Modelling of Complex Real-World Systems
Part C. Applications in Biology

D2. Dynamics of Recurrent Neural Networks

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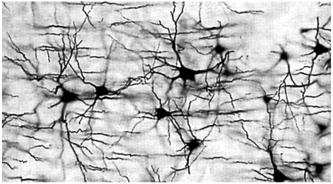
Attractor neural networks

$N \sim 10^{12-14}$ brain cells (neurons),
each connected to $\sim 10^{3-5}$ others



- simplest two-state model neurons

  $\sigma_i = 1$ (i fires electric pulses)
  $\sigma_i = -1$ (i is at rest)

- dynamics

$$\sigma_i(t+1) = sgn\Big[\overbrace{\sum_{j=1}^{N} J_{ij}\sigma_j(t)}^{activation\ signal} + \overbrace{\theta_i + z_i(t)}^{threshold,\ noise} \Big]$$

$\theta_i \in \mathbb{R}$: *firing threshold of neuron i*
$J_{ij} \in \mathbb{R}$: *synaptic connection $j \to i$*

non-local 'distributed' storage of 'program' and 'data'

- learning = adaptation of $\{J_{ij}, \theta_i\}$

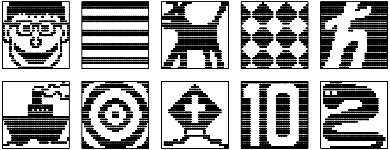
Attractor neural networks

models for *associative memory* in the brain

- represent ‘information patterns’
  as micro-states $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$

$\circ$: $\sigma_i = -1$, $\bullet$: $\sigma_i = 1$

e.g. $N = 400$,
10 patterns:



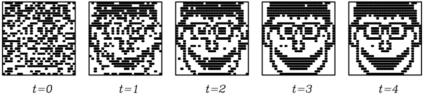
- information storage

  modify synapses $\{J_{ij}, \theta_i\}$ such that $\boldsymbol{\xi}$ is
  stable state (attractor) of the neuronal dynamics

- information recall

  from initial state $\boldsymbol{\sigma}(t=0)$:
  evolution to nearest attractor

  if $\boldsymbol{\sigma}(0)$ close (i.e. similar) to $\boldsymbol{\xi}$:
  $\boldsymbol{\sigma}(t=\infty) = \boldsymbol{\xi}$



Learning rule

recipe for creating attractors
via adaptation of $\{J_{ij}, \theta_i\}$

- create *fixed-point attractor*
  (Hebb, 1949): $\Delta J_{ij} \propto \xi_i \xi_j$

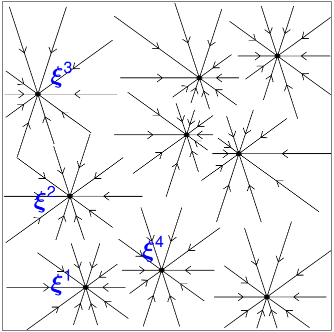
  choose $J_{ij} = \frac{1}{N}\xi_i\xi_j,\ \ \theta_i = 0$:

$$\sigma_i(t+1) = sgn\Big[\sum_{j=1}^N J_{ij}\sigma_j(t) + z_i(t)\Big] = sgn\Big[\xi_i\Big(\overbrace{\frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t)}^{\textit{pattern overlap}}\Big) + z_i(t)\Big]$$

$$= \xi_i\ sgn\Big[\frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t) + \xi_i z_i(t)\Big]$$

if $m(t) = \frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t)$ sufficiently large: $\sigma_i(t+1) = \xi_i$
now: $m(t+1) \geq m(t),\ \ m(t+2) \geq m(t+1)$, ...

continues until $\boldsymbol{\sigma}(\infty) = \boldsymbol{\xi}$ (i.e. $m(\infty) = 1$)



- create p fixed-point attractors $\boldsymbol{\xi}^\mu$: $J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$
  - many alternative flavours, to deal with pattern correlations etc
  - storage capacity for random binary patterns: $p/N < \alpha_c \approx 0.138$ (proved using replica method)
  - works for sequential and parallel dynamics

- more general programming rule:

  if in state $\boldsymbol{\xi}$, go to state $\boldsymbol{\xi}'$: $\Delta J_{ij} \propto \xi_i' \xi_j$

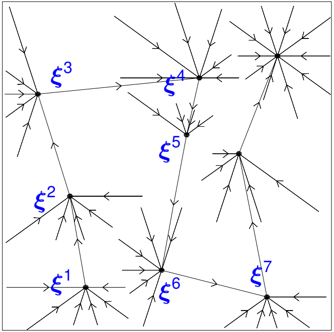
choose $J_{ij} = \frac{1}{N}\xi_i'\xi_j$, $\theta_i = 0$,

define $m(t) = \frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t)$, $m'(t) = \frac{1}{N}\sum_{j=1}^N \xi_j'\sigma_j(t)$

$$\begin{aligned}\sigma_i(t+1) &= sgn\Big[\sum_{j=1}^N J_{ij}\sigma_j(t) + z_i(t)\Big] = sgn\Big[\xi_i'\Big(\overbrace{\frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t)}^{pattern\ overlap}\Big) + z_i(t)\Big] \\ &= \xi_i'\ sgn\Big[\frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t) + \xi_i' z_i(t)\Big]\end{aligned}$$

if $m(t) = \frac{1}{N}\sum_{j=1}^N \xi_j\sigma_j(t)$ sufficiently large: $\sigma_i(t+1) = \xi_i'$

now $m(t+1) \leq m(t)$, $m'(t+1) \geq m'(t)$ ...



- create a *dynamical attractor*, in the form of a *sequence of patterns* $\boldsymbol{\xi}^1 \to \boldsymbol{\xi}^2 \to \ldots \to \boldsymbol{\xi}^p$: $J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^{\mu+1}\xi_j^{\mu}$

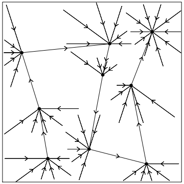
  – many alternative flavours, to deal with pattern correlations etc

  – storage capacity for random patterns: $p/N < \alpha_c' \approx 0.27$ (simulations)

  – works for parallel dynamics only ...

Sequence attractors

Definitions



- neurons $\sigma_i = \pm 1, \ i = 1 \ldots N$
  synapses

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^{\mu+1} \xi_j^{\mu}, \qquad \xi_i^{\mu} \in \{-1, 1\} \ \text{(random)}$$

- parallel dynamics

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W[\boldsymbol{\sigma}, \boldsymbol{\sigma}'] p_t(\boldsymbol{\sigma}'), \qquad W[\boldsymbol{\sigma}, \boldsymbol{\sigma}'] = \prod_i \frac{e^{\beta \sigma_i h_i(\boldsymbol{\sigma}')}}{2 \cosh[\beta h_i(\boldsymbol{\sigma}')]}$$

$$h_i(\boldsymbol{\sigma}) = \sum_j J_{ij} \sigma_j, \qquad h_i(\boldsymbol{\sigma}) = \sum_{\mu=1}^{p} \xi_i^{\mu+1} m_\mu(\boldsymbol{\sigma}), \qquad m_\mu(\boldsymbol{\sigma}) = \frac{1}{N} \sum_i \xi_i^{\mu} \sigma_i$$

- macroscopic probabilities

$$P_t(\mathbf{m}) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \delta[\mathbf{m} - \mathbf{m}(\boldsymbol{\sigma})], \qquad \mathbf{m} = (m_1, \ldots, m_p)$$

