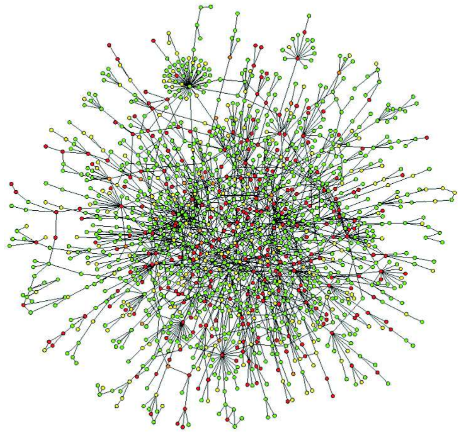
Dynamics on Finitely Connected Random Graphs

ACC Coolen – King's College London



I. General background

Finitely connected random graphs

Spin models on finitely connected random graphs

Equilibrium replica theory

II. Analysis of spin dynamics on random graphs

Spherical models

Generating functional analysis

Dynamical replica theory

III. New directions

with:

JPL Hatchett

T Nikoletopoulos

I Pérez-Castillo

NS Skantzos

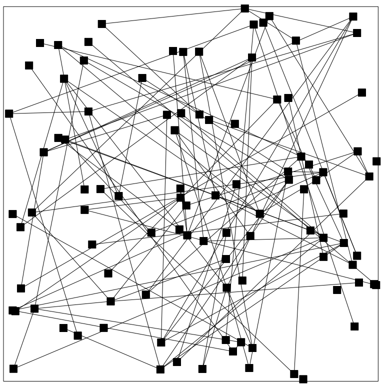
B Wemmenhove

I. General background

Finitely connected random graphs

Spin models on finitely connected random graphs

Equilibrium replica theory

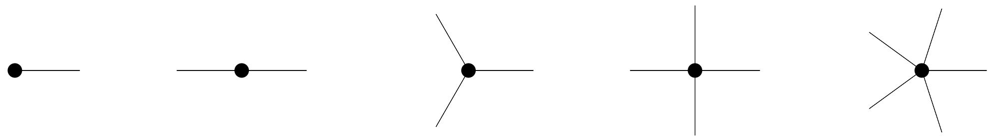


Finitely connected random graphs

- ⋆ *nodes and links:* nodes : $i = 1, \ldots, N$

  links : $c_{ij} \in \{0, 1\}$ $\quad$ $c_{ij} = 1$: link $j \to i$ present; $\quad$ $c_{ij} = 0$: link $j \to i$ absent

- ⋆ *degree k of a node:*
  total nr of links to that node



$k = 1$ $\quad$ $k = 2$ $\quad$ $k = 3$ $\quad$ $k = 4$ $\quad$ $k = 5$

degree distribution: $P(k)$
average connectivity:

$$c = \frac{1}{N}\sum_{i=1}^{N} k_i = \sum_{k\geq 0} P(k)k$$

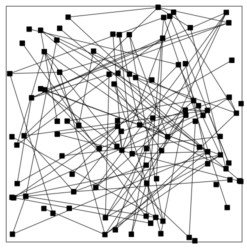
- ⋆ *clustering coeff of node i:*

$$C_i = \frac{\text{actual nr of links amongst the } k_i \text{ neighbours of } i}{\text{possible nr of links amongst the } k_i \text{ neighbours of } i}$$

- ⋆ ℓ_{ij}*: length of shortest path connecting nodes* (i, j)
  distance distribution: $\Pi(\ell)$
  mean path-length:

$$\overline{\ell} = \sum_{\ell\geq 0} \Pi(\ell)\ell$$

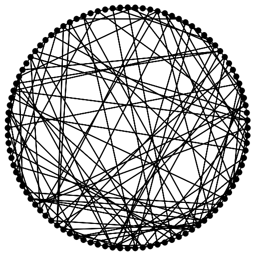
Examples



- *Poissonnian (Erdös-Rényi) random graphs*
  for each pair (i, j): form a link with probability c/N

  k_i random for all i

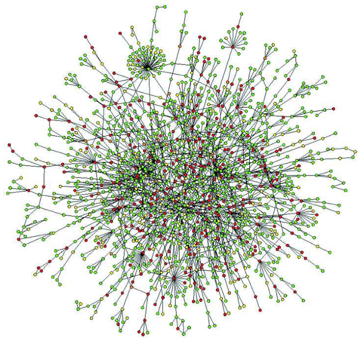
  N large : $P(k) = c^k e^{-c}/k!$ $\bar{\ell} \sim \log(N)$



- *'small-world' networks' (epidemics, etc)*
  nearest neigbours on a ring + Poissonnian random graph

  'small world effect':
  due to even very small ($c \ll 1$) number of random links

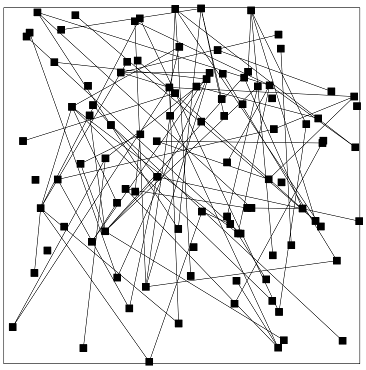
  (i) reduction of distances: $\bar{\ell} \sim \mathcal{O}(N) \;\rightarrow\; \bar{\ell} \sim \mathcal{O}(\log N)$
  (ii) greater robustness of processes against noise
  (iii) phase transitions



- *processes on finitely connected random graphs:*
  spin models, percolation, error correcting codes, K-SAT,
  graph partitioning, graph colouring, social and economic networks,
  internet traffic, neural, proteomic and immune networks, ...

Spin models on random graphs

N spins on random graph, $c_{ij} \in \{0, 1\}$



$N = 100,\ c = 2$

$$H = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j + \sum_i V(\sigma_i)$$

- $P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0}, \quad c = \mathcal{O}(N^0)$
- *indep random bonds* J_{ij}
- *disorder:* $\mathcal{D} = \{c_{ij}, J_{ij}\}$

