Modelling of Complex Real-World Systems
Part B. Tools for Heterogeneous Systems

B2. The replica method

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Self-averaging

Site-heterogeneity

heterogeneity in *single-site properties*, e.g. $J_{ij} \to \xi_i \xi_j / N, \quad \theta_i \to \theta_i$ (Mattis model)

- repeat analysis with $N \to \infty$

$$\text{parallel dynamics :} \qquad m(t+1) = \int \mathrm{d}\xi \mathrm{d}\theta \; p(\xi,\theta)\xi \tanh[\beta(\xi m(t)+\theta)]$$

$$\text{sequential dynamics :} \qquad \frac{\mathrm{d}}{\mathrm{d}t} m = \int \mathrm{d}\xi \mathrm{d}\theta \; p(\xi,\theta)\xi \tanh[\beta(\xi m+\theta) - m$$

$$m(t) = \frac{1}{N}\sum_i \xi_i \sigma_i, \qquad p(\xi,\eta) = \lim_{N\to\infty} \frac{1}{N}\sum_i \delta(\xi-\xi_i)\delta(\theta-\theta_i)$$

(see exercises)

- observations
  - again deterministic macroscopic laws
  - again spontaneous order, but not in terms of $N^{-1}\sum_i \sigma_i$
  - details of $\{\xi_i, \theta_i\}$ not important, only $p(\xi,\theta)$: 'self-averaging'
  - interactions mediated via single observable: $h_i(\boldsymbol{\sigma}) = \xi_i m(\boldsymbol{\sigma}) + \theta_i$

Interaction heterogeneity

site-heterogeneity: $\mathcal{O}(N)$ parameters, harmless
interaction heterogeneity: $\mathcal{O}(N^2)$ parameters ... ?

- simulation examples, $\boldsymbol{\sigma} \in \{-1, 1\}^N$

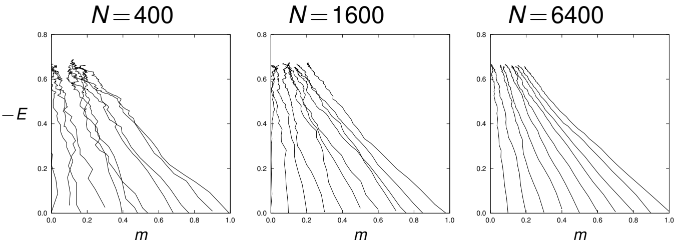
$$J_{ij} = \frac{Jz_{ij}}{\sqrt{N}}(1-\delta_{ij}), \quad z_{ij} = z_{ji}: \text{ drawn indep from } p(z) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}z^2}$$

$$\text{measure}: \quad m(\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \sigma_i, \quad E(\boldsymbol{\sigma}) = -\frac{1}{2N}\sum_{i\neq j} \sigma_i J_{ij} \sigma_j$$

sequential dynamics,

$\beta J = 10$
$\beta\theta = 0$

$E_0 = 0$,
$m_0 = \ell/10$

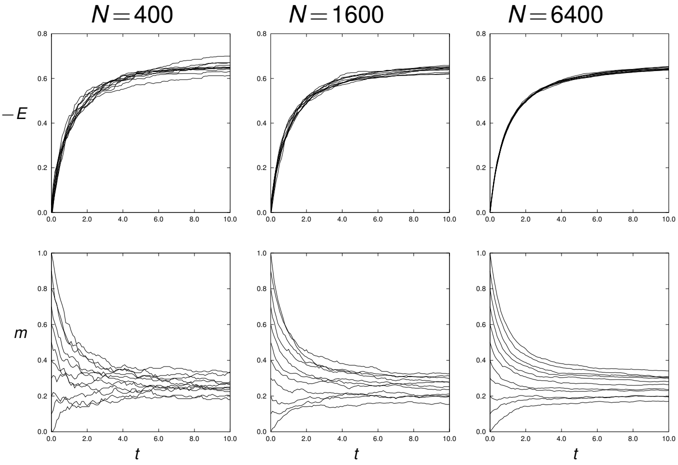


plotted versus time ...

sequential dynamics,

$\beta J = 10$
$\beta\theta = 2$

$E_0 = 0$,
$m_0 = \ell/10$



previous analysis routes no longer work,
but macroscopic dynamics again self-averaging!

