Modelling of Complex Real-World Systems
Part B. Tools for Heterogeneous Systems

B3. Generating functional analysis

Ton Coolen
Department of Biophysics, Radboud University
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1. Disorder averaging in dynamics
2. Illustration
3. Continuous time

Disorder averaging in dynamics

Objective

find *disorder averages of dynamical order parameters*
for models with interaction disorder ...

'generating functional analysis'
'path integral formalism'
'dynamic mean-field theory'

- inspiration from statics,
  why did replicas help?

SK model: $h_i(\boldsymbol{\sigma}) = \sum_{j\neq i} J_{ij}\sigma_j + \theta, \;\; J_{ij} = \frac{J}{\sqrt{N}} z_{ij},$

$z_{ij} = z_{ji}$: drawn indep from $p(z) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}z^2}$

(i) disorder-averaged observables follow from generating function $\overline{f}$
(ii) average over disorder in $\overline{f}$ transformed to averages of the form

$$\overline{\mathrm{e}^{\sum_{i<j} z_{ij} Q_{ij}}} = \prod_{i<j} \overline{\mathrm{e}^{z_{ij} Q_{ij}}} = \ldots = \mathrm{e}^{\frac{1}{2}\sum_{i<j} Q_{ij}^2}, \quad \text{done analytically!}$$

(iii) large N: mapping to homogeneous system

is there an equivalent for dynamics?
especially for systems without detailed balance ...

Strategy

(i) find generating function for disorder-averaged dynamical observables
(ii) transform average over disorder in to averages of the form

$$\overline{\mathrm{e}^{\sum_{i<j} z_{ij} Q_{ij}}} = \prod_{i<j} \overline{\mathrm{e}^{z_{ij} Q_{ij}}} = \ldots = \mathrm{e}^{\frac{1}{2}\sum_{i<j} Q_{ij}^2}, \quad \text{done analytically!}$$

(iii) large N: mapping to homogeneous dynamical system?

consider discrete $\boldsymbol{\sigma} \in \{-1, 1\}^N$
with parallel dynamics

- think in terms of *paths* $\boldsymbol{\sigma}(0) \to \boldsymbol{\sigma}(1) \to \ldots \to \boldsymbol{\sigma}(t_{\max})$,
  with *path probabilities* $\mathcal{P}[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t_{\max})]$
  and averages $\langle \ldots \rangle = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})} \mathcal{P}[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(t_{\max})] \ldots$

- add time-dep perturbations: $h_i(\boldsymbol{\sigma}(t)) \to h_i(\boldsymbol{\sigma}(t)) + \theta_i(t)$
  define *generating functional*:

$$Z[\psi] = \langle \mathrm{e}^{-\mathrm{i} \sum_i \sum_{t=0}^{t_{\max}} \psi_i(t) \sigma_i(t)} \rangle$$

- generating functional $$Z[\psi] = \langle \mathrm{e}^{-\mathrm{i}\sum_i \sum_{t=0}^{t_{\max}} \psi_i(t)\sigma_i(t)} \rangle$$

  generates averages
  and correlation and response functions

$$\langle \sigma_i(t) \rangle = \mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial Z[\boldsymbol{\psi}]}{\partial \psi_i(t)}, \qquad C_{ij}(t,t') = -\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 Z[\boldsymbol{\psi}]}{\partial \psi_i(t)\partial \psi_j(t')}$$

$$G_{ij}(t,t') = \mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 Z[\boldsymbol{\psi}]}{\partial \psi_i(t)\partial \theta_j(t')}$$

- *disorder-averaged* generating functional $$\overline{Z[\psi]} = \overline{\langle \mathrm{e}^{-\mathrm{i}\sum_i \sum_{t=0}^{t_{\max}} \psi_i(t)\sigma_i(t)} \rangle}$$

  generates
  disorder-averaged quantities

$$\overline{\langle \sigma_i(t) \rangle} = \mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial \overline{Z[\boldsymbol{\psi}]}}{\partial \psi_i(t)}, \qquad \overline{C_{ij}(t,t')} = -\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 \overline{Z[\boldsymbol{\psi}]}}{\partial \psi_i(t)\partial \psi_j(t')}$$

$$\overline{G_{ij}(t,t')} = \mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 \overline{Z[\boldsymbol{\psi}]}}{\partial \psi_i(t)\partial \theta_j(t')}$$

objective (i) achieved ✓

Disorder-averaged generating functional

$$\overline{Z[\psi]} = \overline{\langle \mathrm{e}^{-\mathrm{i}\sum_i \sum_t \psi_i(t)\sigma_i(t)} \rangle}$$