(Viana & Bray, 1985)

to calculate:
disorder-averaged free energy per spin

$$\overline{f} = -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\log Z(\mathcal{D})} \qquad Z(\mathcal{D}) = \mathrm{Tr}_{\boldsymbol{\sigma}} e^{-\beta H}$$

replica trick/method

$x^n = 1 + n \log x + \mathcal{O}(n^2)$

$$\overline{\log Z(\mathcal{D})} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n(\mathcal{D})}$$

n integer:

$$Z^n(\mathcal{D}) = \left[\mathrm{Tr}_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}\right]^n = \mathrm{Tr}_{\boldsymbol{\sigma}^1} \ldots \mathrm{Tr}_{\boldsymbol{\sigma}^n} e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}$$

partition function of n independent replicas of system *disorder average:*

$$\overline{f} = -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N n}\log\left[\mathrm{Tr}_{\boldsymbol{\sigma}^1}\dots\mathrm{Tr}_{\boldsymbol{\sigma}^n}e^{-\beta H_{\mathrm{eff}}(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n)}\right]$$

$$H_{\mathrm{eff}}(\boldsymbol{\sigma}^1,\dots,\boldsymbol{\sigma}^n) = -\beta^{-1}\log\overline{e^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}$$

n interacting replicas, no disorder

exchange $N\to\infty$ and $n\to 0$

replica order parameters

	spins	*order param*	*RS*	*1RSB*	*∞RSB*
$c=\mathcal{O}(N)$	*discrete*	$\{q_{\alpha\beta}\}$	q	q_0,q_1	$P(q)$
$c=\mathcal{O}(N)$	*continuous*	$\{q_{\alpha\beta}\}$	Q,q	Q,q_0,q_1	$P(q)$
$1\ll c\ll N$	*discrete*	$\{q_{\alpha\beta}\}$	q	q_0,q_1	$P(q)$
$1\ll c\ll N$	*continuous*	$\{q_{\alpha\beta}\}$	Q,q	Q,q_0,q_1	$P(q)$
$c=\mathcal{O}(1)$	*discrete*	$P(\sigma_1,\dots,\sigma_n)$	$P(h)$	$W[\{P\}]$	
$c=\mathcal{O}(1)$	*continuous*	$P(\sigma_1,\dots,\sigma_n)$	$W[\{P\}]$		

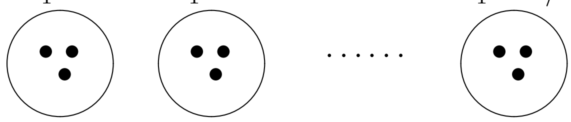
$q_{\alpha\beta}=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i^\alpha\sigma_i^\beta\rangle}$

$P(\sigma_1,\dots,\sigma_n)=$
$\quad\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\delta_{\sigma_\alpha,\sigma_i^\alpha}\dots\delta_{\sigma_n,\sigma_i^n}\rangle}$

1RSB: divide n replicas into subsets of size m

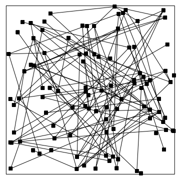
$\alpha\to(\alpha_1,\alpha_2)$	$\alpha_1=1\dots n/m$	subset label
	$\alpha_2=1\dots m$	internal label

$\alpha_1=1$ $\quad$ $\alpha_1=2$ $\quad\quad$ $\alpha_1=n/m$



Example:

Finitely connected Ising system on Poissonnian random graph:



$$H = -\frac{1}{c}\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

Solution, with $P(\boldsymbol{\sigma}) = P(\sigma_1, \ldots, \sigma_n)$*:*

$$\overline{f} = \lim_{n\to 0} \frac{1}{\beta n} \mathrm{extr}_{\{P\}} \left\{ \frac{1}{2} c \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}) P(\boldsymbol{\sigma}') \Big[\int dJ \; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'} - 1 \Big] - \log \sum_{\boldsymbol{\sigma}} e^{c \sum_{\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}')[\int dJ \; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'} - 1]} \right\}$$

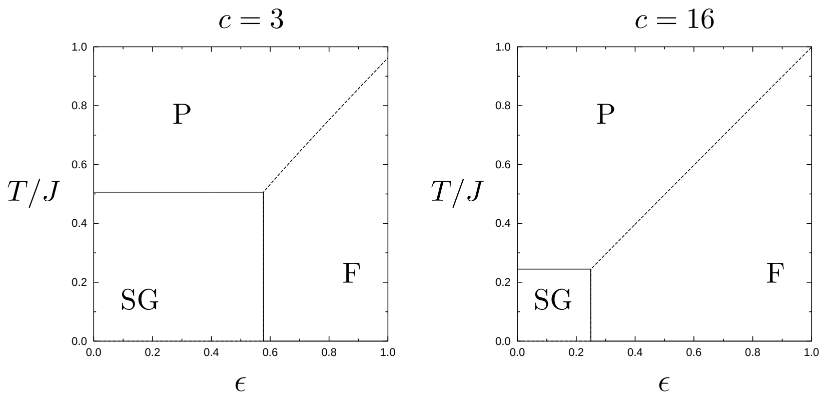
RS ansatz:

$$P_{\mathrm{RS}}(\sigma_1, \ldots, \sigma_n) = \int dh \; W(h) \frac{e^{\beta h \sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h]^n} \qquad W(h):\; \text{effective field distr}$$

$$W(h) = \sum_{k\geq 0} \frac{e^{-c} c^k}{k!} \int \prod_{\ell=1}^{k} [dJ_\ell dh'_\ell P(J_\ell) W(h'_\ell)] \, \delta\Big[h - \frac{1}{\beta} \sum_{\ell \leq k} \mathrm{atanh}[\tanh(\beta h'_\ell) \tanh(\frac{\beta}{c} J_\ell)] \Big]$$

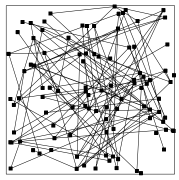
$J_{ij} \in \{-J, J\}$

$P(\pm J) = \frac{1}{2}(1 \pm \epsilon)$



Example:

Finitely connected Ising system on random graph with degree distr $p(k)$



$$H = -\frac{1}{c}\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

Solution, with $P(\boldsymbol{\sigma}) = P(\sigma_1, \ldots, \sigma_n)$*:*

$$\overline{f} = \lim_{n\to 0} \frac{1}{\beta n} \mathrm{extr}_{\{P\}} \left\{ \frac{1}{2} c \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}) P(\boldsymbol{\sigma}') \int dJ \; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'} - \sum_{k\geq 0} p(k) \log \Big[\sum_{\boldsymbol{\sigma}} \Big(\sum_{\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}') \int dJ \; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'} \Big)^k \Big] \right\}$$

RS ansatz:

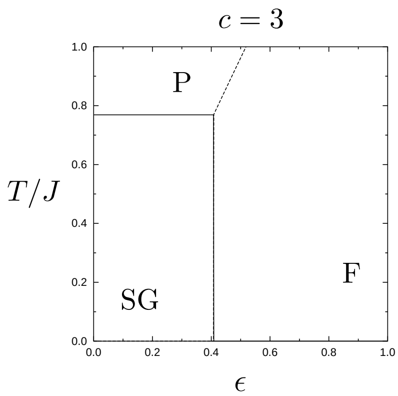
$$P_{\mathrm{RS}}(\sigma_1, \ldots, \sigma_n) = \int dh \; W(h) \frac{e^{\beta h \sum_\alpha \sigma_\alpha}}{[2\cosh(\beta h]^n} \qquad W(h): \text{ effective field distr}$$