Dynamical solution for finite p

- macroscopic dynamics

$$
\begin{aligned}
P_{t+1}(\mathbf{m}) &= \sum_{\boldsymbol{\sigma}} p_{t+1}(\boldsymbol{\sigma})\delta[\mathbf{m}-\mathbf{m}(\boldsymbol{\sigma})] \\
&= \sum_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} \delta[\mathbf{m}-\mathbf{m}(\boldsymbol{\sigma})]\,\frac{\mathrm{e}^{\beta\sum_i \sigma_i \sum_\mu \xi_i^{\mu+1} m_\mu(\boldsymbol{\sigma}')}}{\prod_i 2\cosh[\beta\sum_\mu \xi_i^{\mu+1} m_\mu(\boldsymbol{\sigma}')]}\,p_t(\boldsymbol{\sigma}') \\
&= D(\mathbf{m})\sum_{\boldsymbol{\sigma}'} \frac{\mathrm{e}^{N\beta\sum_\mu m_{\mu+1} m_\mu(\boldsymbol{\sigma}')}}{\prod_i 2\cosh[\beta\sum_\mu \xi_i^{\mu+1} m_\mu(\boldsymbol{\sigma}')]}\,p_t(\boldsymbol{\sigma}')
\end{aligned}
$$

$$
\text{with}\quad D(\mathbf{m}) = \sum_{\boldsymbol{\sigma}} \delta[\mathbf{m}-\mathbf{m}(\boldsymbol{\sigma})]
$$

- insert $1 = \int \mathrm{d}\mathbf{m}'\ \delta[\mathbf{m}'-\mathbf{m}(\boldsymbol{\sigma}')]$

$$
\begin{aligned}
P_{t+1}(\mathbf{m}) &= D(\mathbf{m})\int \mathrm{d}\mathbf{m}'\,\frac{\mathrm{e}^{N\beta\sum_\mu m_{\mu+1} m'_\mu}}{\prod_i 2\cosh[\beta\sum_\mu \xi_i^{\mu+1} m'_\mu]}\sum_{\boldsymbol{\sigma}'} p_t(\boldsymbol{\sigma}')\delta[\mathbf{m}'-\mathbf{m}(\boldsymbol{\sigma}')] \\
&= D(\mathbf{m})\int \mathrm{d}\mathbf{m}'\,\frac{\mathrm{e}^{N\beta\sum_\mu m_{\mu+1} m'_\mu}}{\prod_i 2\cosh[\beta\sum_\mu \xi_i^{\mu+1} m'_\mu]}\,P_t(\mathbf{m}')
\end{aligned}
$$

$$P_{t+1}(\mathbf{m}) = \int \mathrm{d}\mathbf{m}'\ \mathrm{e}^{N\Phi(\mathbf{m},\mathbf{m}')} P_t(\mathbf{m}')$$

$$\Phi(\mathbf{m},\mathbf{m}') = \frac{1}{N}\log D(\mathbf{m}) + \beta\sum_\mu m_{\mu+1}m'_\mu - \frac{1}{N}\sum_i \log[2\cosh(\beta\sum_\mu \xi_i^{\mu+1}m'_\mu)]$$

- $N\to\infty$: steepest descent integration

$$P_{t+1}(\mathbf{m}) = P_t(\mathbf{m}^\star(\mathbf{m})), \qquad \mathbf{m}^\star(\mathbf{m}) = \mathrm{argmax}_{\mathbf{m}'} \lim_{N\to\infty}\Phi(\mathbf{m},\mathbf{m}')$$

$$\text{if } P_t(\mathbf{m}) = \delta[\mathbf{m}-\mathbf{m}(t)] \quad\Rightarrow\quad P_{t+1}(\mathbf{m}) = \delta[\mathbf{m}-\mathbf{m}(t+1)]$$

$\mathbf{m}(\boldsymbol{\sigma})$ evolves *deterministically*

- link between $\mathbf{m}(t)$ and $\mathbf{m}(t+1)$

$$\mathbf{m}(t) = \mathrm{argmax}_{\mathbf{x}}\Big[\beta\sum_\mu m_{\mu+1}(t+1)x_\mu - \langle\log[2\cosh(\beta\sum_\mu \xi^{\mu+1}x_\mu)]\rangle_{\boldsymbol{\xi}}\Big]$$

$$\text{differentiate wrt } x_\mu: \quad m_{\mu+1}(t+1) = \langle\xi^{\mu+1}\tanh(\beta\sum_\nu \xi^{\nu+1}x_\nu)\rangle_{\boldsymbol{\xi}}$$

result

$$m_\mu(t+1) = \langle\xi^\mu\tanh(\beta\sum_\nu \xi^{\nu+1}m_\nu(t))\rangle_{\boldsymbol{\xi}} \qquad \text{(see exercises)}$$

Generalisations

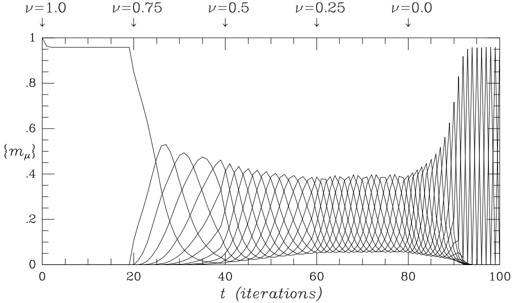
$$J_{ij} = \frac{1}{N}\sum_{\mu\rho=1}^{p} \xi_i^\mu A_{\mu\rho}\xi_j^\rho :$$

$$m_\mu(t+1) = \langle \xi^\mu \tanh(\beta \sum_{\lambda\rho} \xi^\lambda A_{\lambda\rho} m_\rho(t))\rangle_{\boldsymbol{\xi}}$$

- $A_{\lambda\rho} = \nu\delta_{\lambda\rho} + (1-\nu)\delta_{\lambda,\rho+1}$: combine fixed-point and sequence storage

$$J_{ij} = \frac{\nu}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu + \frac{1-\nu}{N}\sum_{\mu=1}^{p} \xi_i^{\mu+1}\xi_j^\mu$$

$$m_\mu(t+1) = \langle \xi^\mu \tanh[\beta \sum_{\rho=1}^{p}(\nu\xi^\rho + (1-\nu)\xi^{\rho+1}) m_\rho(t)]\rangle_{\boldsymbol{\xi}}$$



Simulations

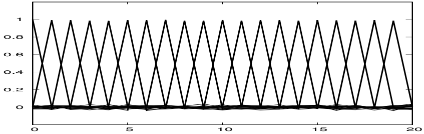
$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p-1}\xi_i^{\mu+1}\xi_j^{\mu}$,
random binary patterns $\{\boldsymbol{\xi}^{\mu}\}$

$N=4000$, $T=0$,
$\alpha=p/N$

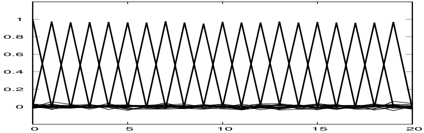
$m_{\mu}=\frac{1}{N}\sum_i \xi_i^{\mu}\sigma_i$
plotted versus time, for $\mu=1\ldots 20$

transition at
$\alpha_c \approx 0.27$?
requires GFA ...