Implications of self-averaging

let ξ be (static) system parameters
drawn from $p(\xi)$

$$\text{notation:} \qquad \overline{g(\xi)} = \int \mathrm{d}\xi \; p(\xi) g(\xi)$$

- $\omega(\boldsymbol{\sigma})$ self-averaging:

$$\text{for all } t \geq 0: \qquad \lim_{N\to\infty} \langle \omega(\boldsymbol{\sigma}(t)) \rangle = \lim_{N\to\infty} \overline{\langle \omega(\boldsymbol{\sigma}(t)) \rangle}$$

- generating functions:

  if $\lim_{N\to\infty} f(\boldsymbol{\lambda})$ exists and is self-averaging

$$\lim_{N\to\infty} \langle \omega_k(\boldsymbol{\sigma}) \rangle = \lim_{N\to\infty} \frac{\partial f}{\partial \lambda_k} = \lim_{N\to\infty} \frac{\partial \overline{f}}{\partial \lambda_k}$$

  - can work with $\overline{f}$ instead of f, for $N\to\infty$
  - fruitful new route to solving complex models
  - generating functions exist for *statics and dynamics*

Previous example

$$h_i(\boldsymbol{\sigma}) = \sum_j J_{ij}\sigma_j + \theta, \quad J_{ij} = \frac{J z_{ij}}{\sqrt{N}}(1-\delta_{ij}), \qquad z_{ij} = z_{ji} : \text{ static random pars}$$

$$\text{observables}: \quad m(\boldsymbol{\sigma}) = \frac{1}{N}\sum_i \sigma_i, \quad E(\boldsymbol{\sigma}) = -\frac{1}{2N}\sum_{i\neq j} \sigma_i J_{ij}\sigma_j$$

Sherrington-Kirkpatrick spin-glass model

- equilibrium

$$p(\boldsymbol{\sigma}) = \frac{1}{Z}\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}, \qquad H(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j - \theta\sum_i \sigma_i$$

$$f = -\frac{1}{\beta N}\log Z = -\frac{1}{\beta N}\log\Big[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\frac{\beta J}{2\sqrt{N}}\sum_{i\neq j} z_{ij}\sigma_i\sigma_j + \beta\theta\sum_i \sigma_i}\Big]$$

large N

$$\overline{\langle E(\boldsymbol{\sigma})\rangle} = J\frac{\partial \bar{f}}{\partial J}, \quad \overline{\langle m(\boldsymbol{\sigma})\rangle} = -\frac{\partial \bar{f}}{\partial \theta} : \qquad \bar{f} = -\frac{1}{\beta N}\overline{\log\Big[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\frac{\beta J}{2\sqrt{N}}\sum_{i\neq j} z_{ij}\sigma_i\sigma_j + \beta\theta\sum_i \sigma_i}\Big]}$$

see also: https://scitechdaily.com/new-whirling-state-of-matter-discovered-self-induced-spin-glass/

The basic replica identity

the replica method

A clever trick that enables the analytical calculation of averages that are normally impossible to do, except numerically

is particularly useful for

Complex heterogeneous systems composed of *many* interacting variables, and with *many* parameters on which we have only statistical information (too large for numerical averages to be computationally feasible)

gives us

Analytical predictions for the behaviour of *macroscopic* quantities in *typical* realisations of the systems under study.

first appearance: Marc Kac 1968
first in physics: Sherrington & Kirkpatrick 1975
first in biology: Amit, Gutfreund & Sompolinksy 1985

since then: computer science, economics, statistics, ...

Disorder-averaged generating functions

- Consider processes with many fixed (pseudo-)random parameters $\boldsymbol{\xi}$ ('disorder'), distributed according to $\mathcal{P}(\boldsymbol{\xi})$

$$p(\boldsymbol{\sigma}|\boldsymbol{\xi}) = \frac{\mathrm{e}^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}}{Z(\boldsymbol{\lambda},\boldsymbol{\xi})}, \qquad Z(\boldsymbol{\lambda},\boldsymbol{\xi}) = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}$$

  - calculating $\langle g(\boldsymbol{\sigma},\boldsymbol{\xi})\rangle$ for *each* realisation of $\boldsymbol{\xi}$ usually impossible
  - we are mostly interested in *typical* values of state averages
  - for $N\to\infty$ macroscopic averages will not depend on $\boldsymbol{\xi}$, only on $\mathcal{P}(\boldsymbol{\xi})$, 'self-averaging': $\lim_{N\to\infty}\langle g(\boldsymbol{\sigma},\boldsymbol{\xi})\rangle$ indep of $\boldsymbol{\xi}$

so focus on

$$\overline{\langle g(\boldsymbol{\sigma},\boldsymbol{\xi})\rangle} = \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi})\langle g(\boldsymbol{\sigma},\boldsymbol{\xi})\rangle = \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma}|\boldsymbol{\xi}) g(\boldsymbol{\sigma},\boldsymbol{\xi})$$