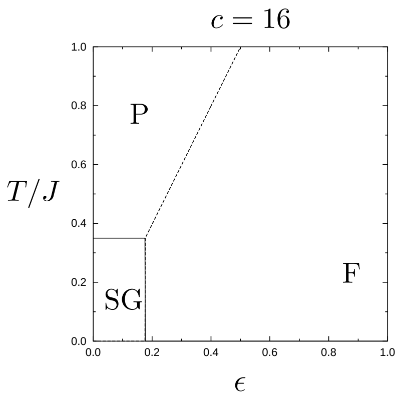
$$W(h) = \sum_{k\geq 0} \frac{p(k+1)(k+1)}{c} \int \prod_{\ell=1}^{k} [dJ_\ell dh'_\ell P(J_\ell) W(h'_\ell)] \, \delta\Big[h - \frac{1}{\beta} \sum_{\ell \leq k} \mathrm{atanh}[\tanh(\beta h'_\ell) \tanh(\frac{\beta}{c} J_\ell)]\Big]$$

$$p(k) = Ce^{-\lambda k}$$

$$J_{ij} \in \{-J, J\}$$

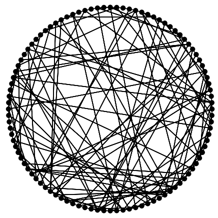
$$P(\pm J) = \tfrac{1}{2}(1 \pm \epsilon)$$





Example:

Ising system on 'small world' (ring + Poissonnian random) graph:



$$H = -J_0 \sum_i \sigma_i \sigma_{i+1} - \frac{1}{c} \sum_{i<j} J_{ij} c_{ij} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

Solution, with $P(\boldsymbol{\sigma}) = P(\sigma_1, \ldots, \sigma_n)$:

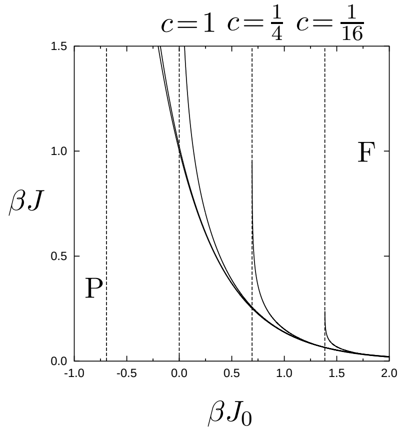
$$\overline{f} = \lim_{n\to 0} \frac{1}{n\beta} \mathrm{extr}_{\{P\}} \left\{ \frac{c}{2} \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} P(\boldsymbol{\sigma}) P(\boldsymbol{\sigma}') \Big[\int dJ \; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\sigma'} - 1 \Big] - \log \lambda_n[P] \right\}$$

$\lambda_n[P]$: largest eigenval of $2^n \times 2^n$ replicated transfer matrix $\quad \boldsymbol{T}_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}[P] = e^{\beta J_0 \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' + c \sum_{\boldsymbol{s}} P(\boldsymbol{s}) \left[\int dJ \;\; P(J) e^{\frac{\beta J}{c} \boldsymbol{\sigma}\cdot\boldsymbol{s}} - 1 \right]}$

RS ansatz:

$$W(h) = \int \mathrm{d}x\mathrm{d}y \, \Phi(x)\Psi(y)\delta[h-x-y] \qquad \Phi(x) = \mathcal{F}_\Phi\Big[x; \Phi, W\Big] \qquad \Psi(x) = \mathcal{F}_\Psi\Big[x; \Psi, W\Big]$$

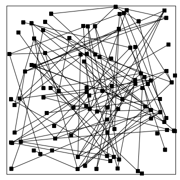
$$P(J_{ij}) = \delta(J_{ij} - J)$$



solid:
P→F transition
for $c = \frac{1}{16}, \frac{1}{4}, 1, 4, 16$

dashed:
$\beta J_0 = \log(1/\sqrt{c})$
for $c = \frac{1}{16}, \frac{1}{4}, 1, 4, 16$

Example:



Finitely connected XY spins on Poissonnian random graph:

$$H = -J\sum_{i<j} c_{ij}\cos(\phi_i-\phi_j-\omega_{ij}), \quad \phi_i\in[0,2\pi], \quad \omega_{ij}\ \text{random},\ P(-\omega)=P(\omega)$$

Solution, with $P(\boldsymbol{\phi})=P(\phi_1,\ldots,\phi_n)$:

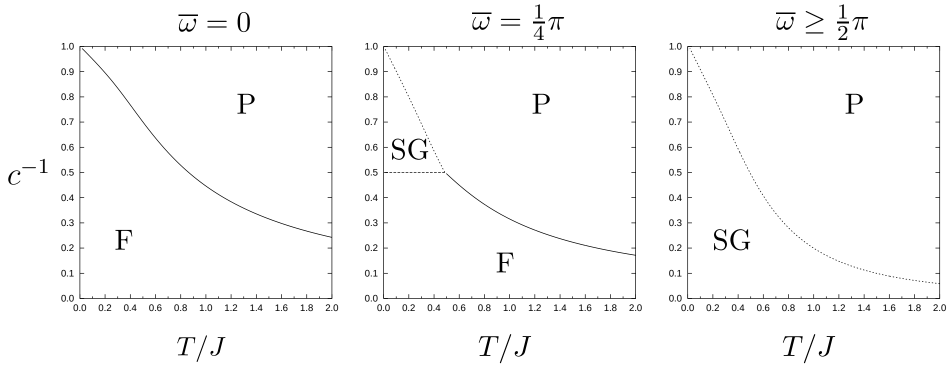
$$\begin{aligned}\overline{f} \;&=\; \lim_{n\to 0}\frac{1}{\beta n}\mathrm{extr}_{\{P\}}\Big\{\frac{1}{2}c\int\! d\boldsymbol{\phi}d\boldsymbol{\phi}'P(\boldsymbol{\phi})P(\boldsymbol{\phi}')\Big[\int\! d\omega P(\omega)e^{\beta J\sum_\alpha \cos(\phi^\alpha-\phi^{\alpha\prime}-\omega)}-1\Big] \\ &\qquad -\log\int\! d\boldsymbol{\phi}\ e^{c\int d\boldsymbol{\phi}' P(\boldsymbol{\phi}')[\int d\omega\ P(\omega)e^{\beta J\sum_\alpha\cos(\phi^\alpha-\phi^{\alpha\prime}-\omega)}-1]}\Big\}\end{aligned}$$

RS ansatz:

$$P_{\mathrm{RS}}(\phi_1,\ldots,\phi_n)=\int\!\{dP\}\ W[\{P\}]\ \prod_{\alpha=1}^n P(\phi_\alpha) \qquad \text{functional measure}\ \ W[\{P\}]$$

$$W[\{P\}]\;=\;\sum_{k\geq 0}\frac{e^{-c}c^k}{k!}\int\prod_{\ell\leq k}[\{dP_\ell\}W[\{P_\ell\}]d\omega_\ell P(\omega_\ell)]\prod_{\phi\in[0,2\pi]}\delta\left[P(\phi)-\frac{\prod_{\ell=1}^k\int d\phi' P_\ell(\phi')e^{\beta J\cos(\phi-\phi'-\omega_\ell)}}{\int d\phi''\prod_{\ell=1}^k\int d\phi' P_\ell(\phi')e^{\beta J\cos(\phi''-\phi'-\omega_\ell)}}\right]$$

$$P(\omega)=\tfrac{1}{2}\delta(\omega-\overline{\omega})+\tfrac{1}{2}\delta(\omega+\overline{\omega})$$

