

The Trick which Became a Theory

A Brief History of the Replica Method

ACC Coolen
King's College London



The Replica Trick for Disordered Systems

Dirty Tricks Should Not Work ...

Alternative Versions & Real Replicas

Physical Meaning of Replica Dimension

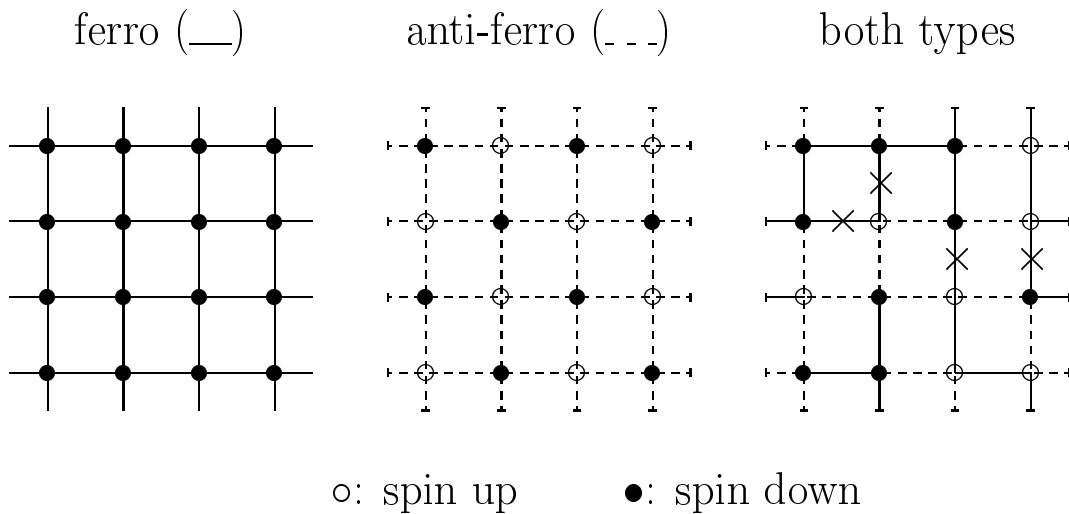
Physics Behind Parisi's RSB Scheme

New Avenues in Replica Theory

I: THE REPLICA TRICK FOR DISORDERED SYSTEMS

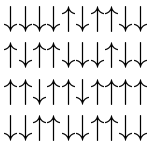
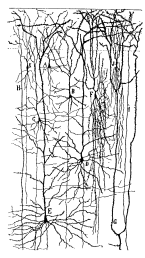

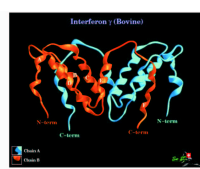
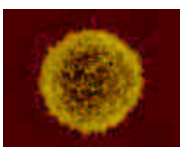

Disordered many-particle systems

- (pseudo-) randomness in microscopic parameters, e.g. interaction energies
- no microscopic periodicity
- high degree of frustration, incompatible forces



- many relevant time-scales, difficult to equilibrate
- many techniques no longer applicable, e.g. transfer matrices, renormalization, ...
- even mean-field (range-free) models are non-trivial

Examples of Disordered Systems

	<i>area</i>	<i>variables</i>	<i>disorder</i>	<i>size</i>
	spin-glasses (impure metals)	spin orientations	RKKY interactions	10^{20}
	neural networks	neuronal firing rates	synapses	10^{10}
	machine learning	programme code	example data	10^4
	protein folding	amino-acid orientations	chemical composition	10^3
	immunology	lymphocyte concentrations	surface shapes	10^8
	market models	trading actions	trading strategies	10^3

‘Harmonic Oscillator’ of Disordered Systems Theory

Sherrington-**K**irkpatrick model (1975):

- N Ising spins $\sigma_i \in \{-1, 1\}$,
indep. random exchange energies J_{ij}

$$H = - \sum_{i < j} J_{ij} \sigma_i \sigma_j, \quad \overline{J_{ij}} = \frac{J_0}{N}, \quad \overline{J_{ij}^2} - \overline{J_{ij}}^2 = \frac{J^2}{N}$$

- Let $\boldsymbol{\sigma} = \{\sigma_i\}$, $\mathbf{J} = \{J_{ij}\}$, $\beta = 1/kT$,
free energy per spin:

$$f[\mathbf{J}] = -\frac{1}{\beta N} \log Z[\mathbf{J}] \quad Z[\mathbf{J}] = \sum_{\boldsymbol{\sigma}} e^{-\beta H}$$