- new generating function

$$\overline{F}(\boldsymbol{\lambda},\mu)=\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\log Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi}),\qquad Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi})=\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\mu\psi(\boldsymbol{\sigma},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}$$

$$\lim_{\mu\to0}\frac{\partial}{\partial\mu}\overline{F}(\boldsymbol{\lambda},\mu)=\lim_{\mu\to0}\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\left\{\frac{\sum_{\boldsymbol{\sigma}}\psi(\boldsymbol{\sigma},\boldsymbol{\xi})\mathrm{e}^{\mu\psi(\boldsymbol{\sigma},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\mu\psi(\boldsymbol{\sigma},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}}\right\}$$

$$=\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\left\{\frac{\sum_{\boldsymbol{\sigma}}\psi(\boldsymbol{\sigma},\boldsymbol{\xi})\mathrm{e}^{\sum_\ell\lambda_\ell\omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\sum_\ell\lambda_\ell\omega_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}}\right\}=\overline{\langle\psi\rangle}$$

- main obstacle in calculating $\overline{F}$:
  the logarithm ...

replica identity : $\overline{\log Z}=\lim_{n\to0}\frac{1}{n}\log\overline{Z^n}$

proof:

$$\lim_{n\to0}\frac{1}{n}\log\overline{Z^n}=\lim_{n\to0}\frac{1}{n}\log\overline{[\mathrm{e}^{n\log Z}]}=\lim_{n\to0}\frac{1}{n}\log\overline{[1+n\log Z+\mathcal{O}(n^2)]}$$

$$=\lim_{n\to0}\frac{1}{n}\log[1+n\overline{\log Z}+\mathcal{O}(n^2)]=\overline{\log Z}$$

Powers by replication

- apply $\overline{\log Z} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n}$ (for simplest case $L=1$)

$$\overline{F}(\lambda) = \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \log \left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\lambda\omega(\boldsymbol{\sigma},\boldsymbol{\xi})} \right] = \lim_{n\to 0} \frac{1}{n} \log \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\lambda\omega(\boldsymbol{\sigma},\boldsymbol{\xi})} \right]^n$$

$$= \lim_{n\to 0} \frac{1}{n} \log \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \left[\sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \mathrm{e}^{\lambda \sum_{\alpha=1}^n \omega(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})} \right]$$

$$= \lim_{n\to 0} \frac{1}{n} \log \left[\sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \mathrm{e}^{\lambda \sum_{\alpha=1}^n \omega(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})} \right]$$

notes

- $\boldsymbol{\xi}$-average converted into doable one ...
- involves n ‘replicas’ $\boldsymbol{\sigma}^\alpha$ of original system,
- but with $n \to 0$ at the end ...
- penultimate step true only for *integer n*, so limit requires *analytical continuation* ...

- imagine a physical system, with
  free energy density as generating function

$$f_N = -\frac{1}{\beta N}\log Z, \qquad Z = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}$$

  replica method

$$\begin{aligned}
\overline{f}_N &= -\frac{1}{\beta N}\overline{\log Z} = -\lim_{n\to 0}\frac{1}{n\beta N}\log \overline{Z^n} \\
&= -\lim_{n\to 0}\frac{1}{\beta n N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n}\overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}}\right] \\
&= -\lim_{n\to 0}\frac{1}{\beta n N}\log Z', \qquad Z' = \sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n}\mathrm{e}^{-\beta H_{\mathrm{eff}}(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)}
\end{aligned}$$

$$H_{\mathrm{eff}}(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n) = -\frac{1}{\beta}\log\left(\overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}}\right)$$

If f self-averaging:

disordered N-variable system with $H(\boldsymbol{\sigma},\boldsymbol{\xi})$ for $N\to\infty$
equivalent to *homogeneous nN-variable system* with $H_{\mathrm{eff}}(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)$

But does the replica identity actually help?

- return to previous example ...

$$
\begin{aligned}
\bar{f}_N &= -\frac{1}{\beta N}\overline{\log\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\frac{\beta J}{\sqrt{N}}\sum_{i<j} z_{ij}\sigma_i\sigma_j+\beta\theta\sum_i \sigma_i}\right]} \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\overline{\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\frac{\beta J}{\sqrt{N}}\sum_{i<j} z_{ij}\sigma_i\sigma_j+\beta\theta\sum_i \sigma_i}\right]^n} \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \overline{\mathrm{e}^{\frac{\beta J}{\sqrt{N}}\sum_{i<j} z_{ij}\sum_{\alpha=1}^n \sigma_i^\alpha\sigma_j^\alpha+\beta\theta\sum_i\sum_{\alpha=1}^n \sigma_i^\alpha}}\right] \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\beta\theta\sum_i\sum_{\alpha=1}^n \sigma_i^\alpha}\prod_{i<j}\int\frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2+\frac{\beta J z}{\sqrt{N}}\sum_{\alpha=1}^n \sigma_i^\alpha\sigma_j^\alpha}\right] \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\beta\theta\sum_i\sum_{\alpha=1}^n \sigma_i^\alpha}\prod_{i<j}\mathrm{e}^{\frac{(\beta J)^2}{2N}\left(\sum_{\alpha=1}^n \sigma_i^\alpha\sigma_j^\alpha\right)^2}\right]
\end{aligned}
$$