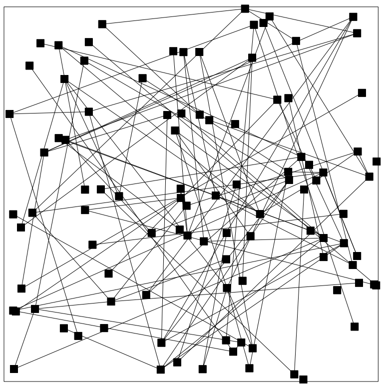


II. Analysis of spin dynamics on random graphs

Spherical models

Generating functional analysis

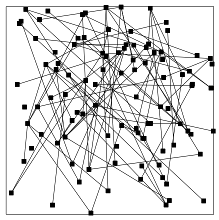
Dynamical replica theory



more information than statics
phase diagrams for non-equilibrium systems

Spherical models

(G Semerjian, L Cugliandolo, Europhys. Lett. 61 2003)



finitely connected graph, real-valued spins constrained on the sphere:

$$\begin{aligned}\frac{d}{dt}\sigma_i(t) &= \sum_j c_{ij}J_{ij}\sigma_j(t) + \mu(t)\sigma_i(t) + \xi_i(t) \\ &\quad \mu(t):\ \ \text{enforces spherical constraint } \langle \boldsymbol{\sigma}^2(t)\rangle = N \\ &\quad \xi_i(t):\ \ \text{Gaussian noise, } \langle \xi_i(t)\rangle = 0,\ \langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t')\end{aligned}$$

Solution:

- *rotate to eigen-basis of* $\{c_{ij}J_{ij}\}$*:*

$$\begin{aligned}\frac{d}{dt}\sigma_\lambda(t) &= (\lambda + \mu(t))\sigma_\lambda(t) + \xi_\lambda(t) \\ &\quad \xi_\lambda(t):\ \ \text{Gaussian noise, } \langle \xi_\lambda(t)\rangle = 0,\ \langle \xi_\lambda(t)\xi_{\lambda'}(t')\rangle = 2T\delta_{\lambda\lambda'}\delta(t-t')\end{aligned}$$

- *solve microscopic eqns:*

$$\sigma_\lambda(t)\sqrt{\Gamma(t)} = \sigma_\lambda(0)e^{\lambda t} + \int_0^t ds\ e^{\lambda(t-s)}\sqrt{\Gamma(s)}\ \xi_\lambda(s) \qquad \Gamma(t) = e^{-2\int_0^t dt'\mu(t')}$$

- *find* $\Gamma(t)$ *via* $\langle \boldsymbol{\sigma}^2(t)\rangle = N$*:*

$$\frac{1}{N}\sum_\lambda \sigma_\lambda^2(0)e^{2\lambda t} + 2T\int d\lambda\ \varrho(\lambda)\int_0^t ds\ e^{2\lambda(t-s)}\Gamma(s) = \Gamma(t)$$

- *calculate observables, e.g.*

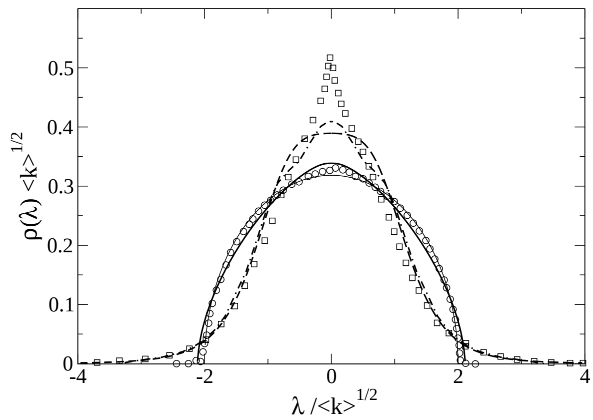
$$C(t,t') = \frac{1}{N}\sum_\lambda \frac{\sigma_\lambda^2(0)e^{\lambda(t+t')}}{\sqrt{\Gamma(t)\Gamma(t')}} + \frac{2T}{\sqrt{\Gamma(t)\Gamma(t')}}\int_0^{\min(t,t')} ds\ \Gamma(s)\int d\lambda\ \varrho(\lambda)e^{\lambda(t+t'-2s)}$$

problem reduced to finding

Eigenvalue spectrum of sparse random matrix

$$\varrho(\lambda) = \lim_{N\to\infty} \frac{1}{N} \sum_i \delta[\lambda - \lambda_i]$$

λ_i : ev of matrix $\boldsymbol{A}$ with entries $A_{ij} = c_{ij} J_{ij}$



Examples: $J_{ij} = 1$ *for all* (i,j)
(Dorogovtsev et al, 2004)

$\circ$: *Erdös-Rényi,* $p(k) = e^{-c}c^k/k!$
$\square$: *scale-free,* $p(k) \sim k^{-3},\ k_{\min} = 5$

methods:

- *cluster expansions for c small (tree-like graphs)*
- *path counting for c small (tree-like graphs)*
- *replica method (general):*

$$\overline{\varrho(\lambda)} = \lim_{N\to\infty} \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im} \frac{\partial}{\partial\lambda} \overline{\log Z(\lambda+i\varepsilon)} \qquad Z(\lambda) = \int d\boldsymbol{\phi}\ e^{-\frac{1}{2}i\boldsymbol{\phi}\cdot(\boldsymbol{A}-\lambda\mathbb{1})\boldsymbol{\phi}}$$

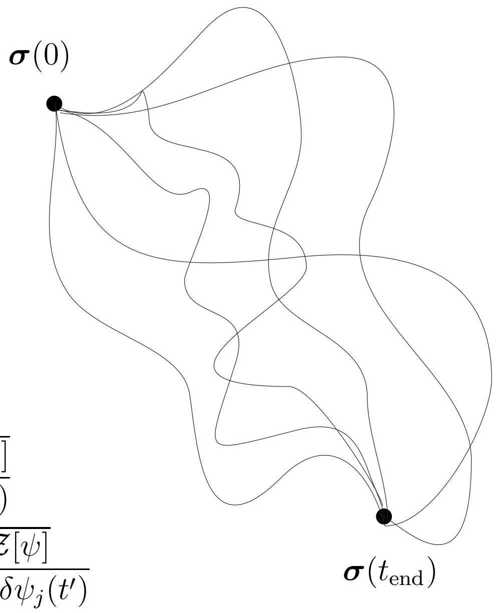
$$\overline{\log Z(\lambda+i\varepsilon)} \ \to\ \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n(\lambda+i\varepsilon)}$$

results for Erdös-Rényi graphs:

symmetric central band, similar to semi-circular law
tails of the form $\sim \exp[-c\lambda^2 \log(\lambda^2)]$ *as* $\lambda \to \pm\infty$

