(work in progress)

ACC Coolen
King's College London

- Brief intro to Minority Games
- Lookup-table versus inner product MGs
- Generalization of GFA for MGs with real histories
- Analysis of macroscopic laws
  - fake history limit
  - true history: random matrix theory

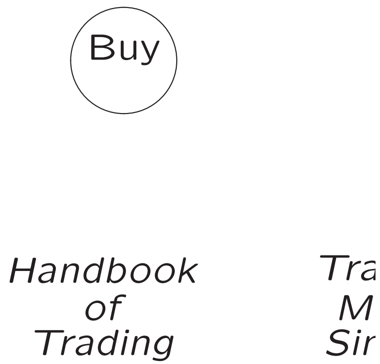
N agents, $i = 1 \ldots N$:

- At each round ℓ:
  all get info $I(\ell) \in \{I_1, \ldots, I_p\}$
  each places a bid $b_i(\ell) \in \mathbb{R}$

- Those in the **minority** group win
  $\sum_j b_j(\ell) > 0$: $\quad b_i(\ell) < 0$ wins
  $\sum_j b_j(\ell) < 0$: $\quad b_i(\ell) > 0$ wins

- Each agent i has S strategies
  converting $I(\ell)$ into bid $b_i(\ell)$

- dynamics:
  choice of strategy $a \in \{1, \ldots, S\}$,
  as a function of time, for all agents,
  based on strategies' performance



Lookup table MGs (Challet, Marsili, Zhang, ...):

info :	m step history,	$\boldsymbol{\lambda}(\ell) = (\mathrm{sgn}[A(\ell-1)], \ldots, \mathrm{s}$
strat :	lookup table,	$\mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p)$
bid :	strategy a, info $\boldsymbol{\lambda}$,	$b_i(\ell) = R_{ia}^{\boldsymbol{\lambda}}$

history window: $\Delta t = \Delta\ell/N = \mathcal{O}(N^{-1}\log N)$

Inner product MGs (Cavagna, Sherrington, Garrahar

info :	p step history,	$\boldsymbol{\lambda}(\ell) = (f[A(\ell-1)], \ldots,$
strat :	linear functional,	$\mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p)$
bid :	strategy a, info $\boldsymbol{\lambda}$,	$b_i(\ell) = p^{-\frac{1}{2}} \sum_{\mu \leq p} R_{ia}^{\mu} \lambda_{\mu}(\ell)$

history window: $\Delta t = \Delta\ell/N = \mathcal{O}(1)$

- Inner product MGs closer to how real agents predict time series (generalized linear models, ARMA, ARIMA, etc)

- Lookup-table MGs analyzed extensively using replica & GFA techniques (including real history version)

- Not true for inner product MGs

- Lookup table: history window $\mathcal{O}(N^{-1}\log N) \to 0$
  Inner product: history window $\mathcal{O}(N^0)$

history strings:

$$\boldsymbol{\lambda}(\ell) = (\mathsf{sgn}[A(\ell-1)], \ldots, \mathsf{sgn}[A(\ell-m)]) \qquad 2^m$$

$$\pi_{\boldsymbol{\lambda}} = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell \leq L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell)} \qquad \varrho(\mathsf{f}) = \lim_{N\to\infty} 2^{-m} \sum_{\boldsymbol{\lambda}} \delta\Big[$$

$\alpha < \alpha_c$

$\alpha > \alpha_c$

$\varrho(\mathsf{f})$

$\varrho(\mathsf{f})$

f

f

- Inner product MGs with real market history
  no such thing as $\varrho(\mathsf{f})$, history string is of length $\mathcal{O}$

(always $S=2$)

- decision noise:

$$\text{sgn}[q_i(\ell)] \;\to\; \sigma[q_i(\ell), z_i(\ell)] \qquad \begin{array}{ll} \text{additive}: & \sigma[q,z] = \text{sgr} \\ \text{multipl}: & \sigma[q,z] = \text{sgr} \end{array}$$

- real vs fake history:

$$A(\ell-\mu) \;\to\; (1-\zeta)A(\ell-\mu) + \zeta Z(\ell,\mu)$$

$$\begin{array}{rl} Z(\ell,\mu): & \text{Gaussian, } \langle Z\rangle = 0 \\ \text{consistent}: & Z(\ell,\mu) = Z(\ell-\mu), \quad \langle Z(\ell)Z(\ell')\rangle = S \\ \text{inconsistent}: & Z(\ell,\mu) \text{ all indep,} \quad \langle Z(\ell,\mu)Z(\ell',\mu')\rangle \end{array}$$

strategy : $\mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p) \qquad 2^m = p$

bid : $b_i(\ell) = R_{ia}^{\boldsymbol{\lambda}}$

Inner product MGs:

info : $\boldsymbol{\lambda}(\ell) = (f[A(\ell-1)], \ldots, f[A(\ell-$

strategy : $\mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p)$

bid : $b_i(\ell) = p^{-\frac{1}{2}} \sum_{\mu \leq p} R_{ia}^{\mu} \lambda_{\mu}(\ell)$