$$\bar{f}_N = -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\beta\theta\sum_i\sum_{\alpha=1}^n \sigma_i^\alpha + \frac{(\beta J)^2}{4N}\sum_{i\neq j}\left(\sum_{\alpha=1}^n \sigma_i^\alpha\sigma_j^\alpha\right)^2}\right]$$

$$= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{N\beta\theta\sum_{\alpha=1}^n\left(\frac{1}{N}\sum_i\sigma_i^\alpha\right) + \frac{N(\beta J)^2}{4}\sum_{\alpha,\beta=1}^n\left(\frac{1}{N}\sum_i\sigma_i^\alpha\sigma_i^\beta\right)^2 + \mathcal{O}(1)}\right]$$

new key macroscopic observables:

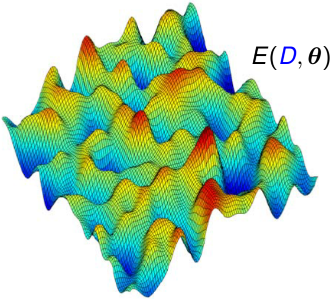
$$m_\alpha = \frac{1}{N}\sum_{i=1}^N \sigma_i^\alpha, \qquad q_{\alpha\beta} = \frac{1}{N}\sum_{i=1}^N \sigma_i^\alpha\sigma_i^\beta, \qquad \alpha,\beta = 1\ldots n$$

- unusual mathematics,
  n-vectors and $n\times n$ matrices with $n\to 0$ ...
- new types of statistical theories
- but results turn out to be *correct*, confirmed by
  computer simulations and alternative (much more tedious) routes

Replicas and minimization

Suppose we have data D, with prob distr $\mathcal{P}(D)$
and an algorithm which minimises an error function $E(D, \boldsymbol{\theta})$

(maximum likelihood, logistic,
Cox & Bayesian regression, machine
learning, discriminant analysis, ...)



- algorithm outcome

$$\boldsymbol{\theta}^{\star}(D) = \arg\min_{\boldsymbol{\theta}} E(D, \boldsymbol{\theta})$$

$$E_{\min}(D) = \min_{\boldsymbol{\theta}} E(D, \boldsymbol{\theta})$$

most computer science analyses
focus on *worst case* performance
(i.e. on most difficult data)

not always useful in practice ...

Typical performance of algorithms

- typical performance

$$\boldsymbol{\theta}^\star = \sum_D \mathcal{P}(D)\boldsymbol{\theta}^\star(D) = \overline{\boldsymbol{\theta}^\star(D)}$$

$$E_{\min} = \sum_D \mathcal{P}(D)E_{\min}(D) = \overline{E_{\min}(D)}$$

- steepest descent identity
  combined with replica trick

$$E_{\min}(D) = \min_{\boldsymbol{\theta}} E(D,\boldsymbol{\theta}) = -\lim_{\beta\to\infty}\frac{1}{\beta}\log\int \mathrm{d}\boldsymbol{\theta}\ \mathrm{e}^{-\beta E(D,\boldsymbol{\theta})}$$

$$E_{\min} = \overline{E_{\min}(D)} = -\lim_{\beta\to\infty}\frac{1}{\beta}\overline{\log\int \mathrm{d}\boldsymbol{\theta}\ \mathrm{e}^{-\beta E(D,\boldsymbol{\theta})}}$$

$$= -\lim_{\beta\to\infty}\lim_{n\to 0}\frac{1}{\beta n}\log\overline{\left[\int \mathrm{d}\boldsymbol{\theta}\ \mathrm{e}^{-\beta E(D,\boldsymbol{\theta})}\right]^n}$$

$$= -\lim_{\beta\to\infty}\lim_{n\to 0}\frac{1}{\beta n}\log\int \mathrm{d}\boldsymbol{\theta}^1\ldots\boldsymbol{\theta}^n\ \overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n E(D,\boldsymbol{\theta}^\alpha)}}$$

