boldsymbol{\sigma}) \Big\{ \frac{2\sigma_i}{N}\frac{\mathrm{d}}{\mathrm{d}m}\delta[m - m(\boldsymbol{\sigma})] + \mathcal{O}(N^{-2}) \Big\} \\
&= \frac{1}{N}\frac{\partial}{\partial m}\sum_i \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})[1 - \sigma_i \tanh(\beta(Jm(\boldsymbol{\sigma}) + \theta))]\sigma_i \delta[m - m(\boldsymbol{\sigma})] + \mathcal{O}(\frac{1}{N}) \\
&= \frac{\partial}{\partial m}\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})[m(\boldsymbol{\sigma}) - \tanh(\beta(Jm(\boldsymbol{\sigma}) + \theta))]\delta[m - m(\boldsymbol{\sigma})] + \mathcal{O}(\frac{1}{N}) \\
&= \frac{\partial}{\partial m}\sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})[m - \tanh(\beta(Jm + \theta))]\delta[m - m(\boldsymbol{\sigma})] + \mathcal{O}(\frac{1}{N}) \\
&= \frac{\partial}{\partial m}\Big\{ P_t(m)\Big[m - \tanh(\beta(Jm + \theta))\Big]\Big\} + \mathcal{O}(\frac{1}{N})
\end{aligned}$$

$N \to \infty$: $\quad \frac{\mathrm{d}}{\mathrm{d}t}P_t(m) = -\frac{\partial}{\partial m}\Big\{P_t(m)F(m)\Big\}, \quad F(m) = \tanh(\beta(Jm + \theta)) - m$

(Liouville equation)

solve Liouville eqn

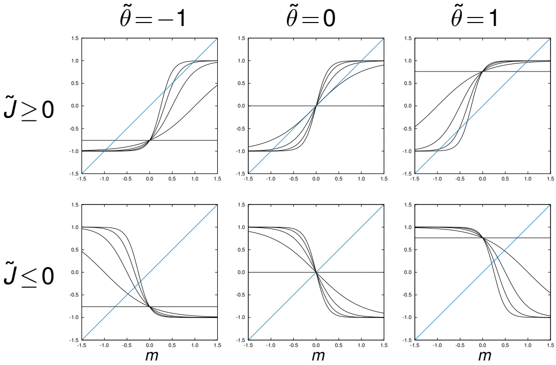
$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(m) = -\frac{\partial}{\partial m}\Big\{P_t(m)F(m)\Big\} \;\;\Rightarrow\;\; P_t(m) = \delta[m-m(t)], \;\; \frac{\mathrm{d}}{\mathrm{d}t}m(t) = F(m(t))$$

(see exercises)

$N\to\infty$: $m(\boldsymbol{\sigma})$ evolves *deterministically*, according to *closed* eqn

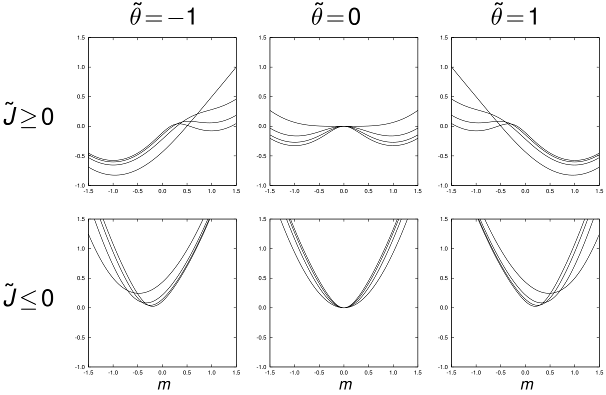
$$\frac{\mathrm{d}}{\mathrm{d}t}m = \tanh(\tilde{J}m+\tilde{\theta}) - m, \qquad \tilde{J}=\beta J=J/T, \;\; \tilde{\theta}=\beta\theta=\theta/T$$

- stationary states:
  $m = \tanh(\tilde{J}m+\tilde{\theta})$



- process minimizes $$\Phi(m) = \frac{1}{2}m^2 - \tilde{J}^{-1}\log\cosh(\tilde{J}m+\tilde{\theta})$$

  proof:
  since $\mathrm{d}m/\mathrm{d}t = -\partial\Phi(m)/\partial m$
  $$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(m) = \frac{\partial\Phi(m)}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}t} = -\Big(\frac{\partial\Phi(m)}{\partial m}\Big)^2 \le 0$$