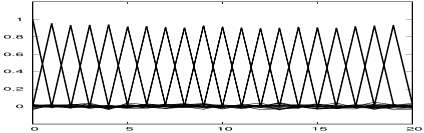
$\alpha=0.15$:



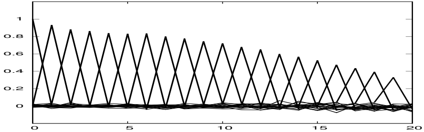
$\alpha=0.20$:



$\alpha=0.25$:

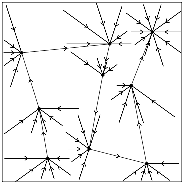


$\alpha=0.30$:



GFA of sequence processing network

Definitions



- neurons $\sigma_i = \pm 1,\ i = 1 \ldots N$
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$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^{\mu+1}\xi_j^{\mu}, \quad \xi_i^{\mu} \in \{-1, 1\} \ \text{(random)}$$

- parallel dynamics

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W_t[\boldsymbol{\sigma}, \boldsymbol{\sigma}'] p_t(\boldsymbol{\sigma}'), \qquad W_t[\boldsymbol{\sigma}, \boldsymbol{\sigma}'] = \prod_i \frac{\mathrm{e}^{\beta \sigma_i h_i(t, \boldsymbol{\sigma}')}}{2\cosh[\beta h_i(t, \boldsymbol{\sigma}')]}$$

$$h_i(t, \boldsymbol{\sigma}) = \sum_j J_{ij}\sigma_j + \theta_i(t),$$

- generating functional

$$\overline{Z[\psi]} = \sum_{\boldsymbol{\sigma}(0,\ldots,\boldsymbol{\sigma}(\tau)} \mathcal{P}[\boldsymbol{\sigma}(0), \ldots, \boldsymbol{\sigma}(\tau)]\, \overline{\mathrm{e}^{-\mathrm{i}\sum_i \sum_t \psi_i(t)\sigma_i(t)}}$$

Work out generating functional

(see earlier section, $\tau = \mathcal{O}(1)$)

$$\overline{Z[\psi]} = \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(\tau)} p_0(\boldsymbol{\sigma}(0)) \int \left[\prod_{t=0}^{\tau-1} \overbrace{\frac{\mathrm{d}\mathbf{h}(t)\mathrm{d}\hat{\mathbf{h}}(t)}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\hat{\mathbf{h}}(t)\cdot[\mathbf{h}(t)-\boldsymbol{\theta}(t)]}}^{\textit{introduction of deltas}} \;\; \overbrace{\mathrm{e}^{-\mathrm{i}\boldsymbol{\psi}(t)\cdot\boldsymbol{\sigma}(t)}}^{\textit{generating fields}} \right]$$

$$\times \underbrace{\left[\prod_{t=0}^{\tau-1} \frac{\mathrm{e}^{\beta\boldsymbol{\sigma}(t+1)\cdot\mathbf{h}(t)}}{\prod_i 2\cosh(\beta h_i(t))} \right]}_{\textit{process dynamics}} \underbrace{\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij} \sum_t \hat{h}_i(t)\sigma_j(t)}}}_{\textit{disorder average}}$$

- disorder average

$$\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij}\sum_t \hat{h}_i(t)\sigma_j(t)}} = \overline{\mathrm{e}^{-\frac{\mathrm{i}}{N}\sum_{ij}\sum_t \hat{h}_i(t)\sum_{\mu\le p}\xi_i^{\mu+1}\xi_j^{\mu}\sigma_j(t)}}$$

$$= \overline{\mathrm{e}^{-\mathrm{i}\sum_{\mu\le p}\sum_t \left[\frac{1}{\sqrt{N}}\sum_i \xi_i^{\mu+1}\hat{h}_i(t)\right]\left[\frac{1}{\sqrt{N}}\sum_j \xi_j^{\mu}\sigma_j(t)\right]}}$$

sequence recall solutions

$$m(t) = \frac{1}{N}\sum_i \xi_i^t \sigma_i(t) = \mathcal{O}(1), \qquad \forall\mu \neq t: \quad \frac{1}{N}\sum_i \xi_i^{\mu}\sigma_i(t) = \mathcal{O}(N^{-\frac{1}{2}})$$

$$k(t) = \frac{1}{N}\sum_i \xi_i^{t+1}\hat{h}_i(t) = \mathcal{O}(1), \qquad \forall\mu \neq t: \quad \frac{1}{N}\sum_i \xi_i^{\mu+1}\hat{h}_i(t) = \mathcal{O}(N^{-\frac{1}{2}})$$

non-condensed patterns

- large N:
  central limit theorem,

$$\mu \neq t: \qquad x_\mu(t) = \frac{1}{\sqrt{N}} \sum_i \xi_i^{\mu+1} \hat{h}_i(t), \qquad y_\mu(t) = \frac{1}{\sqrt{N}} \sum_i \xi_i^{\mu} \sigma_i(t):$$

$$\text{all zero average Gaussian RVs}$$

- covariances with $\mu, \nu \neq t$

$$\overline{x_\mu(t) x_\nu(t')} = \frac{1}{N} \sum_{ij} \hat{h}_i(t) \hat{h}_j(t') \overline{\xi_i^{\mu+1} \xi_j^{\nu+1}} = \delta_{\mu\nu} \Big(\frac{1}{N} \sum_i \hat{h}_i(t) \hat{h}_i(t') \Big)$$

$$\overline{y_\mu(t) y_\nu(t')} = \frac{1}{N} \sum_{ij} \sigma_i(t) \sigma_j(t') \overline{\xi_i^{\mu} \xi_j^{\nu}} = \delta_{\mu\nu} \Big(\frac{1}{N} \sum_i \sigma_i(t) \sigma_i(t') \Big)$$

$$\overline{x_\mu(t) y_\nu(t')} = \frac{1}{N} \sum_{ij} \hat{h}_i(t) \sigma_j(t') \overline{\xi_i^{\mu+1} \xi_j^{\nu}} = \delta_{\mu,\nu-1} \Big(\frac{1}{N} \sum_i \hat{h}_i(t) \sigma_i(t') \Big)$$

- correlations between $\{x_\mu(s), y_\mu(s)\}$ and $\{m(s), k(s)\}$:
  all of order $\mathcal{O}(N^{-\frac{1}{2}})$

- insert

$$1 = \int \frac{\mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{m}}}{(2\pi/N)^{\tau}} \mathrm{e}^{\mathrm{i}N\sum_{t<\tau}\hat{m}(t)[m(t)-\frac{1}{N}\sum_i \xi_i^t \sigma_i(t)]}$$

$$1 = \int \frac{\mathrm{d}\mathbf{k}\mathrm{d}\hat{\mathbf{k}}}{(2\pi/N)^{\tau}} \mathrm{e}^{\mathrm{i}N\sum_{t<\tau}\hat{k}(t)[k(t)-\frac{1}{N}\sum_i \xi_i^{t+1} \hat{h}_i(t)]}$$

$$1 = \int \frac{\mathrm{d}\mathbf{C}\mathrm{d}\hat{\mathbf{C}}}{(2\pi/N)^{\tau^2}} \mathrm{e}^{\mathrm{i}N\sum_{t'<\tau}\hat{C}(t,t')[C(t,t')-\frac{1}{N}\sum_i \sigma_i(t)\sigma_i(t')]}$$

$$1 = \int \frac{\mathrm{d}\mathbf{Q}\mathrm{d}\hat{\mathbf{Q}}}{(2\pi/N)^{\tau^2}} \mathrm{e}^{\mathrm{i}N\sum_{t'<\tau}\hat{Q}(t,t')[Q(t,t')-\frac{1}{N}\sum_i \hat{h}_i(t)\hat{h}_i(t')]}$$

$$1 = \int \frac{\mathrm{d}\mathbf{K}\mathrm{d}\hat{\mathbf{K}}}{(2\pi/N)^{\tau^2}} \mathrm{e}^{\mathrm{i}N\sum_{t'<\tau}\hat{K}(t,t')[K(t,t')-\frac{1}{N}\sum_i \sigma_i(t)\hat{h}_i(t')]}$$

- disorder dependent factor

$$\overline{\mathrm{e}^{-\mathrm{i}\sum_{ij} J_{ij}\sum_t \hat{h}_i(t)\sigma_j(t)}} = \mathrm{e}^{-\mathrm{i}N\sum_{t<\tau} k(t)m(t)+\mathcal{O}(1)}\, \overline{\mathrm{e}^{-\mathrm{i}\sum_{\mu\geq\tau}\sum_{t<\tau}\sum_t x_\mu(t)y_\mu(t)}}$$

so: average only over patterns that are non-condensed,
i.e. those in $\{x_\mu(t), y_\mu(t)\}$, not those in $\{m(t), k(t)\}$