Generating functional analysis ('dynamic mean field theory')

consider all possible paths of N-spin system $\{\boldsymbol{\sigma}(t)\}$ through phase space



target:
generating functional

$$\overline{\mathcal{Z}[\psi]} = \overline{\left\langle e^{i\int_0^t ds \sum_{i=1}^N \psi_i(s)\sigma_i(s)} \right\rangle_{\mathrm{paths}}}$$

'generates' all relevant macroscopic multiple-time observables via (functional) differentiation, e.g.

$$\begin{aligned}\overline{\langle \sigma_i(t)\rangle_{\mathrm{paths}}} &= -i \lim_{\psi\to 0} \frac{\delta \overline{\mathcal{Z}[\psi]}}{\delta\psi_i(t)} \\ \overline{\langle \sigma_i(t)\sigma_j(t')\rangle_{\mathrm{paths}}} &= -\lim_{\psi\to 0} \frac{\delta^2 \overline{\mathcal{Z}[\psi]}}{\delta\psi_i(t)\delta\psi_j(t')}\end{aligned}$$

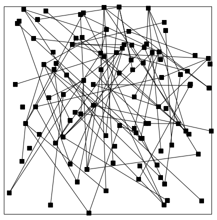
- *theory involving path integrals*
- *$c\to\infty$: closed eqns for correlation and response functions*

$$C(t,t') = \lim_{N\to\infty} \frac{1}{N}\sum_i \overline{\langle \sigma_i(t)\sigma_i(t')\rangle} \qquad G(t,t') = \lim_{N\to\infty} \frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t')}\overline{\langle \sigma_i(t)\rangle}$$

- *N-spin system on random graph $\to$ one non-disordered 'effective' spin*
- *new forces: non-trivial noise, retarded self-interaction*

Synchronous dynamics on finitely connected random graphs

(J Hatchett, B Wemmenhove, I Pérez-Castillo, T Nikoletopoulos, NS Skantzos & ACCC, J. Phys. A37 2004)



finitely connected Ising model with parallel stochastic dynamics (Markov chain):

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W_t[\boldsymbol{\sigma};\boldsymbol{\sigma}']p_t(\boldsymbol{\sigma}') \qquad W_t[\boldsymbol{\sigma};\boldsymbol{\sigma}'] = \prod_i \frac{e^{\beta\sigma_i h_i(\boldsymbol{\sigma}';t)}}{2\cosh[\beta h_i(\boldsymbol{\sigma}';t)]}$$

local fields:

$$h_i(\boldsymbol{\sigma};t) = \frac{1}{c}\sum_{j\neq i} c_{ij}J_{ij}\sigma_j + \theta_i(t)$$

Poissonnian directed graph with controlled symmetry:

$$i<j: \qquad \mathrm{Prob}(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1-\frac{c}{N})\delta_{c_{ij},0}$$
$$i>j: \qquad \mathrm{Prob}(c_{ij}) = \epsilon\delta_{c_{ij},c_{ji}} + (1-\epsilon)\Big\{\frac{c}{N}\delta_{c_{ij},1} + (1-\frac{c}{N})\delta_{c_{ij},0}\Big\}$$

$J_{ij} = J_{ji}$ *random*
detailed balance: $\epsilon = 1$

technicalities in generating functional analysis:

(i) advantage of synchronous dynamics: path sums rather than path integrals
(ii) at the end of derivation: set $\theta_i(t) \to 0$
(iii) no longer closed eqns for C *and* G
(iv) more complicated dynamic order parameters

result of generating functional analysis

$P(\boldsymbol{\sigma}|\boldsymbol{\theta})$: *fraction of sites* i *which exhibit single spin path* $\boldsymbol{\sigma}=(\sigma(0),\sigma(1),\sigma(2),\ldots)$
given a local field perturbation path $\boldsymbol{\theta}=(\theta(0),\theta(1),\theta(2),\ldots)$

to be solved from
effective single spin problem:

$$
\begin{aligned}
P(\boldsymbol{\sigma}|\boldsymbol{\theta}) \;=\; & p_0(\sigma(0))\sum_{k\geq 0}\frac{e^{-c}c^k}{k!}\int\! dJ_1P(J_1)\ldots dJ_kP(J_k)\sum_{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_k} \\
& \times\prod_{\ell=1}^{k}\Big[\epsilon P(\boldsymbol{\sigma}_\ell|\frac{J_\ell}{c}\boldsymbol{\sigma})+(1-\epsilon)P(\boldsymbol{\sigma}_\ell|\mathbf{0})\Big]\;\prod_t\frac{e^{\beta\sigma(t+1)[\theta(t)+\frac{1}{c}\sum_{\ell\leq k}J_\ell\sigma_\ell(t)]}}{2\cosh[\beta[\theta(t)+\frac{1}{c}\sum_{\ell\leq k}J_\ell\sigma_\ell(t)]]}
\end{aligned}
$$

exact, but very hard to solve in practice ...
nr of order parameters grows exponentially with $t_{\max}$ *...*

exception:

- $\epsilon=0$: *asymmetric graphs*
  can be reduced to

$$
p_{t+1}(\sigma)=\sum_{k\geq 0}\frac{e^{-c}c^k}{k!}\int\! dJ_1P(J_1)\ldots dJ_kP(J_k)\sum_{\sigma_1\ldots\sigma_k}p_t(\sigma_1)\ldots p_t(\sigma_\ell)\;\frac{e^{\frac{\beta}{c}\sigma\sum_{0<\ell\leq k}J_\ell\sigma_\ell}}{2\cosh(\frac{\beta}{c}\sum_{0<\ell\leq k}J_\ell\sigma_\ell)}
$$

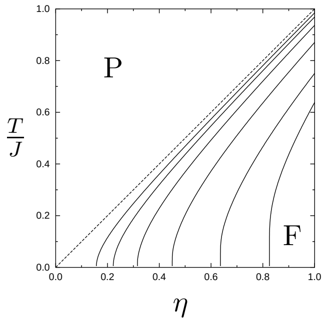
  (as in Derrida, Gardner & Zippelius, 1987)

Some applications:

- *asymmetric Poissonnian graphs,* $\epsilon = 0$
  random bonds, $P(J') = \frac{1}{2}(1+\eta)\delta(J'-J) + \frac{1}{2}(1-\eta)\delta(J'+J)$