Equilibrium solution

$$p(\boldsymbol{\sigma}) = \frac{1}{Z}\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}, \qquad H(\boldsymbol{\sigma}) = -\frac{J}{2N}\sum_{i\neq j}\sigma_i\sigma_j - \theta\sum_i \sigma_i \quad \text{(Curie}-\text{Weiss model)}$$

$$= \frac{1}{2}J - \frac{1}{2}JNm^2(\boldsymbol{\sigma}) - N\theta m(\boldsymbol{\sigma})$$

- generating function
  (free energy density)

$$f = -\frac{1}{\beta N}\log Z, \quad Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})},$$

$$-\frac{\partial f}{\partial \theta} = -\frac{1}{NZ}\sum_{\boldsymbol{\sigma}}\frac{\partial H(\boldsymbol{\sigma})}{\partial \theta}\mathrm{e}^{-\beta H(\boldsymbol{\sigma})} = \frac{1}{Z}\sum_{\boldsymbol{\sigma}} m(\boldsymbol{\sigma})\mathrm{e}^{-\beta H(\boldsymbol{\sigma})} = \langle m(\boldsymbol{\sigma})\rangle$$

- objective: find

$$\lim_{N\to\infty} f = -\lim_{N\to\infty}\frac{1}{\beta N}\log Z, \qquad Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\frac{1}{2}\tilde{J} + \frac{1}{2}\tilde{J}Nm^2(\boldsymbol{\sigma}) + N\tilde{\theta}m(\boldsymbol{\sigma})}$$

main obstacle: $\sum_{\boldsymbol{\sigma}}$

Route A: *Gaussian linearisation*

strategy:

1. use $\int \frac{\mathrm{d}x}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2+ax} = \mathrm{e}^{\frac{1}{2}a^2}$ to linearize the term quadratic in $\boldsymbol{\sigma}$
2. ensure all terms in exponent of integral scale as $\mathcal{O}(N)$
3. use steepest descent integration

1. linearization

$$Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\frac{1}{2}\tilde{J}+\frac{1}{2}\tilde{J}Nm^2(\boldsymbol{\sigma})+N\tilde{\theta}m(\boldsymbol{\sigma})} = \mathrm{e}^{-\frac{1}{2}\tilde{J}}\sum_{\boldsymbol{\sigma}}\int\frac{\mathrm{d}x}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2+xm(\boldsymbol{\sigma})\sqrt{\tilde{J}N}+N\tilde{\theta}m(\boldsymbol{\sigma})}$$

2. scaling with N: $x \to y\sqrt{\tilde{J}N}$

$$\begin{aligned} Z &= \mathrm{e}^{-\frac{1}{2}\tilde{J}}\left(\frac{\tilde{J}N}{2\pi}\right)^{\frac{1}{2}}\sum_{\boldsymbol{\sigma}}\int\mathrm{d}y\;\mathrm{e}^{-\frac{1}{2}\tilde{J}Ny^2+ym(\boldsymbol{\sigma})\tilde{J}N+N\tilde{\theta}m(\boldsymbol{\sigma})} \\ &= \mathrm{e}^{-\frac{1}{2}\tilde{J}}\left(\frac{\tilde{J}N}{2\pi}\right)^{\frac{1}{2}}\int\mathrm{d}y\;\mathrm{e}^{-\frac{1}{2}\tilde{J}Ny^2}\sum_{\boldsymbol{\sigma}}\mathrm{e}^{(y\tilde{J}+\tilde{\theta})\sum_i\sigma_i} \\ &= 2^N\mathrm{e}^{-\frac{1}{2}\tilde{J}}\left(\frac{\tilde{J}N}{2\pi}\right)^{\frac{1}{2}}\int\mathrm{d}y\;\mathrm{e}^{-\frac{1}{2}\tilde{J}Ny^2}\cosh^N(y\tilde{J}+\tilde{\theta}) \end{aligned}$$