Generating functional in saddle point form

choose $p_0(\boldsymbol{\sigma}(0))=\prod_i p_{i0}(\sigma_i(0))$,
$\mathcal{P}(\mathbf{x},\mathbf{y}|\mathbf{C},\mathbf{Q},\mathbf{K})$: Gaussian distr of $\{x_\mu(t),y_\mu(t)\}$,
$p=\alpha N$

$$
\begin{aligned}
\overline{Z[\psi]} = C^N \int & \mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{k}\mathrm{d}\hat{\mathbf{k}}\mathrm{d}\mathbf{C}\mathrm{d}\hat{\mathbf{C}}\mathrm{d}\mathbf{Q}\mathrm{d}\hat{\mathbf{Q}}\mathrm{d}\mathbf{K}\mathrm{d}\hat{\mathbf{K}} \\
& \times \mathrm{e}^{\mathrm{i}N\sum_{t<\tau}[\hat{m}(t)m(t)+\hat{k}(t)k(t)-k(t)m(t)]+\mathrm{i}N\sum_{tt'<\tau}[\hat{C}(t,t')C(t,t')+\hat{Q}(t,t')Q(t,t')+\hat{K}(t,t')K(t,t')]} \\
& \times \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ \mathcal{P}(\mathbf{x},\mathbf{y}|\mathbf{C},\mathbf{Q},\mathbf{K})\ \mathrm{e}^{-\mathrm{i}\sum_{\mu=\tau}^{\alpha N}\sum_{t<\tau}\sum_t x_\mu(t)y_\mu(t)+\mathcal{O}(\sqrt{N})} \\
& \times \prod_i \Bigg\{ \sum_{\sigma(0),\ldots,\sigma(\tau)} p_{i0}(\sigma(0))\mathrm{e}^{-\mathrm{i}[\psi_i(t)+\hat{m}(t)\xi_i^t]\sigma(t)-\mathrm{i}\sum_{tt'<\tau}\hat{C}(t,t')\sigma(t)\sigma(t')} \\
\times \int \prod_{t=0}^{\tau-1} & \frac{\mathrm{d}h(t)\mathrm{d}\hat{h}(t)}{(2\pi)^N}\frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh(\beta h(t))}\mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-\theta_i(t)-\hat{k}(t)\xi_i^{t+1}]-\mathrm{i}\sum_{tt'<\tau}\left[\hat{Q}(t,t')\hat{h}(t)\hat{h}(t')+\hat{K}(t,t')\sigma(t)\hat{h}(t')\right]} \Bigg\}
\end{aligned}
$$

rewrite

$$\overline{Z[\psi]} = C^N \int \mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{k}\mathrm{d}\hat{\mathbf{k}}\mathrm{d}\mathbf{C}\mathrm{d}\hat{\mathbf{C}}\mathrm{d}\mathbf{Q}\mathrm{d}\hat{\mathbf{Q}}\mathrm{d}\mathbf{K}\mathrm{d}\hat{\mathbf{K}}\ \mathrm{e}^{N\left[\Psi[\ldots]+\Phi[\ldots]+\Omega[\ldots]\right]+\mathcal{O}(\sqrt{N})}$$

$$\begin{aligned}
\Psi[\ldots] &= \mathrm{i}\sum_{t<\tau}[\hat{m}(t)m(t) + \hat{k}(t)k(t) - k(t)m(t)] \\
&\quad +\mathrm{i}\sum_{tt'<\tau}[\hat{C}(t,t')C(t,t') + \hat{Q}(t,t')Q(t,t') + \hat{K}(t,t')K(t,t')] \\
\Phi[\ldots] &= \frac{1}{N}\sum_i \log\left\{\sum_{\sigma(0),\ldots,\sigma(\tau)} p_{i0}(\sigma(0))\mathrm{e}^{-\mathrm{i}[\psi_i(t)+\hat{m}(t)\xi_i^t]\sigma(t)-\mathrm{i}\sum_{tt'<\tau}\hat{C}(t,t')\sigma(t)\sigma(t')}\right. \\
&\left.\times\int\prod_{t=0}^{\tau-1}\frac{\mathrm{d}h(t)\mathrm{d}\hat{h}(t)}{(2\pi)^N}\frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh(\beta h(t))}\mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-\theta_i(t)-\hat{k}(t)\xi_i^{t+1}]-\mathrm{i}\sum_{tt'<\tau}\left[\hat{Q}(t,t')\hat{h}(t)\hat{h}(t')+\hat{K}(t,t')\sigma(t)\hat{h}(t')\right]}\right\} \\
\Omega[\ldots] &= \frac{1}{N}\log\int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ \mathcal{P}(\mathbf{x},\mathbf{y}|\mathbf{C},\mathbf{Q},\mathbf{K})\ \mathrm{e}^{-\mathrm{i}\sum_{\mu=\tau}^{\alpha N}\sum_{t<\tau}\sum_t x_\mu(t)y_\mu(t)}
\end{aligned}$$

Solving the saddle point equations

The saddle point equations

- variation of
  $\mathbf{m}, \mathbf{k}, \mathbf{C}, \mathbf{Q}, \mathbf{K}$

$$
\begin{aligned}
\frac{\partial \Psi}{\partial m(t)} &= 0: \qquad && \hat{m}(t) = k(t) \\
\frac{\partial \Psi}{\partial k(t)} &= 0: \qquad && \hat{k}(t) = m(t) \\
\frac{\partial \Psi}{\partial C(t,t')} + \frac{\partial \Omega}{\partial C(t,t')} &= 0: \qquad && \hat{C}(t,t') = \mathrm{i}\frac{\partial \Omega}{\partial C(t,t')} \\
\frac{\partial \Psi}{\partial Q(t,t')} + \frac{\partial \Omega}{\partial Q(t,t')} &= 0: \qquad && \hat{Q}(t,t') = \mathrm{i}\frac{\partial \Omega}{\partial Q(t,t')} \\
\frac{\partial \Psi}{\partial K(t,t')} + \frac{\partial \Omega}{\partial K(t,t')} &= 0: \qquad && \hat{K}(t,t') = \mathrm{i}\frac{\partial \Omega}{\partial K(t,t')}
\end{aligned}
$$

- variation of $\hat{\mathbf{m}}, \hat{\mathbf{k}}, \hat{\mathbf{C}}, \hat{\mathbf{Q}}, \hat{\mathbf{K}}$,
  use $\langle\hat{h}(t)\rangle_i = \langle\hat{h}(t)\hat{h}(t')\rangle_i = 0,\ \langle\sigma(t)\hat{h}(t')\rangle_i = \mathrm{i}\overline{G}_{ii}(t,t')$

$$\frac{\partial\Psi}{\partial\hat{m}(t)} + \frac{\partial\Phi}{\partial\hat{m}(t)} = 0: \qquad m(t) = \lim_{N\to\infty}\frac{1}{N}\sum_i \xi_i^t\langle\sigma(t)\rangle_i$$

$$\frac{\partial\Psi}{\partial\hat{k}(t)} + \frac{\partial\Phi}{\partial\hat{k}(t)} = 0: \qquad k(t) = \lim_{N\to\infty}\frac{1}{N}\sum_i \xi_i^{t+1}\langle\hat{h}(t)\rangle_i = 0$$

$$\frac{\partial\Psi}{\partial\hat{C}(t,t')} + \frac{\partial\Phi}{\partial\hat{C}(t,t')} = 0: \qquad C(t,t') = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle\sigma(t)\sigma(t')\rangle_i$$

$$\frac{\partial\Psi}{\partial\hat{Q}(t,t')} + \frac{\partial\Phi}{\partial\hat{Q}(t,t')} = 0: \qquad Q(t,t') = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle\hat{h}(t)\hat{h}(t')\rangle_i = 0$$