- the process average $\langle \ldots \rangle$

$$\langle \mathrm{e}^{-\mathrm{i}\sum_i \sum_t \psi_i(t)\sigma_i(t)} \rangle = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})} \mathcal{P}[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})] \mathrm{e}^{-\mathrm{i}\sum_i \sum_t \psi_i(t)\sigma_i(t)}$$

$$\text{Markov chain:} \quad \mathcal{P}[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})] = p_0(\boldsymbol{\sigma}(0)) \prod_{t'=0}^{t_{\max}-1} W_{t'}[\boldsymbol{\sigma}(t'+1);\boldsymbol{\sigma}(t')]$$

$$W_{t'}[\boldsymbol{\sigma}(t'+1);\boldsymbol{\sigma}(t')] = \prod_i \frac{\mathrm{e}^{\beta\sigma_i(t'+1)h_i(t',\boldsymbol{\sigma}(t'))}}{2\cosh[\beta h_i(t',\boldsymbol{\sigma}(t'))]}, \quad h_i(t,\boldsymbol{\sigma}(t)) = \sum_j J_{ij}\sigma_j(t) + \theta_i(t)$$

our aim: average $\langle \mathrm{e}^{-\mathrm{i}\sum_i \sum_t \psi_i(t)\sigma_i(t)} \rangle$ over the $\{J_{ij}\}$,
not obvious yet that this is possible ...

way out: use delta functions cleverly

$$\overline{\langle e^{-i\sum_i\sum_t \psi_i(t)\sigma_i(t)}\rangle} = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})} \overline{\mathcal{P}[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})]}\, e^{-i\sum_i\sum_t \psi_i(t)\sigma_i(t)}$$

note: $\{J_{ij}\}$ appear only in $h_i(t,\boldsymbol{\sigma}(t))$

- insert for all (i,t): $1 = \int dh_i(t)\,\delta\Big[h_i(t) - h_i(t,\boldsymbol{\sigma}(t))\Big]$

$$\mathcal{P}[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})] = p_0(\boldsymbol{\sigma}(0))\prod_{it}\left\{\int dh_i(t)\,\delta\Big[h_i(t) - h_i(t,\boldsymbol{\sigma}(t))\Big]\frac{e^{\beta\sigma_i(t+1)h_i(t)}}{2\cosh[\beta h_i(t)]}\right\}$$

$$= p_0(\boldsymbol{\sigma}(0))\prod_{it}\left\{\int\frac{dh_i(t)d\hat{h}_i(t)}{2\pi}e^{i\hat{h}_i(t)[h_i(t)-h_i(t,\boldsymbol{\sigma}(t))]}\frac{e^{\beta\sigma_i(t+1)h_i(t)}}{2\cosh[\beta h_i(t)]}\right\}$$

$$\overline{\mathcal{P}[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})]} = p_0(\boldsymbol{\sigma}(0))\prod_{it}\left\{\int\frac{dh_i(t)d\hat{h}_i(t)}{2\pi}e^{i\hat{h}_i(t)h_i(t)}\frac{e^{\beta\sigma_i(t+1)h_i(t)}}{2\cosh[\beta h_i(t)]}\right\} \times \overline{e^{-i\sum_{it}\hat{h}_i(t)h_i(t,\boldsymbol{\sigma}(t))}}$$

