Modelling of Complex Real-World Systems
Epilogue

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Applications of the replica method

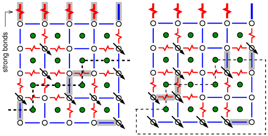
- *spin glasses*

  (i) non-binary variables, $S_i \in \mathbb{R}$ or $\vec{S}_i \in \mathbb{R}^3$

  (ii) higher order interactions,
  $H = -\sum_{ijk} J_{ijk}\sigma_i\sigma_j\sigma_k$, etc
  first order transitions, RS problematic ...

  (iii) quantum spins (spin-$\frac{1}{2}$)

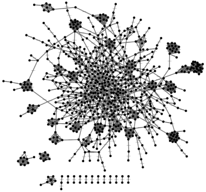
  (iv) sparse interactions and vector spins
  (Guzai expansion)



- *spin systems on random topologies*
  more realistic random graphs,

  (i) constrained degrees and degree correlations

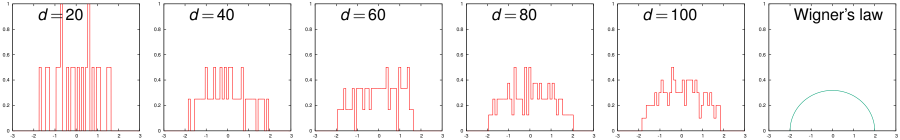
  (ii) graphs with many short loops ...
  (replica theories with $n \in \mathbb{C}$)



- *eigenvalue distribution of random matrices*
  common starting point: Edwards-Jones formula

  $J_{ii} = 0$, $J_{ij} = J_{ji} = z_{ij}/\sqrt{d}$ for $i \neq j$,
  z_{ij} : independent Gaussian, $\langle z_{ij} \rangle = 0$ and $\langle z_{ij}^2 \rangle = 1$
  $d \to \infty$: Wigner's semi-circular law

$$|\mu| > 2: \quad \varrho(\mu) = 0, \qquad |\mu| \leq 2: \quad \varrho(\mu) = \frac{1}{2\pi}\sqrt{4 - \mu^2}$$

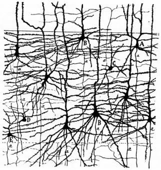


Gaussian symmetric matrices: Wigner's law
covariance matrices: Marchenko-Pastur law

adjacency matrices of Erdos-Renyi graphs: Rodgers-Bray
adjacency matrices of regular graphs: MacKay law
more general degree-constrained graphs: Dorogovtsev
finitely connected loopy random graphs

- *recurrent neural networks*

  (i) information storage capacities (Gardner),
  under various constraints (e.g. degree of symmetry)
  (ii) phase diagrams of hybrid recurrent-layered systems
  (iii) coupled oscillator attractor networks

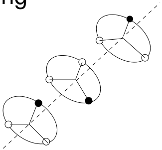


- *adaptive immune system*

  signalling between $\sim 10^8$ T-cell and B-cell clones,
  finitely connected, with heterogeneity in chemical signalling

- *folding of heteropolymers*

  heterogeneity: composition of monomers in the chain



- *satisfiability problems*

  SAT-UNSAT questions: given a set of Boolean expressions,
  do variable assignments exist that satisfy them all?
  (heterogeneity: logical operators in Boolean expressions)

- *mobile communication*

  CDMA protocols, signal sharing, optimal coding and decoding,
  computation of what is possible,
  compressed sensing

- *machine learning*

  teacher machine generates question-answer examples $(\mathbf{x}, y)$,
  to be learned by a student machine

  (i) how many examples needed for given complexity?
  (ii) effects of architecture mismatch?
  (iii) committees of machines
  (iv) sloppy teachers and/or sloppy students