- add bid perturbation term:

$$A(t) = \frac{1}{\sqrt{N}} \sum_i b_i(t) + A_e(t)$$

[ ]: over $\{R_{ia}^{\mu}\}$

$$\overline{Z[\boldsymbol{\psi}]} = \overline{\left\langle e^{-i\sum_{i\ell}\psi_i(\ell)\sigma[q_i(\ell),z_i(\ell)]}\right\rangle}$$

manipulations:

$$\overline{Z[\boldsymbol{\psi}]} = \int \mathcal{D}C\mathcal{D}\widehat{C}\mathcal{D}G\mathcal{D}\widehat{G}\ldots\; e^{N\Psi[C,\widehat{C},G,\widehat{G},\ldots]}$$

saddle-point equations
for dynamic order parameters

$$C(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma[q_i(\ell),z_i(\ell)]\sigma[q_i(\ell'),z_i(\ell'}$$

$$G(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(\ell')}\overline{\langle\sigma[q_i(\ell),z_i(\ell)]\rangle}$$

- Effective single agent process:

$$\frac{d}{dt}q(t) \;=\; \theta(t) - \alpha \int_0^t dt' \; R(t,t') \; \sigma[q(t')] + \sqrt{\alpha} \; \eta(t)$$

Gaussian $\eta(t)$, $\langle \eta(t) \rangle = 0$, $\langle \eta(t)\eta(t')$

$$\sigma[q] = \int Dz \; \sigma[q,z]$$

- closed eqns for order parameters:

$$C(t,t') = \langle \sigma[q(t)]\sigma[q(t')] \rangle_\star \qquad G(t,t') = \frac{\delta}{\delta\theta(t')} \langle \sigma$$

- All complications: in relation between $\{R, \Sigma\}$ and
involves effective overall bid process

$$C(t,t') = C(t-t'), \qquad G(t,t') = G(t-t')$$
$$R(t,t') = R(t-t'), \qquad \Sigma(t,t') = \Sigma(t-t')$$

static order pars:

$$\chi = \int_0^\infty dt\ G(t), \qquad \chi_R = \int_0^\infty dt\ R(t)$$

$$c = \lim_{t\to\infty} C(t), \qquad \phi: \text{ fraction of 'frozen' ag}$$

given by

$$\begin{aligned}
\phi &= 1 - \mathrm{Erf}[u] \\
c &= \sigma^2[\infty]\Big\{1 - \mathrm{Erf}[u] + \frac{1}{2u^2}\mathrm{Erf}[u] - \frac{1}{u\sqrt{\pi}}e^{-u^2} \\
\chi &= \mathrm{Erf}[u]/\alpha\chi_R \\
u &= \sqrt{\alpha}\chi_R\sigma[\infty]/S_0\sqrt{2}, \qquad S_0^2 = \Sigma(\infty)
\end{aligned}$$

- $S_0^2 = \Sigma(\infty)$ and $\chi_R = \int_0^\infty dt \; R(t)$

$$\chi_R \;=\; \lim_{\delta\to 0} \Bigg\{ \overline{W}[0,0;\{A,Z\}] + \sum_{\ell=1}^{\infty} \frac{\partial}{\partial A(0)} \Big\langle\Big\langle \overline{W}[\ell,0;\{A,$$

$$S_0^2 \;=\; \lim_{\delta\to 0} \lim_{L\to\infty} \frac{\tilde{\eta}}{L^2\delta} \sum_{\ell,\ell'=1}^{L} \Big\langle\Big\langle \overline{W}[\ell,\ell';\{A,Z\}]A(\ell)A(\ell')\Big\rangle\Big\rangle$$

- Differences between models: $\overline{W}[\ell,\ell';\{A,Z\}]$
  similarity between histories observed at times ℓ anc

$$\text{lookup table}: \quad \overline{W}[\ell,\ell';\{A,Z\}] = \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}$$

$$\text{inner product}: \quad \overline{W}[\ell,\ell';\{A,Z\}] =$$

$$\frac{1}{\alpha N} \sum_{\lambda\leq p} f\left[(1-\zeta)A(\ell-\lambda) + \zeta Z(\ell,\lambda)\right] f\Big[(1-\zeta)A(\ell'$$

$$\ldots \frac{\ }{2} \sum_{\ell'<\ell} \ldots$$

Gaussian random fields:

$$\langle \phi_\ell \rangle_{\{\phi|A,Z\}} = 0, \qquad \langle \phi_\ell \phi_{\ell'} \rangle_{\{\phi|A,Z\}} \; = \; \frac{1}{2}[1 + C(\ell,\ell')]\overline{W}$$