- disorder average

$$\overline{e^{-i\sum_{it}\hat{h}_i(t)h_i(t,\boldsymbol{\sigma}(t))}} = e^{-i\sum_{it}\hat{h}_i(t)\theta_i(t)}\,\overline{e^{-i\sum_{ij}J_{ij}\sum_t\hat{h}_i(t)\sigma_j(t)}}$$

objective (ii) achieved ✓

Recap

short-hand $\mathbf{x}(t) = \{x_i(t)\}$

$$\overline{Z[\mathbf{0}]} = 1, \qquad \overline{\langle \sigma_i(t) \rangle} = \mathrm{i} \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{\partial \overline{Z[\boldsymbol{\psi}]}}{\partial \psi_i(t)}, \qquad \overline{\langle \sigma_i(t)\sigma_j(t') \rangle} = -\lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{\partial^2 \overline{Z[\boldsymbol{\psi}]}}{\partial \psi_i(t)\partial \psi_j(t')}$$

$$\overline{Z[\boldsymbol{\psi}]} = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})} p_0(\boldsymbol{\sigma}(0)) \int \left[\prod_{t=0}^{t_{\max}-1} \overbrace{\frac{\mathrm{d}\mathbf{h}(t)\mathrm{d}\hat{\mathbf{h}}(t)}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\hat{\mathbf{h}}(t)\cdot[\mathbf{h}(t)-\boldsymbol{\theta}(t)]}}^{\text{introduction of deltas}} \; \overbrace{\mathrm{e}^{-\mathrm{i}\boldsymbol{\psi}(t)\cdot\boldsymbol{\sigma}(t)}}^{\text{generating fields}} \right]$$
$$\times \left[\underbrace{\prod_{t=0}^{t_{\max}-1} \frac{\mathrm{e}^{\beta\boldsymbol{\sigma}(t+1)\cdot\mathbf{h}(t)}}{\prod_i 2\cosh(\beta h_i(t))}}_{\text{process dynamics}} \right] \underbrace{\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij} \sum_t \hat{h}_i(t)\sigma_j(t)}}}_{\text{disorder average}}$$

Illustration

Getting used to GFA: parallel Curie-Weiss model

$$\overline{Z[\psi]} = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_{\max})} p_0(\boldsymbol{\sigma}(0)) \int \Bigg[\prod_{t=0}^{t_{\max}-1} \overbrace{\frac{\mathrm{d}\mathbf{h}(t)\mathrm{d}\hat{\mathbf{h}}(t)}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\hat{\mathbf{h}}(t)\cdot[\mathbf{h}(t)-\boldsymbol{\theta}(t)]}}^{introduction\ of\ deltas} \overbrace{\mathrm{e}^{-\mathrm{i}\boldsymbol{\psi}(t)\cdot\boldsymbol{\sigma}(t)}}^{generating\ fields} \Bigg]$$

$$\times \underbrace{\Bigg[\prod_{t=0}^{t_{\max}-1} \frac{\mathrm{e}^{\beta\boldsymbol{\sigma}(t+1)\cdot\mathbf{h}(t)}}{\prod_i 2\cosh(\beta h_i(t))} \Bigg]}_{process\ dynamics} \underbrace{\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij} \sum_t \hat{h}_i(t)\sigma_j(t)}}}_{disorder\ average}$$

- Curie-Weiss: $J_{ij} = J/N$
  insert $\int \mathrm{d}m\, \delta[m - N^{-1}\sum_i \sigma_i]$

$$\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij} \sum_t \hat{h}_i(t)\sigma_j(t)}} = \mathrm{e}^{-\mathrm{i}J\sum_t \left(\sum_i \hat{h}_i(t)\right)\left(\frac{1}{N}\sum_j \sigma_j(t)\right)}$$

$$= \int \Bigg[\prod_t \frac{\mathrm{d}m(t)\mathrm{d}\hat{m}(t)}{2\pi/N} \mathrm{e}^{\mathrm{i}N\hat{m}(t)m(t) - \mathrm{i}Jm(t)\sum_i \hat{h}_i(t) - \mathrm{i}\hat{m}(t)\sum_i \sigma_i(t)} \Bigg]$$

if $p_0(\boldsymbol{\sigma}) = \prod_i p_i(\sigma_i)$: factorization over sites i!