Alternative forms of the replica identity

suppose we need disorder averages, but for
a $p(\boldsymbol{\sigma}|\boldsymbol{\xi})$ that is not of an exponential form?

$$p(\boldsymbol{\sigma}|\boldsymbol{\xi}) = \frac{W(\boldsymbol{\sigma},\boldsymbol{\xi})}{\sum_{\boldsymbol{\sigma}'} W(\boldsymbol{\sigma}',\boldsymbol{\xi})}, \qquad \overline{\langle g \rangle} = \overline{\sum_{\boldsymbol{\sigma}} p(\boldsymbol{\sigma}|\boldsymbol{\xi}) g(\boldsymbol{\sigma},\boldsymbol{\xi})}$$

- main obstacle: the fraction ...

$$\begin{aligned}
\overline{\langle g \rangle} &= \overline{\left[\frac{\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi}) g(\boldsymbol{\sigma},\boldsymbol{\xi})}{\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi})}\right]} = \overline{\left[\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi}) g(\boldsymbol{\sigma},\boldsymbol{\xi})\right]\left[\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi})\right]^{-1}} \\
&= \lim_{n\to 0} \overline{\left[\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi}) g(\boldsymbol{\sigma},\boldsymbol{\xi})\right]\left[\sum_{\boldsymbol{\sigma}} W(\boldsymbol{\sigma},\boldsymbol{\xi})\right]^{n-1}} \\
&= \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \overline{g(\boldsymbol{\sigma}^1,\boldsymbol{\xi}) W(\boldsymbol{\sigma}^1,\boldsymbol{\xi}) \ldots W(\boldsymbol{\sigma}^n,\boldsymbol{\xi})}
\end{aligned}$$

(again: used integer n, but $n \to 0$ ...)

- equivalence of the two forms of replica identity, if $W(\boldsymbol{\sigma},\boldsymbol{\xi}) = \mathrm{e}^{\sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma},\boldsymbol{\xi})}$

$$
\begin{aligned}
\overline{\langle g \rangle} &= \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \overline{g(\boldsymbol{\sigma}^1,\boldsymbol{\xi}) W(\boldsymbol{\sigma}^1,\boldsymbol{\xi}) \ldots W(\boldsymbol{\sigma}^n,\boldsymbol{\xi})} \\
&= \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \overline{g(\boldsymbol{\sigma}^1,\boldsymbol{\xi})\, \mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{n\to 0} \frac{1}{n} \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \overline{\Big[\sum_{\alpha=1}^n g(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})\Big]\, \mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{n\to 0} \frac{1}{n} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \overline{\mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi}) + \mu \sum_{\alpha=1}^n g(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{n\to 0} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \frac{1}{n} \overline{\sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} \mathrm{e}^{\sum_{\alpha=1}^n \left[\sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi}) + \mu g(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})\right]}} \\
&= \lim_{n\to 0} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \frac{1}{n} \overline{Z^n(\boldsymbol{\lambda},\mu,\boldsymbol{\xi})}, \qquad Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi}) = \sum_{\boldsymbol{\sigma}} \mathrm{e}^{\sum_\ell \lambda_\ell \phi_\ell(\boldsymbol{\sigma},\boldsymbol{\xi}) + \mu g(\boldsymbol{\sigma},\boldsymbol{\xi})} \\
&= \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \overline{\log Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi})}
\end{aligned}
$$

More complicated objects

- let $p(\boldsymbol{\sigma}|\boldsymbol{\xi}) = Z^{-1}(\boldsymbol{\xi})\mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}$

$$\overline{\langle\sigma_i\rangle\langle\sigma_j\rangle} = \overline{\left[\frac{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}\sigma_i}{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}}\right]\left[\frac{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}\sigma_j}{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}}\right]}$$

$$= \lim_{n\to 0} \overline{\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}\sigma_i\right]\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}\sigma_j\right]\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma},\boldsymbol{\xi})}\right]^{n-2}}$$

$$= \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n} \sigma_i^1\sigma_j^2\, \overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}}$$

$$\overline{\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle} = \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n} \sigma_i^1\sigma_j^2\sigma_k^3\, \overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}}$$

- detect macroscopic order,
  not visible in $m = N^{-1}\sum_i\langle\sigma_i\rangle$

$$q_{\mathrm{EA}} = \frac{1}{N}\sum_i \overline{\langle\sigma_i\rangle^2} = \lim_{n\to 0}\sum_{\boldsymbol{\sigma}^1}\cdots\sum_{\boldsymbol{\sigma}^n}\left(\frac{1}{N}\sum_i \sigma_i^1\sigma_i^2\right)\overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha,\boldsymbol{\xi})}}$$

(Edwards-Anderson parameter)