3. steepest descent

$$\lim_{N\to\infty} f = -\lim_{N\to\infty}\frac{1}{\beta N}\log\left\{2^N \mathrm{e}^{-\frac{1}{2}\tilde{J}}\left(\frac{\tilde{J}N}{2\pi}\right)^{\frac{1}{2}}\int \mathrm{d}y\ \mathrm{e}^{-\frac{1}{2}\tilde{J}Ny^2+N\log\cosh(\tilde{J}y+\tilde{\theta})}\right\}$$
$$= -\frac{1}{\beta}\log 2 - \frac{1}{\beta}\mathrm{max}_y\left[\log\cosh(\tilde{J}y+\tilde{\theta}) - \frac{1}{2}\tilde{J}y^2\right]$$
$$= -\frac{1}{\beta}\log 2 + \frac{1}{\beta}\mathrm{min}_y\left[\frac{1}{2}\tilde{J}y^2 - \log\cosh(\tilde{J}y+\tilde{\theta})\right]$$
$$= -\frac{1}{\beta}\log 2 + J\ \mathrm{min}_y\Phi(y), \qquad \Phi(y) = \frac{1}{2}y^2 - \tilde{J}^{-1}\log\cosh(\tilde{J}y+\tilde{\theta})$$

solve $\Phi'(y)=0$ $\qquad y = \tanh(\tilde{J}y+\tilde{\theta}), \qquad \mathrm{soln:}\ \ y^\star$

$$m = \langle m(\boldsymbol{\sigma})\rangle = -\frac{\partial}{\partial\theta}f = -J\frac{\partial}{\partial\theta}\left[\frac{1}{2}y^{\star 2} - \tilde{J}^{-1}\log\cosh(\tilde{J}y^\star+\tilde{\theta})\right]$$
$$= -J\left[\Phi'(y^\star)\frac{\partial y^\star}{\partial\theta} - \beta\tilde{J}^{-1}\tanh(\tilde{J}y^\star+\tilde{\theta})\right] = \tanh(\tilde{J}y^\star+\tilde{\theta}) = y^\star$$

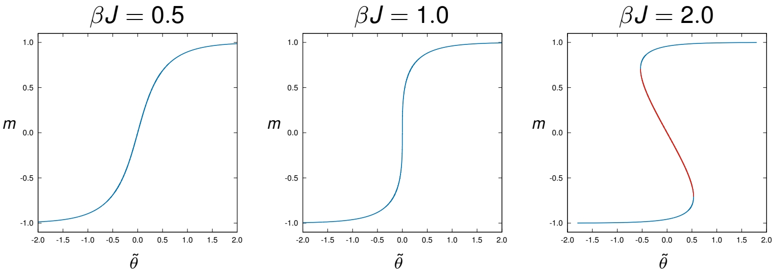
so

$$m = \tanh(\tilde{J}m+\tilde{\theta}), \qquad m = \mathrm{argmin}_y\Phi(y)$$

disadvantage:
works only when H quadratic in $m(\boldsymbol{\sigma})$

Phenomenology

- spontaneous order and remanence



blue: stable solns of $m = \tanh(\tilde{J}m + \tilde{\theta})$
red: unstable solns of $m = \tanh(\tilde{J}m + \tilde{\theta})$

$\tilde{J} = \beta J > 1$: stable solns $m \neq 0$ even when $\tilde{\theta} = 0$
remanence (i.e. memory)

- no external forces, $\tilde{\theta} = 0$:

$\tilde{J} \leq 0$: $$m^2 = m \tanh(\tilde{J}m) = -|m| \tanh(|\tilde{J}m|) \leq 0 \quad \text{so} \quad m = 0$$

$\tilde{J} > 0$: $$m^2 = |m| \tanh(\tilde{J}|m|) = |m| \int_0^{\tilde{J}|m|} \mathrm{d}x \, [1 - \tanh^2(x)] \leq \tilde{J}m^2$$

$$m^2(\tilde{J} - 1) \geq 0 \quad \text{hence} \quad m = 0 \;\; \text{for } \tilde{J} < 1$$

$$\text{near } \tilde{J} = 1: \quad \tilde{J} = 1 + \epsilon, \; |m| \ll 1, \quad \text{use } \tanh(x) = x - \frac{1}{3}x^3 + \mathcal{O}(x^5)$$