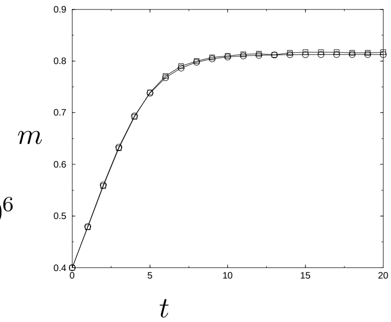
phase diagram

P→F transition
lines for
$c=2, 4, 8, 16, 32, 64, \infty$



dynamics

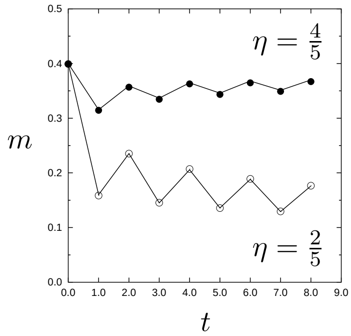
$m = \frac{1}{N}\sum_i \sigma_i$
at $c=5$, $\eta=1$, $T/J=\frac{1}{2}$
simulations: $N=16.10^6$



- *symmetric Poissonnian graphs,* $\epsilon = 1$
  random bonds, $P(J') = \frac{1}{2}(1+\eta)\delta(J'-J) + \frac{1}{2}(1-\eta)\delta(J'+J)$

dynamics

$m = \frac{1}{N}\sum_i \sigma_i$
at $c=2$, $\eta=\frac{2}{5}, \frac{4}{5}$, $T/J=\frac{1}{3}$
simulations: $N=64.10^6$



Dynamical replica theory

objective: closed dynamical laws for macroscopic observables
e.g. for Markov chains:

- *observables:* $\Omega_\mu(\boldsymbol{\sigma}) = \mathcal{O}(1)$

$$\boldsymbol{\Omega} = (\Omega_1, \ldots, \Omega_L): \qquad P_{t+1}(\boldsymbol{\Omega}) = \int d\boldsymbol{\Omega}' \, \mathcal{W}_t(\boldsymbol{\Omega}, \boldsymbol{\Omega}') P_t(\boldsymbol{\Omega}') \qquad \mathcal{W}_t(\boldsymbol{\Omega}, \boldsymbol{\Omega}') = \ldots$$

  macroscopic observables: $\lim_{N\to\infty} P_{t+1}(\boldsymbol{\Omega}) = \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}_t]$

$$\boldsymbol{\Omega}_{t+1} = \lim_{N\to\infty} \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} \boldsymbol{\Omega}(\boldsymbol{\sigma}) W(\boldsymbol{\sigma}, \boldsymbol{\sigma}') p_t(\boldsymbol{\sigma}'|\boldsymbol{\Omega}_t) \qquad p_t(\boldsymbol{\sigma}|\boldsymbol{\Omega}) = \frac{p_t(\boldsymbol{\sigma})\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]}{\sum_{\boldsymbol{\sigma}'} p_t(\boldsymbol{\sigma}')\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma}')]}$$

- *(i) assume for* $N \to \infty$ *that* $\boldsymbol{\Omega}$ *is self-averaging over disorder*
  (ii) assume for $N \to \infty$ *that* $p_t(\boldsymbol{\sigma}) = p_t(\boldsymbol{\sigma}')$ *if* $\boldsymbol{\Omega}(\boldsymbol{\sigma}) = \boldsymbol{\Omega}(\boldsymbol{\sigma}')$*:*

$$p_t(\boldsymbol{\sigma}|\boldsymbol{\Omega}) \;\to\; p(\boldsymbol{\sigma}|\boldsymbol{\Omega}) = \frac{\delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma})]}{\sum_{\boldsymbol{\sigma}'} \delta[\boldsymbol{\Omega} - \boldsymbol{\Omega}(\boldsymbol{\sigma}')]} \qquad \text{closure of macroscopic laws (max entropy)}$$

- *replica identity for disorder average:*

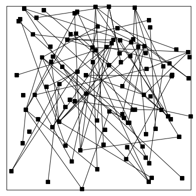
$$\frac{\sum_{\boldsymbol{\sigma}} g(\boldsymbol{\sigma}) w(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}} w(\boldsymbol{\sigma})} = \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^1 \ldots \boldsymbol{\sigma}^n} g(\boldsymbol{\sigma}^1) \prod_{\alpha} w(\boldsymbol{\sigma}^\alpha)$$

End result:
(exact if $\boldsymbol{\Omega}$ *obey closed deterministic laws,*
otherwise: best closed theory in terms of $\boldsymbol{\Omega}$*)*

$$\boldsymbol{\Omega}_{t+1} = \lim_{N\to\infty} \lim_{n\to 0} \sum_{\boldsymbol{\sigma}^0 \ldots \boldsymbol{\sigma}^n} \overline{\boldsymbol{\Omega}(\boldsymbol{\sigma}^0) W(\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1) \prod_{\alpha=1}^{n} \delta[\boldsymbol{\Omega}_t - \boldsymbol{\Omega}(\boldsymbol{\sigma}^\alpha)]}$$

Dynamical replica analysis – Ising ferromagnets on regular random graphs

(G Semerjian, M Weigt, J. Phys. A37 2004)

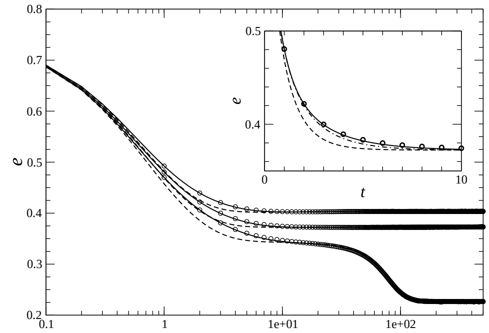


finitely connected Ising model with Glauber dynamics (master eqn):

$$\frac{d}{dt}p_t(\boldsymbol{\sigma}) = \sum_{k=1}^{N}[p_t(F_k\boldsymbol{\sigma})w_k(F_k\boldsymbol{\sigma}) - p_t(\boldsymbol{\sigma})w_k(\boldsymbol{\sigma})]$$

$$w_k(\boldsymbol{\sigma}) = \frac{1}{2}\{1-\sigma_k\tanh[\beta h_k(\boldsymbol{\sigma})]\} \qquad h_i(\boldsymbol{\sigma}) = \sum_{j\neq i} c_{ij}\sigma_j \qquad F_k\boldsymbol{\sigma} = (\sigma_1, \ldots, -\sigma_k, \ldots, \sigma_N)$$

regular random graph: $P(k) = \delta_{k,c}$



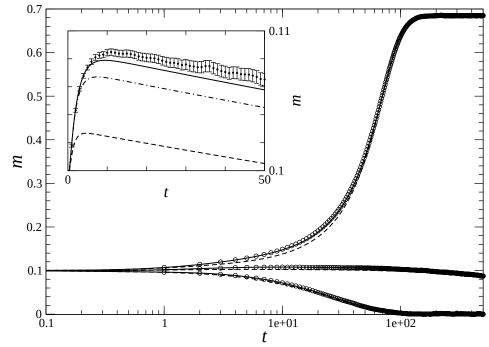
chosen observables $\{\Omega_\mu(\boldsymbol{\sigma})\}$*:*
(analysis with cavity method instead of replicas)