$$\frac{\partial\Psi}{\partial\hat{K}(t,t')} + \frac{\partial\Phi}{\partial\hat{K}(t,t')} = 0: \qquad K(t,t') = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle\sigma(t)\hat{h}(t')\rangle_i = \mathrm{i}G(t,t')$$

after $\psi\to\mathbf{0}$, $\theta_i(t)\to\xi_i^{t+1}\theta(t)$

$$\langle g(\ldots)\rangle_i = \frac{\sum_{\{\sigma\}}\int\{\mathrm{d}h\mathrm{d}\hat{h}\}W_i[\{\sigma,h,\hat{h}\}]\mathrm{e}^{-\mathrm{i}\sum_{tt'<\tau}\left[\hat{C}(t,t')\sigma(t)\sigma(t')+\hat{Q}(t,t')\hat{h}(t)\hat{h}(t')+\hat{K}(t,t')\sigma(t)\hat{h}(t')\right]}g(\ldots)}{\sum_{\{\sigma\}}\int\{\mathrm{d}h\mathrm{d}\hat{h}\}W_i[\{\sigma,h,\hat{h}\}]\mathrm{e}^{-\mathrm{i}\sum_{tt'<\tau}\left[\hat{C}(t,t')\sigma(t)\sigma(t')+\hat{Q}(t,t')\hat{h}(t)\hat{h}(t')+\hat{K}(t,t')\sigma(t)\hat{h}(t')\right]}}$$

$$W_i[\{\sigma,h,\hat{h}\}] = p_{i0}(\sigma(0))\prod_{t<\tau}\left[\frac{1}{2\pi}\frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh(\beta h(t))}\mathrm{e}^{\mathrm{i}\hat{h}(t)[h(t)-(m(t)+\theta(t))\xi_i^{t+1}]}\right], \qquad \sum_{\{\sigma\}}\int\{\mathrm{d}h\mathrm{d}\hat{h}\}W_i[\{\sigma,h,\hat{h}\}] = 1$$

Reduced problem

$$m(t) = \lim_{N\to\infty} \frac{1}{N}\sum_i \xi_i^t \langle\sigma(t)\rangle_i, \qquad C(t,t') = \lim_{N\to\infty} \frac{1}{N}\sum_i \langle\sigma(t)\sigma(t')\rangle_i, \qquad K(t,t') = \mathrm{i}G(t,t')$$

$$\hat{C}(t,t') = \mathrm{i}\frac{\partial\Omega[\mathbf{C},\mathbf{0},\mathbf{K}]}{\partial C(t,t')}, \qquad \hat{Q}(t,t') = \mathrm{i}\frac{\partial\Omega[\mathbf{C},\mathbf{Q},\mathbf{K}]}{\partial Q(t,t')}\Big|_{\mathbf{Q}=\mathbf{0}}, \qquad \hat{K}(t,t') = \mathrm{i}\frac{\partial\Omega[\mathbf{C},\mathbf{0},\mathbf{K}]}{\partial K(t,t')}$$

$z_\mu(t) = x_{\mu-1}(t)$, $S_{\mu\nu} = \delta_{\mu,\nu-1}$:

$$\Omega[\mathbf{C},\mathbf{Q},\mathbf{K}] = \frac{1}{N}\log\int \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\; \mathcal{P}(\mathbf{y},\mathbf{z}|\mathbf{C},\mathbf{Q},\mathbf{K})\; \mathrm{e}^{-\mathrm{i}\sum_{\mu,\nu=\tau}^{\alpha N}\sum_{t<\tau}\sum_t y_\mu(t)S_{\mu\nu}z_\nu(t)}$$

$$\int \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\; \mathcal{P}(\mathbf{y},\mathbf{z}|\mathbf{C},\mathbf{Q},\mathbf{K}) z_\mu(t) z_\nu(t') = \delta_{\mu\nu} Q(t,t')$$

$$\int \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\; \mathcal{P}(\mathbf{y},\mathbf{z}|\mathbf{C},\mathbf{Q},\mathbf{K}) y_\mu(t) y_\nu(t') = \delta_{\mu\nu} C(t,t')$$

$$\int \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\; \mathcal{P}(\mathbf{y},\mathbf{z}|\mathbf{C},\mathbf{Q},\mathbf{K}) y_\mu(t) z_\nu(t') = \delta_{\mu\nu} K(t,t')$$

Gaussian integral Ω

- notation $\mathbf{D}=\mathbf{A}\otimes\mathbf{B}$: $D_{\mu t,\nu t'}=A_{\mu\nu}B(t,t')$, $D^{\mathrm{T}}_{\mu t,\nu t'}=A_{\nu\mu}B(t',t)$

$$\mathcal{P}(\mathbf{y},\mathbf{z}|\mathbf{C},\mathbf{Q},\mathbf{K})=\frac{\mathrm{e}^{-\frac{1}{2}\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{z}\end{smallmatrix}\right)\cdot\mathcal{C}^{-1}\left(\begin{smallmatrix}\mathbf{y}\\ \mathbf{z}\end{smallmatrix}\right)}}{\sqrt{(2\pi)^{2\tau(p-\tau)}\mathrm{Det}\mathcal{C}}},\qquad \mathcal{C}=\begin{pmatrix}(\mathbb{1}\otimes\mathbf{C}) & (\mathbb{1}\otimes\mathbf{K})\\ (\mathbb{1}\otimes\mathbf{K})^{\mathrm{T}} & (\mathbb{1}\otimes\mathbf{Q})\end{pmatrix}$$

$$=\int\frac{\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}}{(2\pi)^{2\tau(p-\tau)}}\,\mathrm{e}^{-\frac{1}{2}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)\cdot\mathcal{C}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)+\mathrm{i}(\mathbf{y}\cdot\mathbf{u}+\mathbf{z}\cdot\mathbf{v})}$$

- now

$$\begin{aligned}\Omega[\mathbf{C},\mathbf{Q},\mathbf{K}] &= \frac{1}{N}\log\left\{\int\frac{\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}}{(2\pi)^{2\tau(p-\tau)}}\,\mathrm{e}^{-\frac{1}{2}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)\cdot\mathcal{C}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)+\mathrm{i}[\mathbf{y}\cdot\mathbf{u}+\mathbf{z}\cdot\mathbf{v}-\mathbf{y}\cdot(\mathbf{S}\otimes\mathbb{1})\mathbf{z}]}\right\}\\ &= \frac{1}{N}\log\left\{\int\frac{\mathrm{d}\mathbf{z}\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}}{(2\pi)^{\tau(p-\tau)}}\,\mathrm{e}^{-\frac{1}{2}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)\cdot\mathcal{C}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)+\mathrm{i}\mathbf{z}\cdot\mathbf{v}}\delta[\mathbf{u}-(\mathbf{S}\otimes\mathbb{1})\mathbf{z}]\right\}\\ &= \frac{1}{N}\log\left\{\mathrm{Det}(\mathbf{S}^{\mathrm{T}}\otimes\mathbb{1})\int\frac{\mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}}{(2\pi)^{\tau(p-\tau)}}\,\mathrm{e}^{-\frac{1}{2}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)\cdot\mathcal{C}\left(\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right)+\mathrm{i}\mathbf{u}\cdot(\mathbf{S}\otimes\mathbb{1})\mathbf{v}}\right\}\end{aligned}$$