- To be calculated: disorder-averaged free energy

$$\begin{aligned} \overline{f} &= \overline{f[\mathbf{J}]} = -\frac{1}{\beta N} \overline{\log Z[\mathbf{J}]} \\ &= -\frac{1}{\beta N} \int \left[\prod_{i < j} \frac{dz_{ij} e^{-\frac{1}{2}z_{ij}^2}}{\sqrt{2\pi}} \right] \log \left[\sum_{\boldsymbol{\sigma}} e^{\frac{\beta J_0}{2N} (\sum_i \sigma_i)^2 + \frac{\beta J}{\sqrt{N}} \sum_{i < j} z_{ij} \sigma_i \sigma_j} + \mathcal{O}\left(\frac{1}{N}\right) \right] \end{aligned}$$

*simple ∞ -range model, but still trouble !
How to carry out the disorder average ?*

The Replica Trick (Marc Kac, 1968)



- $x^n = 1 + n \log x + \mathcal{O}(n^2)$

$$\overline{\log Z[\mathbf{J}]} = \lim_{n \rightarrow 0} \frac{1}{n} \log \overline{Z^n[\mathbf{J}]}$$

- n integer:

$$Z^n[\mathbf{J}] = \left[\sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})} \right]^n = \sum_{\boldsymbol{\sigma}^1} \dots \sum_{\boldsymbol{\sigma}^n} e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}$$

partition function of n *independent* replicas of system

- disorder average:

$$\overline{f} = - \lim_{n \rightarrow 0} \frac{1}{\beta N n} \log \left[\sum_{\boldsymbol{\sigma}^1} \dots \sum_{\boldsymbol{\sigma}^n} e^{-\beta H_{\text{eff}}(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n)} \right]$$

$$H_{\text{eff}}(\boldsymbol{\sigma}^1, \dots, \boldsymbol{\sigma}^n) = -\frac{1}{\beta} \log \overline{e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}$$

n *interacting* replicas, *no* disorder

- exchange limits

$N \rightarrow \infty$ versus $n \rightarrow 0$

Application to the SK Model

$$\lim_{N \rightarrow \infty} \bar{f} = \lim_{n \rightarrow 0} \min \left\{ -\frac{1}{\beta} \log 2 + \frac{\beta J^2}{4n} \sum_{\alpha\gamma} q_{\alpha\gamma}^2 + \frac{J_0}{2n} \sum_{\alpha} m_{\alpha}^2 - \frac{1}{\beta n} \log \sum_{\sigma_1, \dots, \sigma_n} e^{\frac{1}{2} \beta^2 J^2 \sum_{\alpha\gamma} q_{\alpha\gamma} \sigma_{\alpha} \sigma_{\gamma} + \beta J_0 \sum_{\alpha} m_{\alpha} \sigma_{\alpha}} \right\}$$

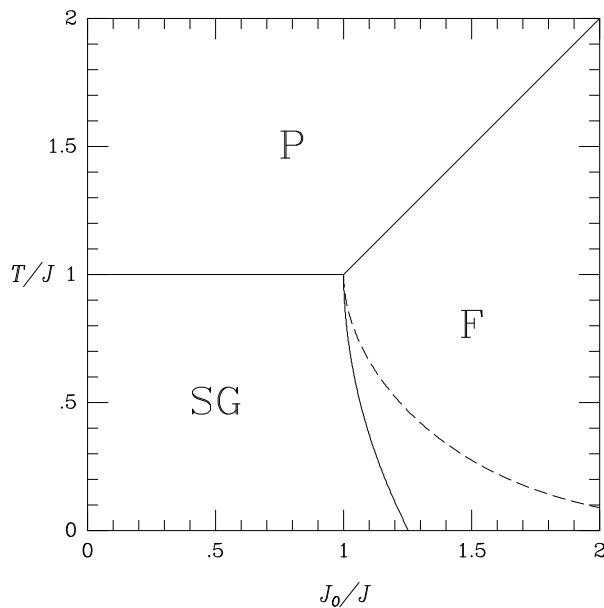
order parameters

$$m_{\alpha} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i^{\alpha} \rangle} \quad q_{\alpha\beta} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i^{\alpha} \sigma_i^{\beta} \rangle}$$

$$\alpha, \beta = 1, \dots, n$$

replica-symmetric (RS) ansatz:

$$m_{\alpha} = m \quad q_{\alpha\beta} = \delta_{\alpha\beta} + q [1 - \delta_{\alpha\beta}]$$



$$m = \lim_{N \rightarrow \infty} N^{-1} \sum_i \overline{\langle \sigma_i \rangle}$$

$$q = \lim_{N \rightarrow \infty} N^{-1} \sum_i \overline{\langle \sigma_i \rangle^2}$$

$$P : m = q = 0$$

$$F : m \neq 0, q > 0$$

$$SG : m = 0, q > 0$$

II: DIRTY TRICKS SHOULD NOT WORK ...