Illustration

- exploit factorization

$$\overline{Z[\psi]} = \int \left[\prod_{t=0}^{t_{\max}-1} \frac{\mathrm{d}m(t)\mathrm{d}\hat{m}(t)}{2\pi/N} \mathrm{e}^{\mathrm{i}N\hat{m}(t)m(t)} \right] \prod_{i=1}^{N} \Bigg\{ \sum_{\sigma(0),\ldots,\sigma(t_{\max})}$$

$$\times \left[\prod_{t=0}^{t_{\max}-1} \int \frac{\mathrm{d}h(t)\mathrm{d}\hat{h}(t)}{2\pi} \mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]-\mathrm{i}[\hat{m}(t)+\psi_i(t)]\sigma(t)} \right] \left[p_i(\sigma(0)) \prod_{t=0}^{t_{\max}-1} \frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]} \right] \Bigg\}$$

- steepest descent form

$$\overline{Z[\psi]} = C_N \int \left[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t) \right] \mathrm{e}^{N\Omega[\{m,\hat{m}\};\psi]}$$

$$\Omega[\{m,\hat{m}\};\psi] = \mathrm{i} \sum_t \hat{m}(t)m(t) + \frac{1}{N} \sum_i \log \Bigg\{ \sum_{\sigma(0),\ldots,\sigma(t_{\max})}$$

$$\times \left[\prod_{t=0}^{t_{\max}-1} \int \frac{\mathrm{d}h(t)\mathrm{d}\hat{h}(t)}{2\pi} \mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]-\mathrm{i}[\hat{m}(t)+\psi_i(t)]\sigma(t)} \right] \left[p_i(\sigma(0)) \prod_{t=0}^{t_{\max}-1} \frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]} \right] \Bigg\}$$

- $\overline{Z[\psi]}$ as generator, for large N, (use $\overline{Z[\mathbf{0}]}=1$)
  $\{m^\star, \hat{m}^\star\}$: extremum of $\Omega[\ldots]$ (paths!)

$$\frac{\partial\overline{Z[\psi]}}{\partial\psi_i(t)}\Big|_{\psi=\mathbf{0}} = \lim_{\psi\to\mathbf{0}} C_N\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}\Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)$$

$$= \lim_{\psi\to\mathbf{0}} \frac{\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}\Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)}{\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}} \to \lim_{N\to\infty}\frac{N\partial\Omega}{\partial\psi_i(t)}\Big|_{\{m^\star,\hat{m}^\star\},\psi=\mathbf{0}}$$

$$\frac{\partial^2\overline{Z[\psi]}}{\partial\psi_i(t)\partial\psi_j(t')}\Big|_{\psi=\mathbf{0}}$$

$$= \lim_{\psi\to\mathbf{0}} C_N\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}\Big[\frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\psi_j(t')}+\Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)\Big(\frac{N\partial\Omega}{\partial\psi_j(t')}\Big)\Big]$$

$$= \lim_{\psi\to\mathbf{0}} \frac{\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}\Big[\frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\psi_j(t')}+\Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)\Big(\frac{N\partial\Omega}{\partial\psi_j(t')}\Big)\Big]}{\int\Big[\prod_t \mathrm{d}m(t)\mathrm{d}\hat{m}(t)\Big]\mathrm{e}^{N\Omega[\ldots]}}$$

$$\to \lim_{N\to\infty}\Big\{\frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\psi_j(t')}+\Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)\Big(\frac{N\partial\Omega}{\partial\psi_j(t')}\Big)\Big\}\Big|_{\{m^\star,\hat{m}^\star\},\psi=\mathbf{0}}$$

- further useful identities, using $\overline{Z[\mathbf{0}]}=1$
  $\{m^\star, \hat{m}^\star\}$: extremum of $\Omega[\ldots]$ (paths!)

$$
\begin{aligned}
0 &= \lim_{N\to\infty} \frac{\partial \overline{Z[\psi]}}{\partial\theta_i(t)}\Big|_{\psi=\mathbf{0}} = \lim_{N\to\infty} \frac{N\partial\Omega}{\partial\theta_i(t)}\Big|_{\{m^\star,\hat{m}^\star\},\psi=\mathbf{0}} \\
0 &= \lim_{N\to\infty} \frac{\partial^2 \overline{Z[\psi]}}{\partial\theta_i(t)\partial\theta_j(t')}\Big|_{\psi=\mathbf{0}} \\
&= \lim_{N\to\infty} \left\{ \frac{N\partial^2\Omega}{\partial\theta_i(t)\partial\theta_j(t')} + \left(\frac{N\partial\Omega}{\partial\theta_i(t)}\right)\left(\frac{N\partial\Omega}{\partial\theta_j(t')}\right) \right\}\Big|_{\{m^\star,\hat{m}^\star\},\psi=\mathbf{0}}
\end{aligned}
$$