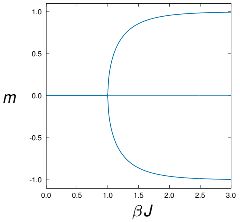
$$m = \tanh(m(1+\epsilon)) = m(1+\epsilon) - \frac{1}{3}m^3 + \mathcal{O}(\epsilon m^3, m^5)$$

$$m = 0 \;\; \text{or} \;\; \epsilon = \frac{1}{3}m^2 + \mathcal{O}(\epsilon m^2, m^4)$$

$$m = 0 \;\; \text{or} \;\; m = \pm(3\epsilon)^{\frac{1}{2}} + \ldots$$

$m \neq 0$ solns emerge at $\tilde{J} = 1$

phase transition



- apparent paradoxes
  - equilibrium analysis: $m = \mathrm{argmin}_y \Phi(y)$,
    $\tilde{J} > 1$ and $\tilde{\theta} \neq 0$: minimum has $m = |m| \mathrm{sgn}(\tilde{\theta})$,

    yet if $|\tilde{\theta}|$ not too big: $\mathrm{sgn}(m(\infty)) = \mathrm{sgn}(m(0))$ (remanence) ...
  - consider $\tilde{\theta} = 0$ for simplicity,
    process is ergodic, all $\boldsymbol{\sigma}$ can in principle be reached,

    yet if $\tilde{J} > 1$: $\mathrm{sgn}[m(\infty)] = \mathrm{sgn}[m(0)]$ ...
  - consider $\tilde{\theta} = 0$ for simplicity,
    $p_\infty(\boldsymbol{\sigma}) = p_\infty(-\boldsymbol{\sigma})$, hence $\langle \sigma_i \rangle = 0$ for all i,

    yet if $\tilde{J} > 1$: $m(\infty) = N^{-1} \sum_i \langle \sigma_i \rangle \neq 0$ ...
- explanation: non-commutation of limits

  equilibrium calculations: $t \to \infty$ *before* $N \to \infty$
  dynamical solution: $N \to \infty$ *before* $t \to \infty$

  spontaneous symmetry breaking,
  timescales required to see ergodicity diverge as $N \to \infty$

Route B: *integral representation of δ-function*

strategy:

1. insert $1 = \int \mathrm{d}m\, \delta[m - m(\boldsymbol{\sigma})]$ to relocate $m(\boldsymbol{\sigma})$
2. use integral representation of δ
3. ensure all terms in exponent of integral scale as $\mathcal{O}(N)$
4. use steepest descent integration

1. more general $H(\boldsymbol{\sigma}) = NE[m(\boldsymbol{\sigma})]$

$$
\begin{aligned}
Z &= \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta NE[m(\boldsymbol{\sigma})]} = \sum_{\boldsymbol{\sigma}} \int \mathrm{d}m\, \delta[m - m(\boldsymbol{\sigma})] \mathrm{e}^{-\beta NE[m(\boldsymbol{\sigma})]} \\
&= \int \mathrm{d}m\, \mathrm{e}^{-\beta NE[m]} \sum_{\boldsymbol{\sigma}} \delta[m - m(\boldsymbol{\sigma})]
\end{aligned}
$$

2. use $\delta(x) = (2\pi)^{-1} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}kx}$

$$
Z = \int \frac{\mathrm{d}m\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}km - \beta NE[m]} \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\mathrm{i}km(\boldsymbol{\sigma})}
$$

3. scaling with N: $k \to Ny$

$$Z = \frac{N}{2\pi}\int \mathrm{d}m\mathrm{d}y\; \mathrm{e}^{\mathrm{i}Nym-\beta NE[m]}\sum_{\boldsymbol{\sigma}}\mathrm{e}^{-\mathrm{i}y\sum_i \sigma_i}$$
$$= 2^N\frac{N}{2\pi}\int \mathrm{d}m\mathrm{d}y\; \mathrm{e}^{\mathrm{i}Nym-\beta NE[m]}\cos^N(y)$$

4. steepest descent

$$\lim_{N\to\infty} f = -\lim_{N\to\infty}\frac{1}{\beta N}\log\left\{2^N\frac{N}{2\pi}\int \mathrm{d}m\mathrm{d}y\; \mathrm{e}^{\mathrm{i}Nym-\beta NE[m]+N\log\cos(y)}\right\}$$
$$= -\frac{1}{\beta}\log 2-\frac{1}{\beta}\mathrm{extr}_{(m,y)}\Big[\mathrm{i}ym-\beta E[m]+\log\cos(y)\Big]$$
$$= -\frac{1}{\beta}\log 2+\Phi(m^\star,y^\star),\qquad \Phi(m,y)=E[m]-\frac{1}{\beta}\Big(\log\cos(y)+\mathrm{i}ym\Big)$$