Problems

- negative entropy at low temperatures
- limits commute in $\bar{f} = \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{-1}{\beta N n} \log \bar{Z}^n$?
- simple identities: $\frac{d}{dn}[n^{-1} \log \bar{Z}^n] \geq 0$, $\frac{d^2}{dn^2} \log \bar{Z}^n \geq 0$
violated at low temperatures
- saddle-point *not* minimum of free energy,
 $\bar{f} = \max_q \min_m f(m, q)$
- AT instability: Hessian of $f(\{q_{\alpha\beta}, m_\alpha\})$ develops negative eigenvalues at low temperatures

Some answers

- f self-averaging, $\lim_{N \rightarrow \infty} \frac{-1}{\beta N} \log Z[\mathbf{J}]$ exists
- limits commute: $\bar{f} = - \lim_{n \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{\beta n} \overline{\log Z^{n/N}}$
- curvature sign changes:

e.g. $f(\{q_{\alpha\beta}\}) = \sum_{\alpha \neq \beta=1}^n q_{\alpha\beta}^2 \rightarrow$ saddle point : $q_{\alpha\beta} = 0$

let $q_{\alpha\beta} = q \quad \forall \alpha \neq \beta$: $f(\dots) = n(n-1)q^2$

$q = 0$: min for $n > 1$, max for $n < 1$

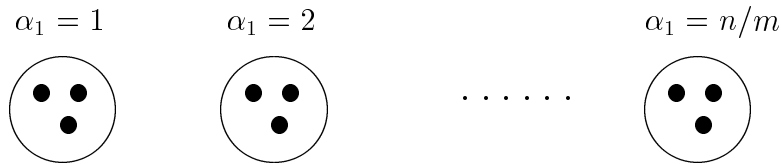
$f(\{q_{\alpha\beta}, m_\alpha\})$ invariant under replica permutations,
saddle-point *not invariant*: Replica Symmetry Breaking

$$q_{\alpha\beta} = \lim_{N \rightarrow \infty} N^{-1} \sum_i \overline{\langle \sigma_i^\alpha \sigma_i^\beta \rangle} \quad \alpha, \beta = 1 \dots n$$

Parisi scheme

- 1-step RSB: divide n replicas into subsets of equal size m :

$$\alpha \rightarrow (\alpha_1, \alpha_2) \quad \begin{array}{ll} \alpha_1 = 1 \dots n/m & \text{subset label} \\ \alpha_2 = 1 \dots m & \text{internal label} \end{array}$$



$$q_{(\alpha_1, \alpha_2), (\beta_1, \beta_2)} = \begin{cases} q_1 & \text{if } \alpha_1 = \beta_1 \quad (\text{same subset}) \\ q_0 & \text{if } \alpha_1 \neq \beta_1 \quad (\text{different subsets}) \end{cases}$$

$$\begin{pmatrix} 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & q & . & . & . \\ . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & q & . & . & . & 1 & . & . \\ . & . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & . & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & q_1 & q_1 & . & . & . & . & . & . \\ q_1 & 1 & q_1 & . & . & . & . & q_0 & . \\ q_1 & q_1 & 1 & . & . & . & . & . & . \\ . & . & . & 1 & q_1 & q_1 & . & . & . \\ . & . & . & q_1 & 1 & q_1 & . & . & . \\ . & . & . & q_1 & q_1 & 1 & . & . & . \\ . & . & . & . & . & . & 1 & q_1 & q_1 \\ . & q_0 & . & . & . & . & q_1 & 1 & q_1 \\ . & . & . & . & . & . & q_1 & q_1 & 1 \end{pmatrix}$$

- insert into $f(\{q_{\alpha\beta}, m_\alpha\})$, let $m \in [0, 1]$

$$\bar{f}_{1\text{RSB}} = \text{extr}_{m \in [0,1], q_0, q_1} f(m, q_0, q_1)$$

- Repeat breaking up of diagonal blocks *ad infinitum*

$$\bar{f}_{\infty\text{RSB}} = \text{extr}_{m_0, m_1, \dots \in [0,1]; q_0, q_1, \dots} f(m_0, m_1, \dots, q_0, q_1, \dots)$$

The case against replicas á la Parisi

1. physical meaning of function $q(x)$?
2. block sizes: $m_i \in \{1, \dots, n\}$ replaced by $m_i \in [0, 1]$...
3. why monotonicity $m_0 \leq m_1 \leq m_2 \leq \dots$?
4. physical meaning of replica dimension n ?
5. one takes limit $n \rightarrow 0$ by first sending $n \rightarrow \infty$...
6. $\{q_{\alpha\beta}\} \in \mathbb{R}^{\frac{1}{2}n(n-1)}$, analysis in space with dimension < 0
7. uniqueness of the scheme ?

language barriers ...