- *magnetization & energy per spin:*

  $m(\boldsymbol{\sigma}) = N^{-1}\sum_i \sigma_i,\; E(\boldsymbol{\sigma}) = -N^{-1}\sum_{i<j} c_{ij}\sigma_i\sigma_j$

- *joint spin-field distribution:*

  $p_{\sigma,h}(\boldsymbol{\sigma}) = N^{-1}\sum_i \delta_{\sigma,\sigma_i}\delta_{h,h_i}$
  $\sigma = \pm 1,\; h \in \{-c, -c+2, \ldots, c-2, c\}$



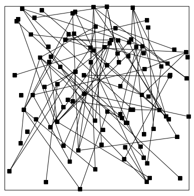
- *joint spin-field distribution for connected pairs:*

  $p_{\sigma,h;\sigma',h'}(\boldsymbol{\sigma}) = (cN)^{-1}\sum_{ij} c_{ij}\delta_{\sigma,\sigma_i}\delta_{h,h_i}\delta_{\sigma',\sigma_j}\delta_{h',h_j}$
  $\sigma, \sigma' = \pm 1,\; h, h' \in \{-c, -c+2, \ldots, c-2, c\}$

Simulations:
$N = 3.10^6$, $\beta = 1, \ln 3, 1.2$

Dynamical replica analysis – bond-disordered Ising systems on arbitrary random graphs

(J Hatchett, I Pérez-Castillo, ACCC, N Skantzos, Phys. Rev. Lett. 95 2005)



finitely connected Ising model with Glauber dynamics (master eqn):

$$\frac{d}{dt}p_t(\boldsymbol{\sigma}) = \sum_{k=1}^{N}[p_t(F_k\boldsymbol{\sigma})w_k(F_k\boldsymbol{\sigma}) - p_t(\boldsymbol{\sigma})w_k(\boldsymbol{\sigma})]$$

$$w_k(\boldsymbol{\sigma}) = \frac{1}{2}\{1-\sigma_k \tanh[\beta h_k(\boldsymbol{\sigma})]\} \qquad h_i(\boldsymbol{\sigma}) = \sum_{j\neq i} J_{ij}c_{ij}\sigma_j \qquad F_k\boldsymbol{\sigma} = (\sigma_1, \ldots, -\sigma_k, \ldots, \sigma_N)$$

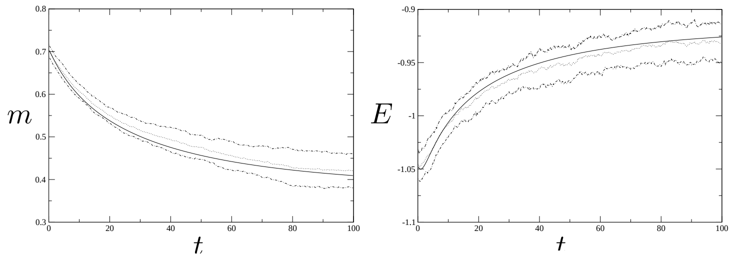
random graph & bonds, degree distr $p(k)$

observables: magnetization & energy per spin:

$m(\boldsymbol{\sigma}) = N^{-1}\sum_i \sigma_i,\ E(\boldsymbol{\sigma}) = -N^{-1}\sum_{i<j} c_{ij}J_{ij}\sigma_i\sigma_j$

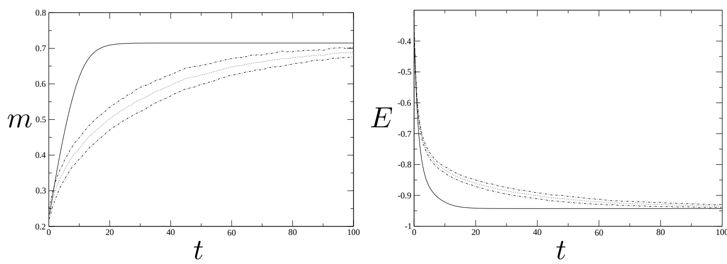
- *regular connectivity, random bonds:*

  $p(k) = \delta_{k3}$
  $P(J) = \eta\delta(J-1)+(1-\eta)\delta(J+1)$
  $\eta = 0.95,\ T = 3/2$



- *Poissonnian graph, uniform bonds:*

  $p(k) = e^{-c}c^k/k!$
  $P(J) = \delta(J-1)$
  $c = 2,\ T/J = 0.75$



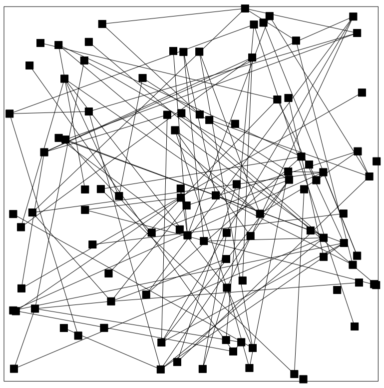
Simulations: $N = 10^4$

III. New directions

Find solutions of the (closed) GFA equations

Construct exact formulation of DRT

Develop alternative dynamical methods



Exact formulation of dynamical replica theory?

Choose p macroscopic observables $\Omega_\mu(\boldsymbol{\sigma})=\mathcal{O}(1)$

- *if deterministic & self-averaging for $N\to\infty$*
- *if $\boldsymbol{\Omega}$ obey closed laws, due to either*
  (a) $p_t(\boldsymbol{\sigma}|\boldsymbol{\Omega})$ dropping out of macroscopic laws, or
  (b) probability equi-partitioning in $\boldsymbol{\Omega}$-subshells, i.e. $p_t(\boldsymbol{\sigma}|\boldsymbol{\Omega})$ depends only on $\boldsymbol{\Omega}$
- *then put $p(\boldsymbol{\sigma}|\boldsymbol{\Omega})\sim\delta[\boldsymbol{\Omega}-\boldsymbol{\Omega}(\boldsymbol{\sigma})]$, and use replica identity for disorder average*

result, for e.g. Markov chains

$$\boldsymbol{\Omega}_{t+1}=\lim_{N\to\infty}\lim_{n\to 0}\sum_{\boldsymbol{\sigma}^0\ldots\boldsymbol{\sigma}^n}\overline{\boldsymbol{\Omega}(\boldsymbol{\sigma}^0)W(\boldsymbol{\sigma}^0,\boldsymbol{\sigma}^1)\prod_{\alpha=1}^{n}\delta[\boldsymbol{\Omega}_t-\boldsymbol{\Omega}(\boldsymbol{\sigma}^\alpha)]}$$

drawback:
theory as good/bad as one's choice of observables ...