$(m^\star, y^\star)$: soln of $\frac{\partial\Phi}{\partial m}=\frac{\partial\Phi}{\partial y}=0$

$$\frac{\mathrm{d}}{\mathrm{d}m}E[m]-\mathrm{i}y/\beta=0,\qquad \tan(y)-\mathrm{i}m=0$$

$$\text{to solve :} \qquad \frac{\mathrm{d}}{\mathrm{d}m}E[m] - \mathrm{i}y/\beta = 0, \quad \tan(y) - \mathrm{i}m = 0$$

- identify correct saddle point

$$E[m] \in \mathbb{R}, \text{ so } y = \mathrm{i}\beta x, \; x \in \mathbb{R} : \qquad \frac{\mathrm{d}}{\mathrm{d}m}E[m] + x = 0, \quad \tanh(\beta x) = m$$

$$\Rightarrow \quad m = \tanh\left(-\beta \frac{\mathrm{d}}{\mathrm{d}m}E[m]\right)$$

$$\lim_{N\to\infty} f = -\frac{1}{\beta}\log 2 + E[m] - \beta^{-1}\log\cosh(\beta x) + xm$$
$$= -\frac{1}{\beta}\log 2 + E[m] + \frac{1}{\beta}c^{\star}(m),$$

$$c^{\star}(m) = \frac{1}{2}\left[(1+m)\log(1+m) + (1-m)\log(1-m)\right]$$

(see exercises)

- present model:

$$E[m] = -\tfrac{1}{2}Jm^2 - \theta m$$

$$m = \tanh(\beta(Jm+\theta)), \qquad \lim_{N\to\infty} f = -\frac{1}{\beta}\log 2 + \frac{1}{\beta}c^{\star}(m) - \frac{1}{2}Jm^2 - \theta m$$

Parallel Markov chain

Dynamical solution

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma};\boldsymbol{\sigma}']p_t(\boldsymbol{\sigma}'), \qquad W[\boldsymbol{\sigma};\boldsymbol{\sigma}'] = \prod_{i=1}^{N} \frac{\mathrm{e}^{\beta\sigma_i[Jm(\boldsymbol{\sigma}')+\theta]}}{2\cosh[\beta(Jm(\boldsymbol{\sigma}')+\theta)]}$$

define

$$m(\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \sigma_i, \qquad P_t(m) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma})\delta[m-m(\boldsymbol{\sigma})]$$

- macroscopic dynamics

$$\begin{aligned} P_{t+1}(m) &= \sum_{\boldsymbol{\sigma}} p_{t+1}(\boldsymbol{\sigma})\delta[m-m(\boldsymbol{\sigma})] \\ &= \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \delta[m-m(\boldsymbol{\sigma})] \frac{\mathrm{e}^{\beta\sum_i \sigma_i[Jm(\boldsymbol{\sigma}')+\theta]}}{2^N\cosh^N[\beta(Jm(\boldsymbol{\sigma}')+\theta)]} p_t(\boldsymbol{\sigma}') \\ &= D(m)\sum_{\boldsymbol{\sigma}'}\left(\frac{\mathrm{e}^{\beta m(Jm(\boldsymbol{\sigma}')+\theta)}}{2\cosh[\beta(Jm(\boldsymbol{\sigma}')+\theta)]}\right)^N p_t(\boldsymbol{\sigma}') \end{aligned}$$

$$\text{with} \quad D(m) = \sum_{\boldsymbol{\sigma}} \delta[m-m(\boldsymbol{\sigma})]$$