'the space of 0×0 matrices is a very large space'

(G. Parisi, 1979)

III: ALTERNATIVE VERSION, REAL REPLICAS

$$\frac{\int dx e^{-\beta H(x)} g(x)}{\int dx e^{-\beta H(x)}} = \lim_{n \rightarrow 0} \int \prod_{\alpha=1}^n [dx_{\alpha} e^{-\beta H(x_{\alpha})}] g(x_1)$$

Meaning of $q(x)$

Imagine two real system copies σ and σ'

$$\begin{aligned} P(q) &= \frac{\sum_{\sigma \sigma'} \delta \left[q - \frac{1}{N} \sum_i \sigma_i \sigma'_i \right] e^{-\beta H(\sigma) - \beta H(\sigma')}}{\sum_{\sigma \sigma'} e^{-\beta H(\sigma) - \beta H(\sigma')}} \\ &= \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \sum_{\sigma^1 \dots \sigma^n} \delta \left[q - \frac{1}{N} \sum_i \sigma_i^{\alpha} \sigma_i^{\beta} \right] \prod_{\gamma=1}^n e^{-\beta H(\sigma^{\gamma})} \\ \overline{P(q)} &= \lim_{n \rightarrow 0} \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \delta [q - q_{\alpha\beta}] \end{aligned}$$

Consequences

- $\overline{P(q)} = dx/dq$
- interpretation of $m_i \in [0, 1]$: consider 1RSB

$$\begin{pmatrix} 1 & q_1 & q_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_1 & 1 & q_1 & \cdot & \cdot & \cdot & \cdot & q_0 & \cdot \\ q_1 & q_1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & q_1 & q_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & q_1 & 1 & q_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & q_1 & q_1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & q_1 & q_1 \\ \cdot & q_0 & \cdot & \cdot & \cdot & \cdot & q_1 & 1 & q_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & q_1 & q_1 & 1 \end{pmatrix}$$

$$\begin{aligned} \overline{P(q)} &= \lim_{n \rightarrow 0} \\ &\frac{n-m}{n-1} \delta[q - q_0] + \frac{m-1}{n-1} \delta[q - q_1] \end{aligned}$$

$$\frac{n-m}{n-1} \geq 0 \quad \rightarrow \quad m \geq 0$$

$$\frac{m-1}{n-1} \geq 0 \quad \rightarrow \quad m \leq 1$$

- replicas and ergodicity:

assume L ergodic sectors,

with measures $p_\ell(\boldsymbol{\sigma})$ and probabilities w_ℓ

$$\begin{aligned}
 P(q) &= \lim_{N \rightarrow \infty} \langle \langle \delta[q - \frac{1}{N} \sum_i \sigma_i \sigma'_i] \rangle \rangle \\
 &= \lim_{N \rightarrow \infty} \sum_{\ell \ell'=1}^L w_\ell w_{\ell'} \sum_{\boldsymbol{\sigma} \boldsymbol{\sigma}'} p_\ell(\boldsymbol{\sigma}) p_{\ell'}(\boldsymbol{\sigma}') \delta[q - \frac{1}{N} \sum_i \sigma_i \sigma'_i]
 \end{aligned}$$

mean field model:

$$\frac{1}{N} \sum_i \sigma_i \sigma'_i = Q_{\ell \ell'} + \mathcal{O}(N^{-\frac{1}{2}})$$

$$\overline{P(q)} = \sum_{\ell \ell'=1}^L \overline{w_\ell w_{\ell'} \delta[q - Q_{\ell \ell'}]}$$

Hence:

ergodicity = replica symmetry

finite nr of ergodic sectors: $\overline{P(q)}$ sum of δ -peaks

infin nr of ergodic sectors: $\overline{P(q)}$ with continuous pieces

IV: PHYSICAL MEANING OF REPLICA DIMENSION n

Adiabatically slow bond dynamics

$$\frac{d}{dt}J_{ij} = \frac{1}{N}\langle\sigma_i\sigma_j\rangle_{\mathbf{J}} + \frac{K}{N} - \mu J_{ij} + \xi_{ij}$$

$$\langle\xi_{ij}(t)\rangle = 0, \quad \langle\xi_{ij}(t)\xi_{kl}(t')\rangle = \frac{2}{\tilde{\beta}N}\delta_{(i,j),(k,\ell)}\delta(t-t')$$

each bond J_{ij} :

stochastic motion in harmonic potential
+ frustration removing force $\langle\sigma_i\sigma_j\rangle_{\mathbf{J}}$