Inner product MG with inconsistent fake market I

- Put $\zeta \to 1$,
  calculate kernels (grammatically),
  define $\kappa = \int Dz \ f^2[Sz]$

- Inner product and lookup table MGs
  differ only in characteristic amplitude,
  that can be transformed away:

$$G_{\text{IP}}(t,t') = \kappa^{-1} G_{\text{LU}}(t,t'), \quad \theta_{\text{IP}}(t) = \kappa \theta_{\text{LU}}(t), \quad \sigma^2_{\text{IF}}$$

- define random matrix $\mathbf{B}(\ell)$:

$$B_{\lambda\lambda'}(\ell) = \mathcal{F}_\lambda[\ell, A, Z]\mathcal{F}_{\lambda'}[\ell, A, Z]$$

$$\text{LU}: \qquad \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\ \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \qquad \boldsymbol{\lambda}$$

$$\text{IP}: \quad \mathcal{F}_\lambda[\ell, A, Z] = f[(1-\zeta)A(\ell-\lambda) + \zeta Z(\ell,\lambda)] \quad \lambda$$

- $\{R, \boldsymbol{\Sigma}\}$ can be written in terms of

$$\Delta_{r+1}(\ell_0, \ldots, \ell_r) = \frac{1}{p}\ \text{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\Big\rangle\Big\rangle_{\{A}$$

$$\tilde{\Delta}_{r+r'+2}(\ell_0, \ldots, \ell_r; \ell'_0, \ldots, \ell'_{r'}) =$$

$$\frac{1}{p}\ \text{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1)\mathbf{B}(\ell$$

$$\tilde{\tilde{\Delta}}_{r+r'+1}(\ell_0, \ell_1 \ldots, \ell_r; \ell_0, \ell'_1, \ldots, \ell'_{r'}) =$$

$$\frac{1}{p}\ \text{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1)\Big\rangle\Big\rangle_{\{}$$

$$\prod_{i=1}^{r} \mathbf{B}(\ell_i) \ \rightarrow \ \mathbf{B}^r(A,Z) \qquad\qquad \mathbf{B}(A,Z) = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell \leq L}$$

$$\mu_i(A,Z) \text{ ev of } \mathbf{B}(A,Z): \qquad \varrho(\mu) = \langle\langle \frac{1}{p} \sum_{i=1}^{p} \delta[\mu - \mu_i(A, Z$$

$$\Delta_{r+1}(\ell_0, \ldots, \ell_r) \ \rightarrow \ \int_0^\infty d\mu \ \varrho(\mu) \ \mu^{r+}$$

$$\tilde{\Delta}_{r+r'+2}(\ell_0, \ldots, \ell_r; \ell'_0, \ldots, \ell'_{r'}) \ \rightarrow \ \int_0^\infty d\mu \ \varrho(\mu) \ \mu^{r+}$$

$$\tilde{\tilde{\Delta}}_{r+r'+1}(\ell_0, \ell_1 \ldots, \ell_r; \ell_0, \ell'_1, \ldots, \ell'_{r'}) \ \rightarrow \ \int_0^\infty d\mu \ \varrho(\mu) \ \mu^{r+}$$

$$\chi_R = \int_0^\infty d\mu \ \varrho(\mu) \frac{\mu}{1+\mu\chi} \qquad\qquad S_0^2 = (1+c) \int_0^\infty d\mu \ \varrho(\mu)$$

eigenvalue distribution $\varrho(\mu)$ of random matrix $\mathbf{B}(A$
takes over role of history frequency distribution $\varrho($f
in lookup table MGs

- link with lookup table MG theory:

$$B_{\lambda\lambda'}(A,Z) = p\delta_{\boldsymbol{\lambda\lambda'}}\pi_{\boldsymbol{\lambda}}(A,Z)$$

$$\varrho(\mu) = \lim_{p\to\infty}\frac{1}{p}\sum_{\boldsymbol{\lambda}}\langle\langle\delta\left[\mu - p\pi_{\boldsymbol{\lambda}}(A,Z)\right]\rangle\rangle_{\{A,Z\}}$$

(recover the history frequency distr)

- Generation functional analysis of MGs can be gene
  to include lookup table & inner product as special
- Exact macroscopic theory in terms of effective sing
  and (in case of real histories) an effective overall bi
- Fake histories:
  inner product and lookup table MGs differ only in
  simple rescaling of observables, phase diagrams ide
- Real histories:
  - Short history correlation time ansatz can be gen
  - Role of history frequency distribution in lookup t
    taken over by eigenvalue spectrum of a random
    (randomness: generated by the effective bid pro
- To be finished:
  calculation of random matrix eigenvalue spectrum