- insert $1=\int \mathrm{d}m' \; \delta[m'-m(\boldsymbol{\sigma}')]$

$$P_{t+1}(m) \;=\; D(m)\int \mathrm{d}m' \Big(\frac{\mathrm{e}^{\beta m(Jm'+\theta)}}{2\cosh[\beta(Jm'+\theta)]}\Big)^N \sum_{\boldsymbol{\sigma}'} p_t(\boldsymbol{\sigma}')\delta[m'-m(\boldsymbol{\sigma}')]$$
$$\;=\; D(m)\int \mathrm{d}m' \Big(\frac{\mathrm{e}^{\beta m(Jm'+\theta)}}{2\cosh[\beta(Jm'+\theta)]}\Big)^N P_t(m')$$

equivalently

$$P_{t+1}(m) \;=\; \int \mathrm{d}m' \; \mathrm{e}^{N\Phi(m,m')} P_t(m')$$
$$\Phi(m,m') \;=\; \frac{1}{N}\log D(m) + \log\Big(\frac{\mathrm{e}^{\beta m(Jm'+\theta)}}{2\cosh[\beta(Jm'+\theta)]}\Big)$$

- $N\to\infty$: steepest descent integration

$$P_{t+1}(m) = P_t(m^\star(m)),$$
$$m^\star(m) = \mathrm{argmax}_{m'} \lim_{N\to\infty}\Phi(m,m')$$
$$= \mathrm{argmax}_x \Big[m(\tilde{J}x+\tilde{\theta}) - \log\cosh(\tilde{J}x+\tilde{\theta})\Big]$$

$N \to \infty$: if $P_t(m) = \delta[m - m(t)] \quad \Rightarrow \quad P_{t+1}(m) = \delta[m - m(t+1)]$

$m(\boldsymbol{\sigma})$ evolves *deterministically*

- link between $m(t)$ and $m(t+1)$,
  solve

$$m(t) = \mathrm{argmax}_x \Big[m(t+1)[\tilde{J}x + \tilde{\theta}] - \log\cosh(\tilde{J}x + \tilde{\theta}) \Big]$$

$$\text{differentiate wrt } x: \qquad m(t+1) = \tanh(\tilde{J}x + \tilde{\theta})$$

  result

$$m(t+1) = \tanh(\tilde{J}m(t) + \tilde{\theta})$$

- phenomenology
  - fixed-point eqn: same as for sequential dynamics
  - stability of fixed-points: different when $\tilde{J} < 0$!

    $\tilde{\theta} = 0$:

    $\tilde{J} \in [-1, 0)$: $m = 0$ stable

    $\tilde{J} \in (-\infty, -1)$: $m = 0$ unstable, system entering oscillation

    large times: $m(t) = (-1)^t m^\star, \quad m^\star = \tanh(|\tilde{J}|m^\star)$

Fokker-Planck equation

Dynamical solution

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma}) = -\sum_i \frac{\partial}{\partial \sigma_i}\left[p_t(\boldsymbol{\sigma})f_i(\boldsymbol{\sigma})\right] + T\sum_i \frac{\partial^2}{\partial \sigma_i^2}p_t(\boldsymbol{\sigma}), \qquad f_i(\boldsymbol{\sigma}) = Jm(\boldsymbol{\sigma}) + \theta + g(\sigma_i)$$

assume $p_t(\boldsymbol{\sigma})$ decays sufficiently fast for $|\boldsymbol{\sigma}| \to \infty$,
so that we need not worry about boundary terms

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle &= \int \mathrm{d}\boldsymbol{\sigma}\; G(\boldsymbol{\sigma})\frac{\mathrm{d}}{\mathrm{d}t}p_t(\boldsymbol{\sigma}) \\
&= \sum_i \int \mathrm{d}\boldsymbol{\sigma}\; G(\boldsymbol{\sigma})\left[-\frac{\partial}{\partial \sigma_i}\left[p_t(\boldsymbol{\sigma})f_i(\boldsymbol{\sigma})\right] + T\frac{\partial^2}{\partial \sigma_i^2}p_t(\boldsymbol{\sigma})\right] \\
&= \sum_i \int \mathrm{d}\boldsymbol{\sigma}\; \left[p_t(\boldsymbol{\sigma})f_i(\boldsymbol{\sigma})\frac{\partial}{\partial \sigma_i}G(\boldsymbol{\sigma}) + Tp_t(\boldsymbol{\sigma})\frac{\partial^2}{\partial \sigma_i^2}G(\boldsymbol{\sigma})\right] \\
&= \sum_i \left\langle f_i \frac{\partial}{\partial \sigma_i}G\right\rangle + T\sum_i \left\langle \frac{\partial^2}{\partial \sigma_i^2}G\right\rangle
\end{aligned}$$

no longer closed eqn for $m(\boldsymbol{\sigma})$ ...