Note:

$$\frac{d}{dt}J_{ij} = -\frac{\partial}{\partial J_{ij}}\mathcal{H}[\mathbf{J}] + \xi_{ij}$$

$$\mathcal{H}[\mathbf{J}] = \frac{1}{N\beta}\log Z[\mathbf{J}] - \frac{K}{N}\sum_{i<j}J_{ij} + \frac{1}{2}\mu\sum_{i<j}J_{ij}^2$$

Bond equilibrium:

$$\tilde{f} = -\lim_{N\rightarrow\infty}\frac{1}{\tilde{\beta}N}\log\int d\mathbf{J} e^{-\tilde{\beta}\mathcal{H}[\mathbf{J}]}$$

transform parameters:

$$K \rightarrow \mu J_0, \quad \mu \rightarrow 1/\tilde{\beta} J^2, \quad \tilde{\beta} \rightarrow n\beta$$

$$P(J_{ij}) = (2\pi J^2/N)^{-\frac{1}{2}} e^{-\frac{1}{2}N(J_{ij}-J_0/N)^2/J^2}$$

slowly evolving bonds:

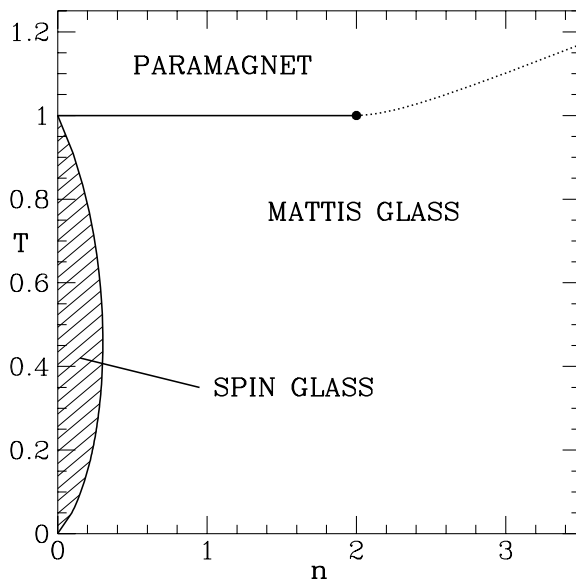
$$\bar{f} = - \lim_{N \rightarrow \infty} \frac{1}{nN\beta} \log \int \prod_{i < j} [dJ_{ij} P(J_{ij})] Z^n[\mathbf{J}] + \text{const}$$

random bonds (SK):

$$\tilde{f} = - \lim_{n \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{nN\beta} \log \int \prod_{i < j} [dJ_{ij} P(J_{ij})] Z^n[\mathbf{J}] + \text{const}$$

$$n = T/\tilde{T}, \quad \text{ratio of temperatures}$$

(i.e. ratio of energies, or ratio of time-scales)



phase diagram for

$$J = 1, \quad J_0 = 0$$

IV: PHYSICS BEHIND PARISI'S RSB SCHEME

Boltzmann's \mathcal{H} -function

$$\mathcal{H}(t) = \sum_{\boldsymbol{\sigma}} p_t(\boldsymbol{\sigma}) \{H(\boldsymbol{\sigma}) + T \log p_t(\boldsymbol{\sigma})\} \quad \frac{d\mathcal{H}}{dt} \leq 0, \quad \mathcal{H} \geq F$$

Fast and slow spins

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s) \quad p(\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s) = p(\boldsymbol{\sigma}_f|\boldsymbol{\sigma}_s)p(\boldsymbol{\sigma}_s)$$

$$\mathcal{H}(t) = \sum_{\boldsymbol{\sigma}_s} p_t(\boldsymbol{\sigma}_s) \{H_{\text{eff}}(\boldsymbol{\sigma}_s) + T \log p_t(\boldsymbol{\sigma}_s)\}$$

$$H_{\text{eff}}(\boldsymbol{\sigma}_s) = \sum_{\boldsymbol{\sigma}_f} p_t(\boldsymbol{\sigma}_f|\boldsymbol{\sigma}_s) \{H(\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s) + T \log p_t(\boldsymbol{\sigma}_f|\boldsymbol{\sigma}_s)\}$$

disparate time-scales:

- First, $p(\boldsymbol{\sigma}_f|\boldsymbol{\sigma}_s)$ will evolve to minimize $H_{\text{eff}}(\boldsymbol{\sigma}_s)$:

$$p(\boldsymbol{\sigma}_f|\boldsymbol{\sigma}_s) \rightarrow Z_f^{-1}(\boldsymbol{\sigma}_s) e^{-\beta H(\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s)} \quad Z_f(\boldsymbol{\sigma}_s) = \sum_{\boldsymbol{\sigma}_f} e^{-\beta H(\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_s)}$$