- use $\mathbf{K} = \mathrm{i}\mathbf{G}$ and integrate over $(\mathbf{u}, \mathbf{v})$

$$\begin{aligned}\Omega[\mathbf{C},\mathbf{Q},\mathrm{i}\mathbf{G}] &= \frac{1}{N}\log\mathrm{Det}(\mathbf{S}^{\mathrm{T}}\otimes\mathbb{I}) - \frac{1}{2N}\log\mathrm{Det}(\mathbb{I}\otimes\mathbf{C}) \\ &\quad -\frac{1}{2N}\log\mathrm{Det}\Big[\mathbb{I}\otimes\mathbf{Q}+[(\mathbb{I}\otimes\mathbf{G})-(\mathbf{S}\otimes\mathbb{I})]^{\mathrm{T}}(\mathbb{I}\otimes\mathbf{C})^{-1}[(\mathbb{I}\otimes\mathbf{G})-(\mathbf{S}\otimes\mathbb{I})]\Big]\end{aligned}$$

(see exercises)

- small $\mathbf{Q}$:
  use $\log\mathrm{Det}(\mathbf{M}+\boldsymbol{\epsilon}) = \log\mathrm{Det}(\mathbf{M})+\mathrm{Tr}(\mathbf{M}^{-1}\boldsymbol{\epsilon})+\mathcal{O}(\boldsymbol{\epsilon}^2)$

$$\begin{aligned}\Omega[\mathbf{C},\mathbf{Q},\mathrm{i}\mathbf{G}] &= -\frac{1}{2N}\log\mathrm{Det}\Big[[(\mathbf{S}^{\mathrm{T}}\otimes\mathbb{I})^{-1}(\mathbb{I}\otimes\mathbf{G}^{\mathrm{T}})-\mathbb{I}\otimes\mathbb{I}][(\mathbf{S}\otimes\mathbb{I})^{-1}(\mathbb{I}\otimes\mathbf{G})-\mathbb{I}\otimes\mathbb{I}]\Big] \\ &\quad -\frac{1}{2N}\mathrm{Tr}\Big[[(\mathbb{I}\otimes\mathbf{G})-(\mathbf{S}\otimes\mathbb{I})]^{-1}(\mathbb{I}\otimes\mathbf{C})[(\mathbb{I}\otimes\mathbf{G}^{\mathrm{T}})-(\mathbf{S}^{\mathrm{T}}\otimes\mathbb{I})]^{-1}(\mathbb{I}\otimes\mathbf{Q})\Big]+\mathcal{O}(\mathbf{Q}^2)\end{aligned}$$

(see exercises)

final result for $\Omega[\ldots]$, using $\mathbf{S}^{\mathrm{T}}\mathbf{S}=\mathbb{1}$

$$\Omega[\mathbf{C},\mathbf{Q},\mathrm{i}\mathbf{G}]=-\frac{1}{N}\log\mathrm{Det}(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G}) -\frac{1}{2N}\mathrm{Tr}\left[(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}(\mathbb{1}\otimes\mathbf{C})(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1}(\mathbb{1}\otimes\mathbf{Q})\right]+\mathcal{O}(\mathbf{Q}^2)$$

- order parameter eqns, use $\log\mathrm{Det}\mathbf{A}=\mathrm{Tr}\log\mathbf{A}$

$$\hat{C}(t,t')=\mathrm{i}\frac{\partial\Omega[\mathbf{C},\mathbf{0},\mathrm{i}\mathbf{G}]}{\partial C(t,t')}=0$$

$$\hat{K}(t,t')=\frac{\partial\Omega[\mathbf{C},\mathbf{0},\mathrm{i}\mathbf{G}]}{\partial G(t,t')}=-\frac{1}{N}\frac{\partial}{\partial G(t,t')}\log\mathrm{Det}(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})$$
$$=-\frac{1}{N}\mathrm{Tr}\frac{\partial\log(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})}{\partial G(t,t')}=\frac{1}{N}\mathrm{Tr}\left[(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}\frac{\partial(\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})}{\partial G(t,t')}\right]$$

$$\hat{Q}(t,t')=\mathrm{i}\frac{\partial\Omega[\mathbf{C},\mathbf{Q},\mathrm{i}\mathbf{G}]}{\partial Q(t,t')}\bigg|_{\mathbf{Q}=\mathbf{0}}$$
$$=-\frac{\mathrm{i}}{2N}\mathrm{Tr}\left[(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}(\mathbb{1}\otimes\mathbf{C})(\mathbb{1}\otimes\mathbb{1}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1}\frac{\partial(\mathbb{1}\otimes\mathbf{Q})}{\partial Q(t,t')}\right]$$

- next hurdle

$$\left[\frac{\partial(\mathbb{I}\otimes\mathbf{Q})}{\partial Q(t,t')}\right]_{\mu s,\nu s'}=\delta_{\mu\nu}\delta_{st}\delta_{s't'}\qquad\left[\frac{\partial(\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})}{\partial G(t,t')}\right]_{\mu s,\nu s'}=S_{\nu\mu}\delta_{st}\delta_{s't'}$$

$$\frac{1}{N}\mathrm{Tr}\left[\mathbf{A}\frac{\partial(\mathbb{I}\otimes\mathbf{Q})}{\partial Q(t,t')}\right]=\frac{1}{N}\sum_{\mu\nu ss'}A_{\mu s,\nu s'}\delta_{\mu\nu}\delta_{st}\delta_{s't'}=\frac{1}{N}\sum_{\mu}A_{\mu t,\mu t'}$$

$$\frac{1}{N}\mathrm{Tr}\left[\mathbf{A}\frac{\partial(\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})}{\partial G(t,t')}\right]=\frac{1}{N}\sum_{\mu\nu ss'}A_{\mu s,\nu s'}S_{\nu\mu}\delta_{st}\delta_{s't'}=\frac{1}{N}\sum_{\mu\nu}A_{\mu t,\nu t'}S_{\nu\mu}$$

- result

$$\begin{aligned}\hat{K}(t,t') &= \frac{1}{N}\mathrm{Tr}\left[(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}\frac{\partial(\mathbf{S}^{\dagger}\otimes\mathbf{G})}{\partial G(t,t')}\right]\\ &= \frac{1}{N}\sum_{\mu\nu}(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}_{\mu t,\nu t'}S_{\mu\nu}\\ \hat{Q}(t,t') &= -\frac{\mathrm{i}}{2N}\mathrm{Tr}\left[(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}(\mathbb{I}\otimes\mathbf{C})(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1}\frac{\partial(\mathbb{I}\otimes\mathbf{Q})}{\partial Q(t,t')}\right]\\ &= -\frac{\mathrm{i}}{2N}\sum_{\mu}\left[(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}(\mathbb{I}\otimes\mathbf{C})(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1}\right]_{\mu t',\mu t}\end{aligned}$$

geometric series: $(1-z)^{-1} = \sum_{\ell\geq 0} z^\ell$
apply to matrices

$$(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1} = \sum_{\ell\geq 0}(\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^\ell = \sum_{\ell\geq 0}(\mathbf{S}^\ell)\otimes(\mathbf{G}^\ell)^{\mathrm{T}}$$

$$(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1} = \sum_{\ell\geq 0}(\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^\ell = \sum_{\ell\geq 0}(\mathbf{S}^\ell)^{\mathrm{T}}\otimes(\mathbf{G}^\ell)$$

now, with $\mathbf{S}^{\mathrm{T}}\mathbf{S} = \mathbb{I}$

$$\begin{aligned}
\hat{K}(t,t') &= \frac{1}{N}\sum_{\mu\nu}(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}_{\mu t,\nu t'}S_{\mu\nu} = \frac{1}{N}\sum_{\ell\geq 0}\sum_{\mu\nu}[(\mathbf{S}^\ell)^{\mathrm{T}}\otimes(\mathbf{G}^\ell)]_{\mu t,\nu t'}S_{\mu\nu} \\
&= \frac{1}{N}\sum_{\ell\geq 0}\left[\sum_{\mu=\tau}^{\alpha N}(\mathbf{S}^{\ell+1})^{\mathrm{T}}_{\mu\mu}\right](\mathbf{G}^\ell)(t,t') = \frac{1}{N}\sum_{\ell\geq 0}\left[\sum_{\mu=\tau}^{\alpha N}(\mathbf{S}^{\ell+1})_{\mu\mu}\right](\mathbf{G}^\ell)(t,t') \\
&= \frac{1}{N}\sum_{\ell\geq 0}\left[\sum_{\mu=\tau}^{\alpha N}\delta_{\mu,\mu-\ell-1}\right](\mathbf{G}^\ell)(t,t') \;=\; 0
\end{aligned}$$