Macroscopic dynamics

observables $\mathbf{\Omega}(\phi)=(\Omega_1(\boldsymbol{\sigma}),\ldots,\Omega_n(\boldsymbol{\sigma}))$,
macroscopic, so $\partial\Omega_\mu(\boldsymbol{\sigma})/\partial\sigma_i=\mathcal{O}(N^{-1})$
$P_t(\mathbf{\Omega})=\int\mathrm{d}\boldsymbol{\sigma}\; p_t(\boldsymbol{\sigma})\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]=\langle\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]\rangle$

- evolution of $P_t(\mathbf{\Omega})$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(\mathbf{\Omega}) = \sum_i\Big\langle f_i(\boldsymbol{\sigma})\frac{\partial\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\sigma_i}\Big\rangle + T\sum_i\Big\langle\frac{\partial^2\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\sigma_i^2}\Big\rangle$$

$$\frac{\partial\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\sigma_i}=-\sum_{\mu=1}^n\frac{\partial\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\Omega_\mu}\frac{\partial\Omega_\mu(\boldsymbol{\sigma})}{\partial\sigma_i}$$

$$\frac{\partial^2\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\sigma_i^2}=\sum_{\mu=1}^n\Big[\sum_{\nu=1}^n\frac{\partial^2\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\Omega_\mu\partial\Omega_\nu}\overbrace{\frac{\partial\Omega_\mu(\boldsymbol{\sigma})}{\partial\sigma_i}\frac{\partial\Omega_\nu(\boldsymbol{\sigma})}{\partial\sigma_i}}^{\mathcal{O}(N^{-2})}-\frac{\partial\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]}{\partial\Omega_\mu}\overbrace{\frac{\partial^2\Omega_\mu(\boldsymbol{\sigma})}{\partial\sigma_i^2}}^{\mathcal{O}(N^{-1})}\Big]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t(\mathbf{\Omega})=-\sum_{\mu=1}^n\frac{\partial}{\partial\Omega_\mu}\Big\langle\Big[\sum_i f_i(\boldsymbol{\sigma})\frac{\partial\Omega_\mu(\boldsymbol{\sigma})}{\partial\sigma_i}+T\sum_i\frac{\partial^2\Omega_\mu(\boldsymbol{\sigma})}{\partial\sigma_i^2}\Big]\delta[\mathbf{\Omega}-\mathbf{\Omega}(\boldsymbol{\sigma})]\Big\rangle+\mathcal{O}(\frac{1}{N})$$

- $N \to \infty$: Liouville eqn

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t(\mathbf{\Omega}) = -\sum_{\mu=1}^{n} \frac{\partial}{\partial \Omega_\mu} \Big\langle \Big[\sum_i f_i(\boldsymbol{\sigma}) \frac{\partial \Omega_\mu(\boldsymbol{\sigma})}{\partial \sigma_i} + T \sum_i \frac{\partial^2 \Omega_\mu(\boldsymbol{\sigma})}{\partial \sigma_i^2} \Big] \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] \Big\rangle$$
$$= -\sum_{\mu=1}^{n} \frac{\partial}{\partial \Omega_\mu} \left[F_\mu(\mathbf{\Omega}, t) P_t(\mathbf{\Omega}) \right]$$

with

$$F_\mu(\mathbf{\Omega}, t) = \lim_{N \to \infty} \Big\langle \sum_i f_i(\boldsymbol{\sigma}) \frac{\partial \Omega_\mu(\boldsymbol{\sigma})}{\partial \sigma_i} + T \sum_i \frac{\partial^2 \Omega_\mu(\boldsymbol{\sigma})}{\partial \sigma_i^2} \Big\rangle_{\mathbf{\Omega}, t}$$
$$\langle G(\phi) \rangle_{\mathbf{\Omega}, t} = \frac{\int \mathrm{d}\boldsymbol{\sigma}\ p_t(\boldsymbol{\sigma}) \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})] G(\boldsymbol{\sigma})}{\int \mathrm{d}\boldsymbol{\sigma}\ p_t(\boldsymbol{\sigma}) \delta[\mathbf{\Omega} - \mathbf{\Omega}(\boldsymbol{\sigma})]}$$

deterministic soln,
$P_t(\mathbf{\Omega}) = \delta[\mathbf{\Omega} - \mathbf{\Omega}(t)]$,
in which $\mathbf{\Omega}(t)$ is soln of