- Second: postulate stationarity of entropy of slow vars, minimize $\mathcal{H}(t)$ with $S_s = -\sum_{\boldsymbol{\sigma}_s} p(\boldsymbol{\sigma}_s) \log p(\boldsymbol{\sigma}_s)$ constant

$$p(\boldsymbol{\sigma}_s) \rightarrow Z_s^{-1} e^{-\tilde{\beta} H_{\text{eff}}(\boldsymbol{\sigma}_s)} \quad Z_s = \sum_{\boldsymbol{\sigma}_s} e^{-\tilde{\beta} H_{\text{eff}}(\boldsymbol{\sigma}_s)}$$

- $\tilde{\beta} = \tilde{m}\beta$:

$$Z_s = \sum_{\boldsymbol{\sigma}_s} [Z_f(\boldsymbol{\sigma}_s)]^{\tilde{m}} \quad \tilde{m} : \tilde{m}^2 (\partial F_s / \partial \tilde{m}) = S_s$$

Generalization to hierarchy of time-scales

$$\begin{aligned} \text{time scales :} & \quad \infty > \tau_0 \gg \dots \gg \tau_\ell \gg \dots \gg \tau_L > 0 \\ \text{temperatures :} & \quad \infty > T_0 \geq \dots \geq T_\ell \geq \dots \geq T_L > 0 \end{aligned}$$

ordering of $\{T_\ell\}$: thermodynamic stability
(if not: fast levels become heat baths for slow ones)

- ergodic equilibration at each scale, constrained S_ℓ :

$$Z_L = \sum_{\boldsymbol{\sigma}_L} e^{-\beta_L H(\{\boldsymbol{\sigma}\})} \quad \ell < L : \quad Z_\ell = \sum_{\boldsymbol{\sigma}_\ell} [Z_{\ell+1}]^{\tilde{m}_{\ell+1}}$$

- $\beta_L = \beta$, $\tilde{m}_\ell = T_\ell/T_{\ell-1}$,

$$\tilde{m}_\ell : \quad \beta_{\ell+1} \tilde{m}_{\ell+1}^2 \frac{\partial F_\ell}{\partial \tilde{m}_{\ell+1}} = S_\ell, \quad F_\ell = -T_\ell \log Z_\ell$$

- F_0 : generator of observables

$$H(\{\boldsymbol{\sigma}\}) \rightarrow H(\{\boldsymbol{\sigma}\}) + \lambda \psi(\{\boldsymbol{\sigma}\}) : \quad \lim_{\lambda \rightarrow 0} \frac{\partial}{\partial \lambda} F_0 = \langle \psi(\{\boldsymbol{\sigma}\}) \rangle$$

$$\tilde{m}_\ell < 1 : S_\ell > S_{\text{equil}},$$

standard Boltzmann state: $\tilde{m}_\ell = 1$ for all ℓ

Application to SK model

- divide spins in $L + 1$ mobility groups of size $|I_\ell| = \epsilon_\ell N$:

$$\{1, \dots, N\} = \bigcup_{\ell=0}^L I_\ell$$

system can choose $\{\epsilon_\ell\}$

- disorder-averaged slowest free energy:

$$\bar{f}_0 = - \lim_{N \rightarrow \infty} \frac{1}{\beta_0 N} \overline{\log Z_0} = - \lim_{N \rightarrow \infty} \lim_{\tilde{n} \rightarrow 0} \frac{1}{N \tilde{n} \beta_0} \log \overline{Z_0^{\tilde{n}}}$$

- nested set of $\tilde{n} \prod_{\ell=1}^L \tilde{m}_\ell$ replicas
spin at level ℓ : carries replica indices $\{a_0, \dots, a_\ell\}$

$a_0 \in \{1, \dots, \tilde{n}\}$: from disorder average

$a_k \in \{1, \dots, \tilde{m}_k\}$: from constrained equilibrations

- minimize \bar{f}_0 wrt $\{\epsilon_\ell\}$:

$$\epsilon_L = 1, \quad \epsilon_\ell = 0 \quad \forall \ell < L \quad (\text{vanishing fraction of slow spins})$$

Result:

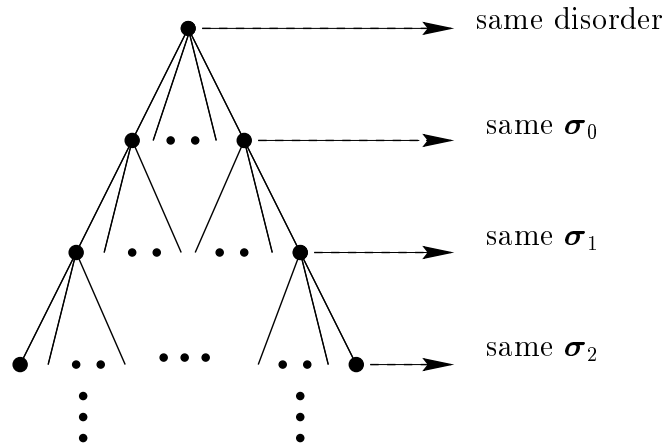
Parisi's L -level RSB scheme,

$$n = \tilde{n} \prod_{\ell=1}^L \tilde{m}_\ell, \quad m_\ell = \prod_{k=\ell}^L \tilde{m}_k = T/T_{\ell-1}$$