- *parameter inference in statistics*

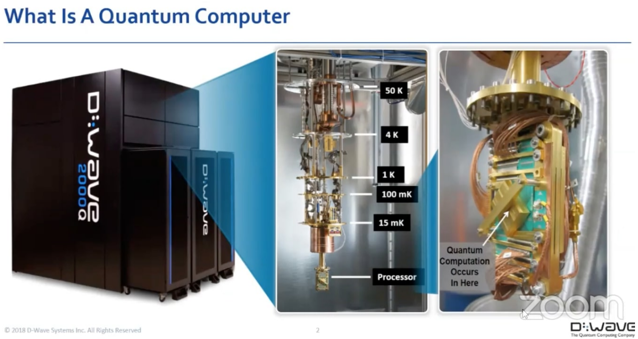
  model high-dimensional inference (e.g. survival analysis in medicine),
  overfitting decontamination methods

- *Agent-based models of financial markets*

  El Farol problem and minority games:
  understand instabilities of financial markets,
  resulting from (partly irrational) decision-making of interacting agents

- *quantum computing*

  analysis of quantum annealing (quantum spin system)



Further development of the replica method

- *replica symmetry breaking*

  $P(q) = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\alpha\neq\beta} \delta[q - q_{\alpha\beta}]$

  RS: $P(q) = \delta[q - q_{\mathrm{EA}}]$

  Imagine: ergodicity is broken, phase space divided into ergodic sectors $\ell = 1 \ldots L$,
  each containing a fraction w_ℓ of all system states $\boldsymbol{\sigma}$

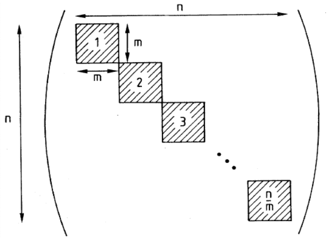
$$
\begin{aligned}
P(q) &= \Big\langle\Big\langle \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i'] \Big\rangle\Big\rangle \\
&= \sum_{\ell\ell'} w_\ell w_{\ell'} \Big\langle\Big\langle \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i'] \Big\rangle_\ell\Big\rangle_{\ell'} \\
&= \sum_\ell w_\ell^2 \Big\langle\Big\langle \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i'] \Big\rangle_\ell\Big\rangle_\ell + \sum_{\ell\neq\ell'} w_\ell w_{\ell'} \Big\langle\Big\langle \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i'] \Big\rangle_\ell\Big\rangle_{\ell'}
\end{aligned}
$$

- if sectors and their mutual similarities equivalent:

$$P(q) = W\delta[q-q_0] + (1-W)\delta[q-q_1], \qquad W = \sum_\ell w_\ell^2$$

Hence:
two off-diagonal values for the $\{q_{\alpha\beta}\}$

(i) SK model: Parisi's RSB scheme



1-step, 2-step, ... ∞-step RSB
nontrivial function $P(q)$

(ii) derivation of Paris scheme from hierarchy of equilibrating subsystems

(iii) RSB in finite connected systems?

(iv) link between RSB and violation of FDT in dynamics

(v) experimental evidence for RSB in lattice spin systems? (critical dimension?)

- *replicas to do dynamics*

  dynamical replica method, as alternative to GFA
  (controlled approximation, simpler to use, especially in finitely connected systems)

- *finite n replica method for systems with adiabatically separated timescales*

  (i) spin glasses with slowly moving impurities
  (ii) self-programming neural networks
  (iii) protein folding – genetic selection versus operation

- *alternative derivations in support of the replica method*

  (i) stochastic stability, Guerra's interpolation method
  (ii) link with the cavity method (in tree-like systems)
  (iii) link with message passing algorithms from computer science (in tree-like systems)

Generating functional analysis

- *link between GFA and replicas*

  (i) SK model: derive RS equations from GFA, using FDT
  (ii) SK model: derive RSB equations from GFA, via broken FDT
  (iii) finitely connected systems: not yet achieved ...

- *dynamics of 'glassy' systems*

  (i) spin glass dynamics
  (ii) dynamics of actual glasses
  (iii) algorithms in computer science

- *systems without detailed balance*

  (i) neural systems with nonsymmetric synapses
  (ii) gene regulation
  (iii) heterogeneous predator-prey ecologies
  (iv) minority games and other market models
  (v) dynamics of learning in machine learning