notation

$$\{\sigma\} = (\sigma(0), \ldots, \sigma(t_{\max}))$$
$$\{h\} = (h(0), \ldots, h(t_{\max}-1))$$

$$\mathcal{P}_i[\{\sigma\}|\{h\}] = p_i(\sigma(0)) \prod_{t=0}^{t_{\max}-1} \frac{e^{\beta\sigma(t+1)h(t)}}{2\cosh[\beta h(t)]}$$

left to do

(i) compute derivatives of Ω w.r.t. $\{\psi_i(t)\}$
(ii) compute paths $\{m^\star, \hat{m}^\star\}$ by extremization of Ω

$$\Omega = \mathrm{i}\sum_t \hat{m}(t)m(t) + \frac{1}{N}\sum_i \log\left\{\sum_{\{\sigma\}}\int \frac{\{\mathrm{d}h\mathrm{d}\hat{h}\}}{(2\pi)^{t_{\max}}}\mathcal{P}_i[\{\sigma\}|\{h\}]\, e^{\mathrm{i}\sum_t\left[\hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]-[\hat{m}(t)+\psi_i(t)]\sigma(t)\right]}\right\}$$

more notation ...

$$\Big\langle f(\ldots)\Big\rangle_i = \frac{\sum_{\{\sigma\}}\int\frac{\{\mathrm{d}h\mathrm{d}\hat{h}\}}{(2\pi)^{t_{\max}}}\mathcal{P}_i[\{\sigma\}|\{h\}]\, e^{\mathrm{i}\sum_t\left[\hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]-[\hat{m}(t)+\psi_i(t)]\sigma(t)\right]} f(\ldots)}{\sum_{\{\sigma\}}\int\frac{\{\mathrm{d}h\mathrm{d}\hat{h}\}}{(2\pi)^{t_{\max}}}\mathcal{P}_i[\{\sigma\}|\{h\}]\, e^{\mathrm{i}\sum_t\left[\hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]-[\hat{m}(t)+\psi_i(t)]\sigma(t)\right]}}$$

- derivatives wrt $\{\psi_i(t), \theta_i(t)\}$

$$\frac{N\partial\Omega}{\partial\psi_i(t)} = -\mathrm{i}\langle\sigma(t)\rangle_i, \qquad \frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\psi_j(t')} = -\delta_{ij}\Big[\langle\sigma(t)\sigma(t')\rangle_i - \langle\sigma(t)\rangle_i\langle\sigma(t')\rangle_i\Big]$$

$$\frac{N\partial\Omega}{\partial\theta_i(t)} = -\mathrm{i}\langle\hat{h}(t)\rangle_i, \qquad \frac{N\partial^2\Omega}{\partial\theta_i(t)\partial\theta_j(t')} = -\delta_{ij}\Big[\langle\hat{h}(t)\hat{h}(t')\rangle_i - \langle\hat{h}(t)\rangle_i\langle\hat{h}(t')\rangle_i\Big]$$

$$\frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\theta_j(t')} = -\delta_{ij}\Big[\langle\sigma(t)\hat{h}(t')\rangle_i - \langle\sigma(t)\rangle_i\langle\hat{h}(t')\rangle_i\Big]$$

for $N\to\infty$:

$$\begin{aligned}
\overline{\langle\sigma_i(t)\rangle} &= \mathrm{i}\frac{N\partial\Omega}{\partial\psi_i(t)}\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}} = \langle\sigma(t)\rangle_i \\
\overline{\langle\sigma_i(t)\sigma_j(t')\rangle} &= -\Big\{\frac{N\partial^2\Omega}{\partial\psi_i(t)\partial\psi_j(t')} + \Big(\frac{N\partial\Omega}{\partial\psi_i(t)}\Big)\Big(\frac{N\partial\Omega}{\partial\psi_j(t')}\Big)\Big\}\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}} \\
&= \Big\{\delta_{ij}\langle\sigma(t)\sigma(t')\rangle_i + (1-\delta_{ij})\langle\sigma(t)\rangle_i\langle\sigma(t')\rangle_j\Big\}\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}} \\
0 &= \frac{N\partial\Omega}{\partial\theta_i(t)}\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}} = -\mathrm{i}\langle\hat{h}(t)\rangle_i\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}}
\end{aligned}$$

$$0 = \Big\{\frac{N\partial^2\Omega}{\partial\theta_i(t)\partial\theta_j(t')} + \Big(\frac{N\partial\Omega}{\partial\theta_i(t)}\Big)\Big(\frac{N\partial\Omega}{\partial\theta_j(t')}\Big)\Big\}\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}} = -\delta_{ij}\langle\hat{h}(t)\hat{h}(t')\rangle_i\Big|_{\{m^\star,\hat{m}^\star\},\boldsymbol{\psi}=\mathbf{0}}$$