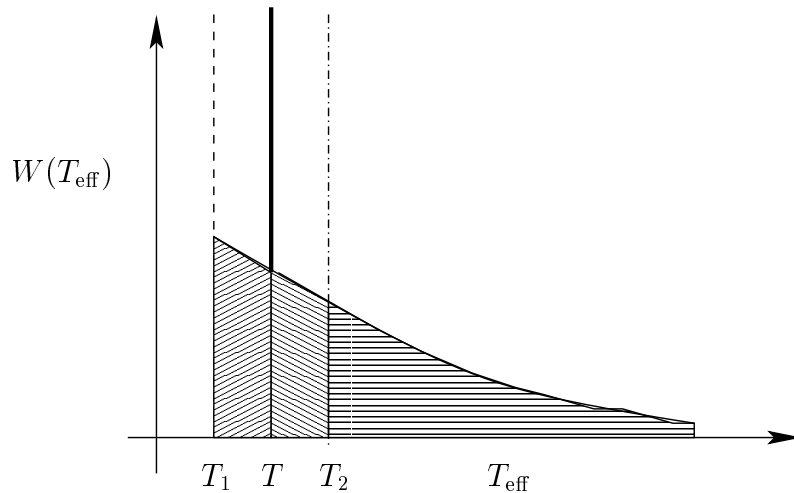
$$q_\ell = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_j \overline{\langle \dots \langle \langle \dots \langle \sigma_j \rangle_L \dots \rangle_{\ell+1} \rangle_\ell^2 \rangle_{\ell-1} \dots \rangle_0}$$

$$\epsilon_\ell = 0 \quad \rightarrow \quad S_\ell/N = 0 \quad \rightarrow \quad \partial \bar{f}_0 / \partial m_\ell = 0$$

Explanations:



- Ultra-metric organization of states in RSB:
simple consequence of hierarchy of spin clusters
- $n = \tilde{n} \prod_{\ell=1}^L \tilde{m}_\ell$: explains why $n \rightarrow 0$ was taken via $n \rightarrow \infty$
- $m_\ell = T/T_{\ell-1}$: explains $0 \leq m_0 \leq m_1 \leq m_2 \leq \dots \leq 1$
(thermodynamic stability)
- memory effects, thermo-cycling experiments:



- cool to $T_1 < T$, then heat back to T :
states of spins with $T_{\text{eff}} > T$ unchanged (memory effects)
- heat to $T_2 > T$, then cool back to T :
states of spins with $T \leq T_{\text{eff}} \leq T_2$ erased

V: NEW AVENUES IN REPLICA THEORY

- Replicas in dynamics: based on

$$\frac{\int dx w_t(x)g(x)}{\int dx w_t(x)} = \lim_{n \rightarrow 0} \int \prod_{\alpha=1}^n [dx_{\alpha} w_t(x_{\alpha})] g(x_1)$$

- replica theory of disordered quantum systems
(replicated quantum operators, Trotter formula, etc)
- Finite dimensional disordered systems:
replica field theory (not for the faint-hearted ...)
- Systems on finitely connected random graphs

Order parameter types

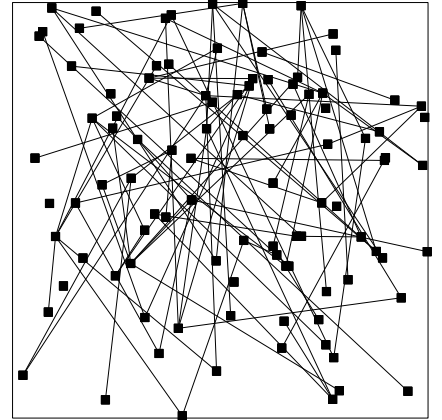
Dependence on connectivity c
(average number of bonds/spin)

<i>connectivity</i>	<i>variables</i>	<i>order param</i>	<i>RS ansatz</i>
$c = N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = \mathcal{O}(1)$	discrete	$P(\sigma_1, \dots, \sigma_n)$	functions, $P(h)$
$c = \mathcal{O}(1)$	continuous	$P(\sigma_1, \dots, \sigma_n)$	functionals, $W[\{P\}]$