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{\Omega} = \lim_{N \to \infty} \Big\langle \sum_i f_i(\boldsymbol{\sigma}) \frac{\partial \mathbf{\Omega}(\boldsymbol{\sigma})}{\partial \sigma_i} + T \sum_i \frac{\partial^2 \mathbf{\Omega}(\boldsymbol{\sigma})}{\partial \sigma_i^2} \Big\rangle_{\mathbf{\Omega}, t}$$

Choice of macroscopic observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$

- key macroscopic object

$$\varrho(\sigma|\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \delta(\sigma-\sigma_i), \qquad \sigma\in S, \quad S=\{\sigma|\ \sigma=\ell\epsilon,\ \ell\in\mathbb{Z},\ |\ell|<1/\epsilon^2\}$$

$$\epsilon\downarrow 0 \ \text{ at end of calculation}$$

$$\frac{\partial}{\partial\sigma_i}\varrho(\sigma|\boldsymbol{\sigma}) = -\frac{1}{N}\delta'(\sigma-\sigma_i), \qquad \frac{\partial^2}{\partial\sigma_i^2}\varrho(\sigma|\boldsymbol{\sigma}) = \frac{1}{N}\delta''(\sigma-\sigma_i)$$

dynamics

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\varrho(\sigma) &= \lim_{N\to\infty}\Big\langle \sum_i f_i(\boldsymbol{\sigma})\frac{\partial\varrho(\sigma|\boldsymbol{\sigma})}{\partial\sigma_i} + T\sum_i \frac{\partial^2\varrho(\sigma|\boldsymbol{\sigma})}{\partial\sigma_i^2}\Big\rangle_{\varrho,t} \\
&= -\lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle f_i(\boldsymbol{\sigma})\delta'(\sigma-\sigma_i)\Big\rangle_{\varrho,t} + T\lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle\delta''(\sigma-\sigma_i)\Big\rangle_{\varrho,t} \\
&= -\frac{\partial}{\partial\sigma}\lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle f_i(\boldsymbol{\sigma})\delta(\sigma-\sigma_i)\Big\rangle_{\varrho,t} + T\frac{\partial^2}{\partial\sigma^2}\varrho(\sigma)
\end{aligned}$$

- remaining term

$$\lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle f_i(\boldsymbol{\sigma})\delta(\sigma-\sigma_i)\Big\rangle_{\varrho,t} = \lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle\Big[Jm(\boldsymbol{\sigma})+\theta+g(\sigma_i)\Big]\delta(\sigma-\sigma_i)\Big\rangle_{\varrho,t}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_i\Big\langle\Big[J\int\mathrm{d}\sigma'\sigma'\varrho(\sigma'|\boldsymbol{\sigma})+\theta+g(\sigma)\Big]\delta(\sigma-\sigma_i)\Big\rangle_{\varrho,t}$$

$$= \lim_{N\to\infty}\Big[J\int\mathrm{d}\sigma'\sigma'\varrho(\sigma')+\theta+g(\sigma)\Big]\Big\langle\frac{1}{N}\sum_i\delta(\sigma-\sigma_i)\Big\rangle_{\varrho,t}$$

$$= \Big[J\int\mathrm{d}\sigma'\sigma'\varrho(\sigma')+\theta+g(\sigma)\Big]\varrho(\sigma)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_t(\sigma) = -\frac{\partial}{\partial\sigma}\Big[\Big(J\int\mathrm{d}\sigma'\sigma'\varrho_t(\sigma')+\theta+g(\sigma)\Big)\varrho_t(\sigma)\Big] + T\frac{\partial^2}{\partial\sigma^2}\varrho_t(\sigma)$$

- stationary soln, let $g(\sigma) = G'(\sigma)$

$$\varrho(\sigma)=\frac{1}{Z}\mathrm{e}^{\beta[\sigma(Jm+\theta)+G(\sigma)]},\quad m=\int\mathrm{d}\sigma\ \sigma\varrho(\sigma),\quad Z=\int\mathrm{d}\sigma\ \mathrm{e}^{\beta[\sigma(Jm+\theta)+G(\sigma)]}$$

(see exercises)