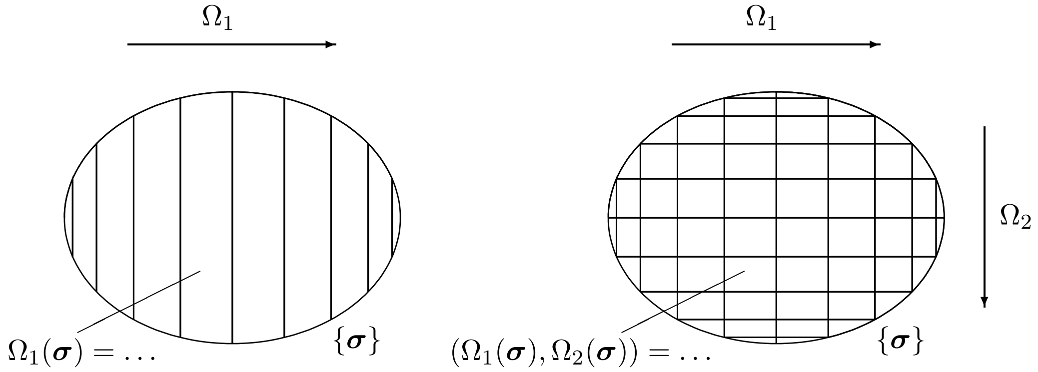
Relevant questions:

1. *can DRT in principle be exact?*
   i.e. does a set of observables $\boldsymbol{\Omega}(\boldsymbol{\sigma})$ exist that for $N\to\infty$ obey closed laws?
2. *If yes: can we devise a method with which to construct such a set?*
3. *If yes: can we define and construct the simplest such set?*
   (since if there is one, there is an infinite number ...)

tentative answers:

1. *yes*
2. *yes*
3. *working on at the moment ...*

Partitioning phase phase according to the value of macroscopic observables:



information theory picture of probability equi-partitioning in $\boldsymbol{\Omega}$*-subshells*

- *adding one observable:*

$$p(\boldsymbol{\sigma}|\boldsymbol{\Omega}) \sim \prod_{\mu=1}^{p} \delta[\Omega_\mu - \Omega_\mu(\boldsymbol{\sigma})] \quad \rightarrow \quad p_{\rm new}(\boldsymbol{\sigma}|\boldsymbol{\Omega}) \sim \prod_{\mu=1}^{p+1} \delta[\Omega_\mu - \Omega_\mu(\boldsymbol{\sigma})]$$

  canonical instead of micro-canonical version (exponential families):

$$p(\boldsymbol{\sigma}) \sim e^{N\sum_{\mu=1}^{p} \Lambda_\mu \Omega_\mu(\boldsymbol{\sigma})} \quad \rightarrow \quad p_{\rm new}(\boldsymbol{\sigma}|\boldsymbol{\Omega}) \sim e^{N\sum_{\mu=1}^{p+1} \Lambda_\mu \Omega_\mu(\boldsymbol{\sigma})}$$

- *equipartitioning exact for* $\boldsymbol{\Omega} = (\Omega, \ldots, \Omega_p)$, *for* $t = 1 \ldots t_{\max}$*:*

$$\forall t \leq t_{\max}: \quad p_t \ \text{ in manifold } \ \mathcal{M}_p = \Big\{ p | \ p(\boldsymbol{\sigma}) \sim e^{N\sum_{\mu=1}^{p} \Lambda_\mu \Omega_\mu(\boldsymbol{\sigma})}, \ \ (\Lambda_1, \ldots, \Lambda_p) \in {\rm I\!R}^p \Big\}$$

  our problem: find $\mathcal{M}_p$

- *existence & construction of $\mathbf{\Omega}(\boldsymbol{\sigma})$ such that probability equi-partitioning is exact:*

$$\Omega_0(\boldsymbol{\sigma}) = \frac{1}{N}\log p_0(\boldsymbol{\sigma}) \qquad \Omega_{\mu+1}(\boldsymbol{\sigma}) = \frac{1}{N}\log\sum_{\boldsymbol{\sigma}'} W(\boldsymbol{\sigma},\boldsymbol{\sigma}')e^{N\Omega_\mu(\boldsymbol{\sigma}')}$$

(nr of required order parameters grows as $\sim t$)

- *finding the simplest set:*
develop formalism to 'project' new probability measures onto previous ones

linear algebra of probability measures on phase space Γ:

$$\mathcal{H} = \Big\{p:\Gamma\to\mathbb{R}^+|\ p(\boldsymbol{\sigma})>0\ \forall\boldsymbol{\sigma}\in\Gamma,\ \ \sum_{\boldsymbol{\sigma}\in\Gamma}p(\boldsymbol{\sigma})=1\Big\}$$

$$(p{+}q)(\boldsymbol{\sigma}) = \frac{p(\boldsymbol{\sigma})q(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}'\in\Gamma}p(\boldsymbol{\sigma}')q(\boldsymbol{\sigma}')} \qquad (\lambda p)(\boldsymbol{\sigma}) = \frac{p^\lambda(\boldsymbol{\sigma})}{\sum_{\boldsymbol{\sigma}'\in\Gamma}p^\lambda(\boldsymbol{\sigma}')} \qquad (0.p)(\boldsymbol{\sigma}) = 2^{-N}$$

null-element $0\in\mathcal{H}$:
$p_0(\boldsymbol{\sigma}) = |\Gamma|^{-1}$ *for all* $\boldsymbol{\sigma}\in\Gamma$

inner product on $\mathcal{H}$:

$$\langle p|q\rangle = \frac{1}{|\Gamma|}\sum_{\boldsymbol{\sigma}\in\Gamma}\Big[\log p(\boldsymbol{\sigma}) - \frac{1}{|\Gamma|}\sum_{\boldsymbol{\sigma}'\in\Gamma}\log p(\boldsymbol{\sigma}')\Big]\Big[\log q(\boldsymbol{\sigma}) - \frac{1}{|\Gamma|}\sum_{\boldsymbol{\sigma}'\in\Gamma}\log q(\boldsymbol{\sigma}')\Big]$$

$$p(\boldsymbol{\sigma}) = e^{\phi(\boldsymbol{\sigma})},\ q(\boldsymbol{\sigma}) = e^{\psi(\boldsymbol{\sigma})}: \qquad \langle p|q\rangle = \langle\phi\psi\rangle_0 - \langle\phi\rangle_0\langle\psi\rangle_0 \qquad \langle\ldots\rangle_0 = \frac{1}{|\Gamma|}\sum_{\boldsymbol{\sigma}\in\Gamma}\ldots$$

Summary

- *Spin models on finitely connected random graphs:*
  interesting & nontrivial, even in equilibrium
- *In the last three years attention has turned towards their dynamics*
- *Three main approaches:*
  - *pseudo-linear spherical models*
    (easily to solve, but artificial spin variables)
  - *generating functional analysis*
    (exact, but rather nasty macroscopic laws, nr of order parameters grows as $\sim e^t$*)*
  - *dynamical replica theory*
    (relatively easy, similar to equil calculations, but not yet manifestly exact)

    exact formulation possible? (nr of order parameters would grow as $\sim t$*)*

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