- extremize Ω over $\{m, \hat{m}\}$

$$\frac{\partial\Omega}{\partial m(t)} = 0: \qquad \mathrm{i}\hat{m}(t) - \frac{\mathrm{i}J}{N}\sum_i \langle\hat{h}(t)\rangle_i = 0 \quad \Rightarrow \quad \hat{m}(t) = \frac{J}{N}\sum_i \langle\hat{h}(t)\rangle_i = 0$$

$$\frac{\partial\Omega}{\partial \hat{m}(t)} = 0: \qquad \mathrm{i}m(t) - \frac{\mathrm{i}}{N}\sum_i \langle\sigma(t)\rangle_i = 0 \quad \Rightarrow \quad m(t) = \frac{1}{N}\sum_i \langle\sigma(t)\rangle_i$$

- summarize

$$\begin{aligned}
\lim_{N\to\infty} \overline{\langle\sigma_i(t)\rangle} &= \langle\sigma(t)\rangle_i \\
\lim_{N\to\infty} \overline{C_{ij}}(t,t') &= \delta_{ij}\langle\sigma(t)\sigma(t')\rangle_i + (1-\delta_{ij})\langle\sigma(t)\rangle_i\langle\sigma(t')\rangle_j \\
\lim_{N\to\infty} \overline{G_{ij}}(t,t') &= -\mathrm{i}\delta_{ij}\langle\sigma(t)\hat{h}(t')\rangle_i
\end{aligned}$$

macroscopic observables

$$\begin{aligned}
m(t) &= \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma_i(t)\rangle} = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle\sigma(t)\rangle_i \\
C(t,t') &= \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma_i(t)\sigma_i(t')\rangle} = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle\sigma(t)\sigma(t')\rangle_i \\
G(t,t') &= \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t')}\overline{\langle\sigma_i(t)\rangle} = -\mathrm{i}\lim_{N\to\infty}\frac{1}{N}\sum_i \langle\sigma(t)\hat{h}(t')\rangle_i
\end{aligned}$$

- put it all together, send $\psi \to \mathbf{0}$

$$\left\langle f(\{\sigma\})\right\rangle_i = \frac{\sum_{\{\sigma\}} \int \frac{\{\mathrm{d}h\mathrm{d}\hat{h}\}}{(2\pi)^{t_{\max}}} \mathcal{P}_i[\{\sigma\}|\{h\}]\; \mathrm{e}^{\mathrm{i}\sum_t \hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]} f(\{\sigma\})}{\sum_{\{\sigma\}} \int \frac{\{\mathrm{d}h\mathrm{d}\hat{h}\}}{(2\pi)^{t_{\max}}} \mathcal{P}_i[\{\sigma\}|\{h\}]\; \mathrm{e}^{\mathrm{i}\sum_t \hat{h}(t)[h(t)-Jm(t)-\theta_i(t)]}}$$

$$= \sum_{\{\sigma\}} f(\{\sigma\})\mathcal{P}_i[\{\sigma\}|\{Jm+\theta_i\}]$$

$$= \sum_{\{\sigma\}} f(\{\sigma\})p_i(\sigma(0)) \prod_{t'=0}^{t_{\max}-1} \frac{\mathrm{e}^{\beta\sigma(t'+1)[Jm(t')+\theta_i(t')]}}{2\cosh[\beta(Jm(t')+\theta_i(t'))]}$$

looking ahead: what changes in presence of interaction disorder?
e.g. symmetric Gaussian $J_{ij} = (1-\delta_{ij})Jz_{ij}/\sqrt{N}$, insert

$$\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij} \sum_t \hat{h}_i(t)\sigma_j(t)}} = \prod_{i<j} \int \frac{\mathrm{d}z}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}z^2 - \frac{\mathrm{i}Jz}{\sqrt{N}} \sum_t [\hat{h}_i(t)\sigma_j(t)+\hat{h}_j(t)\sigma_i(t)]}$$