$$\begin{aligned}
\hat{Q}(t,t') &= -\frac{\mathrm{i}}{2N}\sum_{\mu}\left[(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}^{\mathrm{T}}\otimes\mathbf{G})^{-1}(\mathbb{I}\otimes\mathbf{C})(\mathbb{I}\otimes\mathbb{I}-\mathbf{S}\otimes\mathbf{G}^{\mathrm{T}})^{-1}\right]_{\mu t',\mu t} \\
&= -\frac{\mathrm{i}}{2N}\sum_{\mu}\left[\Big(\sum_{\ell\geq 0}(\mathbf{S}^{\ell})^{\mathrm{T}}\otimes(\mathbf{G}^{\ell})\Big)(\mathbb{I}\otimes\mathbf{C})\Big(\sum_{\ell'\geq 0}(\mathbf{S}^{\ell'})\otimes(\mathbf{G}^{\ell'})^{\mathrm{T}}\Big)\right]_{\mu t',\mu t} \\
&= -\frac{\mathrm{i}}{2N}\sum_{\ell,\ell'\geq 0}\left[\sum_{\mu=\tau}^{\alpha N}\Big((\mathbf{S}^{\ell})^{\mathrm{T}}(\mathbf{S}^{\ell'})\Big)_{\mu\mu}\right](\mathbf{G}^{\ell}\mathbf{C}\mathbf{G}^{\mathrm{T}\ell'})(t',t) \\
&= -\frac{\mathrm{i}}{2N}\sum_{\ell,\ell'\geq 0}\left[\sum_{\mu=\tau}^{\alpha N}\delta_{\ell\ell'}\right](\mathbf{G}^{\ell}\mathbf{C}\mathbf{G}^{\mathrm{T}\ell'})(t',t) \\
&= -\mathrm{i}\frac{\alpha N-\tau}{2N}\sum_{\ell\geq 0}(\mathbf{G}^{\ell}\mathbf{C}\mathbf{G}^{\dagger\ell})(t',t)
\end{aligned}$$

hence

$$\hat{Q}(t,t') \to -\frac{1}{2}\alpha\mathrm{i}\sum_{\ell\geq 0}(\mathbf{G}^{\dagger\ell}\mathbf{C}\mathbf{G}^{\ell})(t,t') \quad \text{for } N\to\infty$$

The sequence recall transition

Final saddle point equations

$$m(t)=\lim_{N\to\infty}\frac{1}{N}\sum_i \xi_i^t\langle\sigma(t)\rangle_i,\qquad C(t,t')=\lim_{N\to\infty}\frac{1}{N}\sum_i\langle\sigma(t)\sigma(t')\rangle_i$$

$$G(t,t')=\lim_{N\to\infty}\frac{1}{N}\sum_i \xi_i^{t'+1}\frac{\partial\langle\sigma(t)\rangle_i}{\partial\theta(t')}$$

let $p_{i0}(\sigma(0))=p_0(\sigma(0))$

$$\langle g(\ldots)\rangle_i=\frac{\sum_{\{\sigma\}}\int\{\mathrm{d}h\mathrm{d}\hat{h}\}W[\{\sigma\}|\{h\}]\mathrm{e}^{\mathrm{i}\sum_t\hat{h}(t)[h(t)-(m(t)+\theta(t))\xi_i^{t+1}]-\frac{1}{2}\alpha\sum_{tt'}R(t,t')\hat{h}(t)\hat{h}(t')}g(\ldots)}{\sum_{\{\sigma\}}\int\{\mathrm{d}h\mathrm{d}\hat{h}\}W[\{\sigma\}|\{h\}]\mathrm{e}^{\mathrm{i}\sum_t\hat{h}(t)[h(t)-(m(t)+\theta(t))\xi_i^{t+1}]-\frac{1}{2}\alpha\sum_{tt'}R(t,t')\hat{h}(t)\hat{h}(t')}}$$

$$W[\{\sigma\}|\{h\}]=p_0(\sigma(0))\prod_t\frac{\mathrm{e}^{\beta\sigma(t+1)h(t)}}{2\cosh(\beta h(t))},\qquad R(t,t')=\sum_{\ell\geq 0}(\mathbf{G}^{\mathrm{T}\ell}\mathbf{C}\mathbf{G}^{\ell})(t,t')$$

we can now do sums over $\{\sigma\}$ ...

- focus on $m(t)$ and $G(t,t')$,
  relevant sums:

$$\sum_{\{\sigma\}} W[\{\sigma\}|\{h\}] = 1, \qquad \sum_{\{\sigma\}} \sigma(t) W[\{\sigma\}|\{h\}] = \tanh(\beta h(t-1))$$

(see exercises)

hence

$$\langle\sigma(t)\rangle_i = \frac{\int\{\mathrm{d}h\mathrm{d}\hat{h}\}\mathrm{e}^{\mathrm{i}\sum_s \hat{h}(s)[h(s)-(m(s)+\theta(s))\xi_i^{s+1}]-\frac{1}{2}\alpha\sum_{ss'}R(s,s')\hat{h}(s)\hat{h}(s')}\tanh(\beta h(t-1))}{\int\{\mathrm{d}h\mathrm{d}\hat{h}\}\mathrm{e}^{\mathrm{i}\sum_s \hat{h}(s)[h(s)-(m(s)+\theta(s))\xi_i^{s+1}]-\frac{1}{2}\alpha\sum_{ss'}R(s,s')\hat{h}(s)\hat{h}(s')}}$$

- transform:

$$h(s)=\xi_i^{s+1}[m(s)+\theta(s)+\sqrt{\alpha}v(s)], \quad \hat{h}(s)=\xi_i^{s+1}w(s)/\sqrt{\alpha}$$

$$\langle\sigma(t)\rangle_i = \xi_i^t \int \frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}\xi_i^{s+1}\xi_i^{s'+1}R(s,s')w(s)w(s')} \times \tanh[\beta(m(t-1)+\theta(t-1)+\sqrt{\alpha}v(t-1))]$$

- eqns for $m(t)$ and $G(t,t')$:

$$m(t) = \lim_{N\to\infty} \frac{1}{N}\sum_i \xi_i^t \langle \sigma_i(t)\rangle_i$$
$$= \lim_{N\to\infty} \frac{1}{N}\sum_i \int \frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau} \mathrm{e}^{\mathrm{i}\sum_s v(s)w(s) - \frac{1}{2}\sum_{ss'} \xi_i^{s+1}\xi_i^{s'+1} R(s,s')w(s)w(s')}$$
$$\times \tanh[\beta(m(t-1)+\theta(t-1)+\sqrt{\alpha}v(t-1))]$$

$t > t'$:

$$G(t,t') = \lim_{N\to\infty} \frac{1}{N}\sum_i \xi_i^{t'+1} \frac{\partial}{\partial\theta(t')}\langle \sigma_i(t)\rangle$$
$$= \delta_{t',t-1} \lim_{N\to\infty} \frac{1}{N}\sum_i \int \frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau} \mathrm{e}^{\mathrm{i}\sum_s v(s)w(s) - \frac{1}{2}\sum_{ss'} \xi_i^{s+1}\xi_i^{s'+1} R(s,s')w(s)w(s')}$$
$$\times \beta\Big[1 - \tanh^2[\beta(m(t-1)+\theta(t-1)+\sqrt{\alpha}v(t-1))]\Big]$$
$$= \delta_{t',t-1}\beta\Big[1 - \lim_{N\to\infty} \frac{1}{N}\sum_i \int \frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau} \mathrm{e}^{\mathrm{i}\sum_s v(s)w(s) - \frac{1}{2}\sum_{ss'} \xi_i^{s+1}\xi_i^{s'+1} R(s,s')w(s)w(s')}$$
$$\times \tanh^2[\beta(m(t-1)+\theta(t-1)+\sqrt{\alpha}v(t-1))]\Big]$$