Soft spin finite connectivity problems

N spins on random graph, $c_{ij} \in \{0, 1\}$

$$H = - \sum_{i < j} c_{ij} J_{ij} \sigma_i \sigma_j + \sum_i V(\sigma_i)$$



$N = 100, c = 2$

- $P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + (1 - \frac{c}{N}) \delta_{c_{ij},0}, \quad c = \mathcal{O}(N^0)$
- indep random interaction energies distributed according to $P[J]$
- disorder: $\{c_{ij}, J_{ij}\}$

$$\begin{aligned} \bar{f} &= - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{\beta N n} \log \bar{Z}^n \\ &= \dots\dots\dots \\ &= \lim_{n \rightarrow 0} \frac{1}{\beta n} \text{extr} \left\{ \frac{1}{2} c \int d\boldsymbol{\sigma} d\boldsymbol{\sigma}' P(\boldsymbol{\sigma}) P(\boldsymbol{\sigma}') [\int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'} - 1] \right. \\ &\quad \left. - \log \int d\boldsymbol{\sigma} e^{c \int d\boldsymbol{\sigma}' P(\boldsymbol{\sigma}') [\int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'} - 1] - \beta \sum_{\alpha} V(\sigma_{\alpha})} \right\} \end{aligned}$$

order parameter:

$$P(\sigma_1, \dots, \sigma_n) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\prod_{\alpha} \delta[\sigma_{\alpha} - \sigma_i^{\alpha}]}$$

note:

$$m_{\alpha} = \int d\boldsymbol{\sigma} P(\boldsymbol{\sigma}) \sigma_{\alpha} \quad q_{\alpha\beta} = \int d\boldsymbol{\sigma} P(\boldsymbol{\sigma}) \sigma_{\alpha} \sigma_{\beta}$$

saddle-point equation:

$$P(\boldsymbol{\sigma}) = \frac{e^c \int d\boldsymbol{\sigma}' P(\boldsymbol{\sigma}') [\int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}' - 1}]^{-\beta \sum_{\alpha} V(\sigma_{\alpha})}}{\int d\boldsymbol{\sigma}' e^c \int d\boldsymbol{\sigma}'' P(\boldsymbol{\sigma}'') [\int dJ P(J) e^{\beta J \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'' - 1}]^{-\beta \sum_{\alpha} V(\sigma'_{\alpha})}}$$

Replica symmetry

$$P_{\text{RS}}(\boldsymbol{\sigma}) = \int \{dP\} W[\{P\}] \prod_{\alpha} P(\sigma_{\alpha})$$

RS order parameter: functional $W[\{P\}]$

insert into eqns, take $n \rightarrow 0$:

RS saddle-point eqn:

$$W[\{P\}] = \sum_{\ell \geq 0} \frac{c^{\ell}}{\ell!} e^{-c} \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] dJ_k P(J_k)] \\ \times \prod_{\sigma} \delta \left[P(\sigma) - \frac{e^{-\beta V(\sigma)} \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma \sigma'}}{\int d\sigma'' e^{-\beta V(\sigma'')} \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma'' \sigma'}} \right]$$

RS free energy:

$$\bar{f}_{\text{RS}} = \frac{c}{2\beta} \int \{dP_1 dP_2\} W[\{P_1\}] W[\{P_2\}] \int dJ P(J) \\ \times \log \left[\int d\sigma d\sigma' P_1(\sigma) P_2(\sigma') e^{\beta J \sigma \sigma'} \right] \\ - \frac{1}{\beta} \sum_{\ell \geq 0} \frac{c^{\ell}}{\ell!} e^{-c} \int \prod_{k=1}^{\ell} [\{dP_k\} W[\{P_k\}] dJ_k P(J_k)] \\ \times \log \left[\int d\sigma e^{-\beta V(\sigma)} \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma \sigma'} \right]$$

Textbooks

- replica theory a la Parisi and its applications, with many reprints of original papers:
M Mezard, G Parisi and MA Virasoro 1987 *Spin Glass Theory and Beyond* (World Scientific)
- spin glass theory, including replica theory and generating functional analysis:
Fisher KH and Hertz J 1991 *Spin Glasses* (Cambridge UP)
- interdisciplinary applications of replica theory:
H Nishimori 2001 *Statistical Physics of Spin Glasses and Information Processing* (Oxford UP)

More recent papers

- physical meaning of replica dimension:
ACC Coolen, RW Penney and D Sherrington 1993 *Phys. Rev. B* **48** 16116
- physics behind Parisi scheme:
J Van Mourik and ACC Coolen 2001 *J. Phys. A* **34** L111
- dynamics with replicas:
ACC Coolen, SN Loughton and D Sherrington 1996 *Phys. Rev. B* **53** 8184
- finite connectivity replica theory:
I Kanter I and H Sompolinsky 1987 *Phys. Rev. Lett.* **58** 164