$$= \mathrm{e}^{-\frac{1}{2}\frac{J^2}{N}\sum_{t,t'}\sum_{i\neq j}[\hat{h}_i(t)\sigma_j(t)+\hat{h}_j(t)\sigma_i(t)][\hat{h}_i(t')\sigma_j(t')+\hat{h}_j(t')\sigma_i(t')]}$$

expressions with: $\frac{1}{N}\sum_i \hat{h}_i(t)\hat{h}_i(t')$, $\frac{1}{N}\sum_i \sigma_i(t)\sigma_i(t')$, $\frac{1}{N}\sum_i \sigma_i(t)\hat{h}_i(t')$

Continuous time

Disorder-averaged generating functional

$$\overline{Z[\psi]} = \overline{\langle e^{-i\sum_i \int dt\ \psi_i(t)\sigma_i(t)}\rangle} = \sum_{\{\boldsymbol{\sigma}\}} \overline{\mathcal{P}[\{\boldsymbol{\sigma}\}]} e^{-i\sum_i \int dt\ \psi_i(t)\sigma_i(t)}, \qquad \{\boldsymbol{\sigma}\} = \{\sigma_1(t), \ldots, \sigma_N(t)\}$$

N paths, with $t \geq 0$

- discretize: $t = \ell\Delta$, $\Delta\downarrow 0$, insert for all i and t:

$$1 = \int dh_i(t)\ \delta\left[h_i(t) - h_i(t, \boldsymbol{\sigma}(t))\right]$$

$$\overline{\mathcal{P}[\{\boldsymbol{\sigma}\}]} = \lim_{\Delta\downarrow 0} \overline{\left[\prod_{it} \int \frac{dh_i(t) d\hat{h}_i(t)}{2\pi/\Delta} e^{i\Delta \hat{h}_i(t)[h_i(t) - h_i(t,\boldsymbol{\sigma}(t))]}\right] \mathcal{P}[\{\boldsymbol{\sigma}\}|\{\mathbf{h}\}]}$$

$$= \overbrace{\int \{d\mathbf{h} d\hat{\mathbf{h}}\}}^{\text{path integrals}} e^{i\sum_i \int dt\ \hat{h}_i(t)[h_i(t) - \theta_i(t)]}\ \overline{e^{-i\sum_{ij} J_{ij} \int dt\ \hat{h}_i(t)\sigma_j(t)}}\ \mathcal{P}[\{\boldsymbol{\sigma}\}|\{\mathbf{h}\}]$$

with

$$\mathcal{P}[\{\boldsymbol{\sigma}\}|\{\mathbf{h}\}]: \quad \text{soln of} \quad \frac{d}{dt} p_t(\boldsymbol{\sigma}) = \sum_i \frac{p_t(F_i\boldsymbol{\sigma}) e^{\beta\sigma_i h_i(t)} - p_t(\boldsymbol{\sigma}) e^{-\beta\sigma_i h_i(t)}}{\cosh(\beta h_i(t))}$$

- if factorising initial conditions

$$p_0(\boldsymbol{\sigma}) = \prod_i \left[\frac{1}{2}[1+m_i(0)]\delta_{\sigma_i,1} + \frac{1}{2}[1-m_i(0)]\delta_{\sigma_i,-1}\right]$$

then in $\mathcal{P}[\{\boldsymbol{\sigma}\}|\{\mathbf{h}\}]$:

$$\forall t \geq 0 : \quad p_t(\boldsymbol{\sigma}) = \prod_i \left[\frac{1}{2}[1+m_i(t)]\delta_{\sigma_i,1} + \frac{1}{2}[1-m_i(t)]\delta_{\sigma_i,-1}\right]$$

with

$$\forall (i,t) : \quad \frac{\mathrm{d}}{\mathrm{d}t} m_i(t) = \tanh[\beta h_i(t)] - m_i(t)$$

$$\forall (i,t) : \quad m_i(t) = m_i(0)\mathrm{e}^{-t} + \int_0^t \mathrm{d}t'\ \mathrm{e}^{-(t-t')} \tanh[\beta h_i(t')]$$

(see exercises)