Stationary state

- set $\theta(t)=0$, send $\tau\to\infty$ and initial time to $-\infty$, look for time-translation-invariant solutions,

$$m(t)=m, \qquad G(t,t')=\beta\delta_{t,t'+1}(1-\tilde{q}), \qquad C(t,t')=C(t-t')$$

now

$$m=\int \mathrm{d}x\;\mathcal{P}(x)\tanh[\beta(m+x\sqrt{\alpha})], \qquad \tilde{q}=\int \mathrm{d}x\;\mathcal{P}(x)\tanh^2[\beta(m+x\sqrt{\alpha})]$$

$$\begin{aligned}
\mathcal{P}(x) &= \lim_{N\to\infty}\frac{1}{N}\sum_i\int\frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}\xi_i^{s+1}\xi_i^{s'+1}R(s,s')w(s)w(s')}\delta[x-v(0)] \\
&= \lim_{N\to\infty}\frac{1}{N}\sum_i\int\frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}R(s,s')w(s)w(s')}\delta[x-\xi_i^1 v(0)] \\
&= \int\frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}R(s,s')w(s)w(s')}\Big(\lim_{N\to\infty}\frac{1}{N}\sum_i\delta[x-\xi_i^1 v(0)]\Big) \\
&= \int\frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}R(s,s')w(s)w(s')}\Big(\frac{1}{2}\delta[x-v(0)]+\frac{1}{2}\delta[x+v(0)]\Big) \\
&= \int\frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau}\mathrm{e}^{\mathrm{i}\sum_s v(s)w(s)-\frac{1}{2}\sum_{ss'}R(s,s')w(s)w(s')}\delta[x-v(0)]
\end{aligned}$$

- $\mathcal{P}(x)$: zero average Gaussian, with covariance

$$\int \mathrm{d}x\, \mathcal{P}(x)x^2 \;=\; \int \frac{\{\mathrm{d}v\mathrm{d}w\}}{(2\pi)^\tau} \mathrm{e}^{\mathrm{i}\sum_s v(s)w(s) - \frac{1}{2}\sum_{ss'} R(s,s')w(s)w(s')} v^2(0)$$
$$= \int \frac{\{\mathrm{d}v\}}{\sqrt{(2\pi)^\tau \mathrm{Det}\mathbf{R}}} \mathrm{e}^{-\frac{1}{2}\sum_{ss'} v(s)(\mathbf{R}^{-1})(s,s')v(s')} v^2(0) \;=\; R(0,0)$$

- $\mathbf{G}^\ell(t,t') = \beta^\ell(1-\tilde{q})^\ell \delta_{t,t'+\ell}$, $\mathbf{G}^{\mathrm{T}\ell}(t,t') = \beta^\ell(1-\tilde{q})^\ell \delta_{t,t'-\ell}$, hence

$$\begin{aligned} R(0,0) &= \sum_{\ell\geq 0}\sum_{ss'} \mathbf{G}^{\mathrm{T}\ell}(0,s)C(s-s')\mathbf{G}^\ell(s',0) \\ &= \sum_{\ell\geq 0} \beta^{2\ell}(1-\tilde{q})^{2\ell} \sum_{ss'} \delta_{0,s-\ell} C(s-s')\delta_{s',\ell} \\ &= C(0)\sum_{\ell\geq 0}\beta^{2\ell}(1-\tilde{q})^{2\ell} \;=\; \frac{1}{1-\beta^2(1-\tilde{q})^2} \end{aligned}$$

- final result

$$m = \int \mathrm{D}x\, \tanh\left(\beta m + \frac{\beta x\sqrt{\alpha}}{\sqrt{1-\beta^2(1-\tilde{q})^2}}\right)$$
$$\tilde{q} = \int \mathrm{D}x\, \tanh^2\left(\beta m + \frac{\beta x\sqrt{\alpha}}{\sqrt{1-\beta^2(1-\tilde{q})^2}}\right)$$

The recall transition

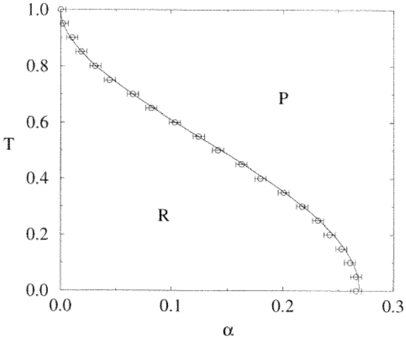
- recall phase (R): $m \neq 0$, stable sequence recall
  paramagnetic phase (P): $m = 0$, no sequence recall

$$\text{P phase:} \qquad \tilde{q} = \int \mathrm{D}x \ \tanh^2\left(\frac{\beta x\sqrt{\alpha}}{\sqrt{1-\beta^2(1-\tilde{q})^2}}\right)$$

- P→R transition:
  turns out to be
  discontinuous

  $\alpha = p/N, \ \ T = 1/\beta$

  markers:
  simulations with $N = 10^4$



Storage capacity for sequences

maximum α: found for $T=0$

- define $\beta(1-\tilde{q})=u$, assume $m>0$

$$m=\int \mathrm{D}x\ \tanh\left(\beta m+\frac{\beta x\sqrt{\alpha}}{\sqrt{1-u^2}}\right),\qquad u=\int \mathrm{D}x\ \beta\left[1-\tanh^2\left(\beta m+\frac{\beta x\sqrt{\alpha}}{\sqrt{1-u^2}}\right)\right]$$

take $\beta\to\infty$

$$m=J(m,u),\qquad u=\frac{\partial}{\partial m}J(m,u),\qquad J(m,u)=\int \mathrm{D}x\ \mathrm{sgn}\left(m+\frac{x\sqrt{\alpha}}{\sqrt{1-u^2}}\right)$$

- compute $J(m,u)$, abbreviate $\sqrt{1-u^2}=y\sqrt{2\alpha}/m$

$$\begin{aligned}J(m,u) &= \int \mathrm{D}x\ \mathrm{sgn}\left(1+\frac{x}{y\sqrt{2}}\right)=\int_{-y\sqrt{2}}^{\infty}\frac{\mathrm{d}x}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}-\int_{-\infty}^{-y\sqrt{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}\\ &= 2\int_0^{y\sqrt{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}=\frac{2}{\sqrt{\pi}}\int_0^y \mathrm{d}x\ \mathrm{e}^{-x^2}=\mathrm{Erf}(y)\end{aligned}$$

- final eqns for $T=0$

$$m=\mathrm{Erf}(y), \qquad u=\frac{\partial y}{\partial m}\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{Erf}(y)$$

$$m=\mathrm{Erf}(y), \qquad \pm\sqrt{1-\frac{2\alpha y^2}{m^2}}=\frac{y}{m}\frac{2}{\sqrt{\pi}}\mathrm{e}^{-y^2}$$

eliminate m:

$$\mathrm{Erf}^2(y)-2\alpha y^2=\frac{4y^2}{\pi}\mathrm{e}^{-2y^2} \qquad \Rightarrow \qquad y\sqrt{2\alpha}=\sqrt{\mathrm{Erf}^2(y)-\frac{4y^2}{\pi}\mathrm{e}^{-2y^2}}$$

solve numerically for y,
find largest α for which
nontrivial solns exist

$\alpha_c \approx 0.269$