Generating Functional Analysis of Minority Games with Real Histories

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Introduction to Minority Games

Real versus Fake Histories
GFA of Games with Fake Histories

GFA of MGs with Real Histories
Resulting Dynamical 'Effective Agent' Theory

Role and Calculation of History Statistics
Final Theory vs Simulations

Econophysics

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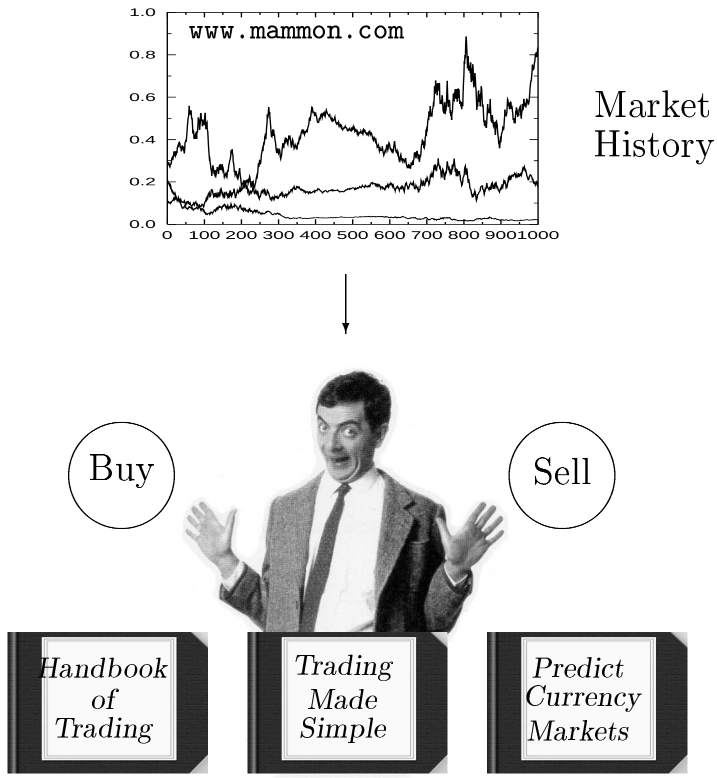
Physicists' thinking and quantitative methods applied to problems in economics and finance

- Economics largely axiomatic, and focused on optimimal stationary states in deterministic microscopic economic models
- Real world economic decisions *not* based on precise deductive reasoning and perfect information, but on *inductive* reasoning and imperfect information
- Ergo: economists do not understand the fluctuations in financial time series
- Statistical mechanical models of interacting agents in simplified 'markets', example: Minority Game (1997)

Minority Game



N agents, $i = 1 \dots N$:

- At each round ℓ:
  - all agents receive market information $I(\ell) \in \{I_1, \dots, I_p\}$
  - each takes a binary decision $\sigma_i(\ell) \in \{-1, 1\}$
- Those in the **minority** win !

$$\Sigma_j\, \sigma_j(\ell) > 0 : \quad \text{the } \sigma_i(\ell) = -1 \text{ win}$$

$$\Sigma_j\, \sigma_j(\ell) < 0 : \quad \text{the } \sigma_i(\ell) = 1 \text{ win}$$

- Each agent i has S strategies (look-up tables):

$$\boldsymbol{R}_{ia} = (R_{ia}^1, \dots, R_{ia}^p) \in \{-1, 1\}^p, \qquad a = 1 \dots S$$

Strategy a used, and $I(\ell) = I_\mu$: $\quad \sigma_i(\ell) = R_{ia}^\mu$

MG Dynamics

Choice of strategy $a \in \{1, \ldots, S\}$, as a function of time, for all agents

Agents learn to select profitable strategies, by keeping track of strategies' performance p_{ia}

$$\sum_j \sigma_j(\ell) > 0 : \qquad R_{ia}^{\mu(\ell)} = -1 \;\; \text{winning strategy}$$
$$\sum_j \sigma_j(\ell) < 0 : \qquad R_{ia}^{\mu(\ell)} = 1 \;\; \text{winning strategy}$$

Hence:

$$p_{ia}(\ell + 1) = p_{ia}(\ell) \; - \; \eta \; R_{ia}^{\mu(\ell)} \sum_j \sigma_j(\ell)$$

Choice:

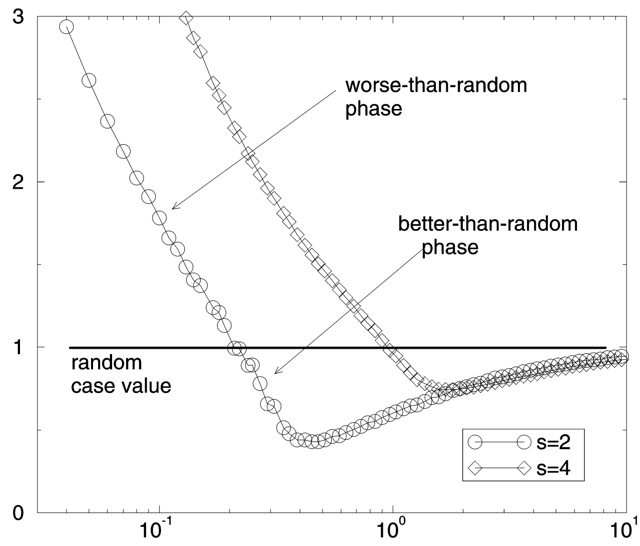
$$\tilde{a}_i(\ell) = \text{arg max }_a\{p_{ia}(\ell)\}$$

(would have performed best so far)

- External information $I(\ell)$: history of the market
- Closed microscopic equations for the $\{p_{ia}\}$ (non-Markovian !)
- Quenched disorder: realization of strategies $\{\boldsymbol{R}_{ia}\}$
- Competition & frustration: most agents must inevitably lose ...

Market volatility

$$\sigma^2 = \left\langle \left[\frac{1}{\sqrt{N}} \sum_i \sigma_i(\ell) \right]^2 \right\rangle \qquad \sigma$$



$$\alpha = p/N$$

- Non-trivial behaviour; phase-transition at α_c
- Value of S of no qualitative relevance
- Agents appear to understand & predict the market !

Simplest MG: $S = 2$ (strategies/agent)

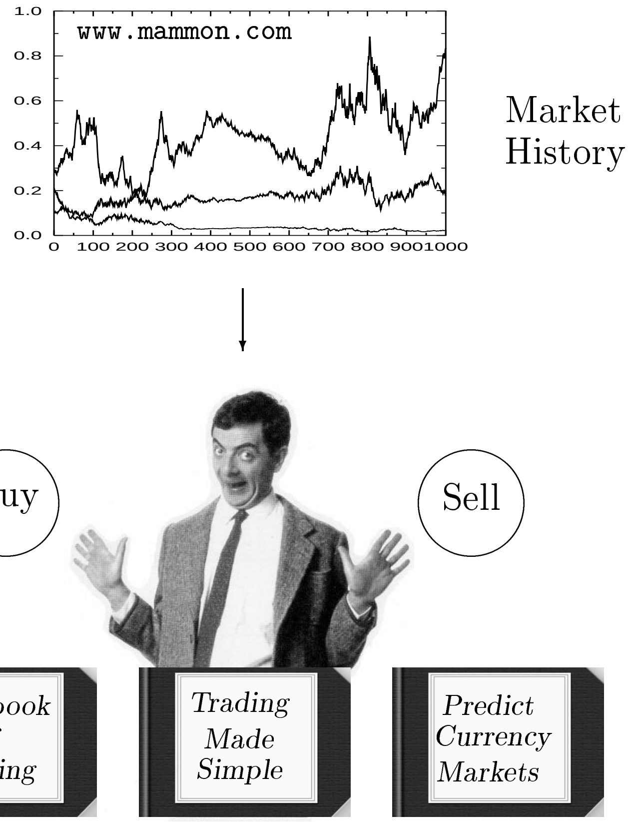
$$q_i = \frac{1}{2}[p_{i1} - p_{i2}], \qquad \xi_i^\mu = \frac{1}{2}[R_{i1}^\mu - R_{i2}^\mu], \qquad \Omega_\mu = \frac{1}{2\sqrt{N}} \sum_j [R_{j1}^\mu + R_{j2}^\mu]$$

then

$$q_i(\ell+1) = q_i(\ell) - \frac{\eta}{\sqrt{N}} \xi_i^{\mu(\ell)} \left[\Omega_{\mu(\ell)} + \frac{1}{\sqrt{N}} \sum_j \xi_j^{\mu(\ell)} \mathrm{sgn}[q_j(\ell)] \right]$$

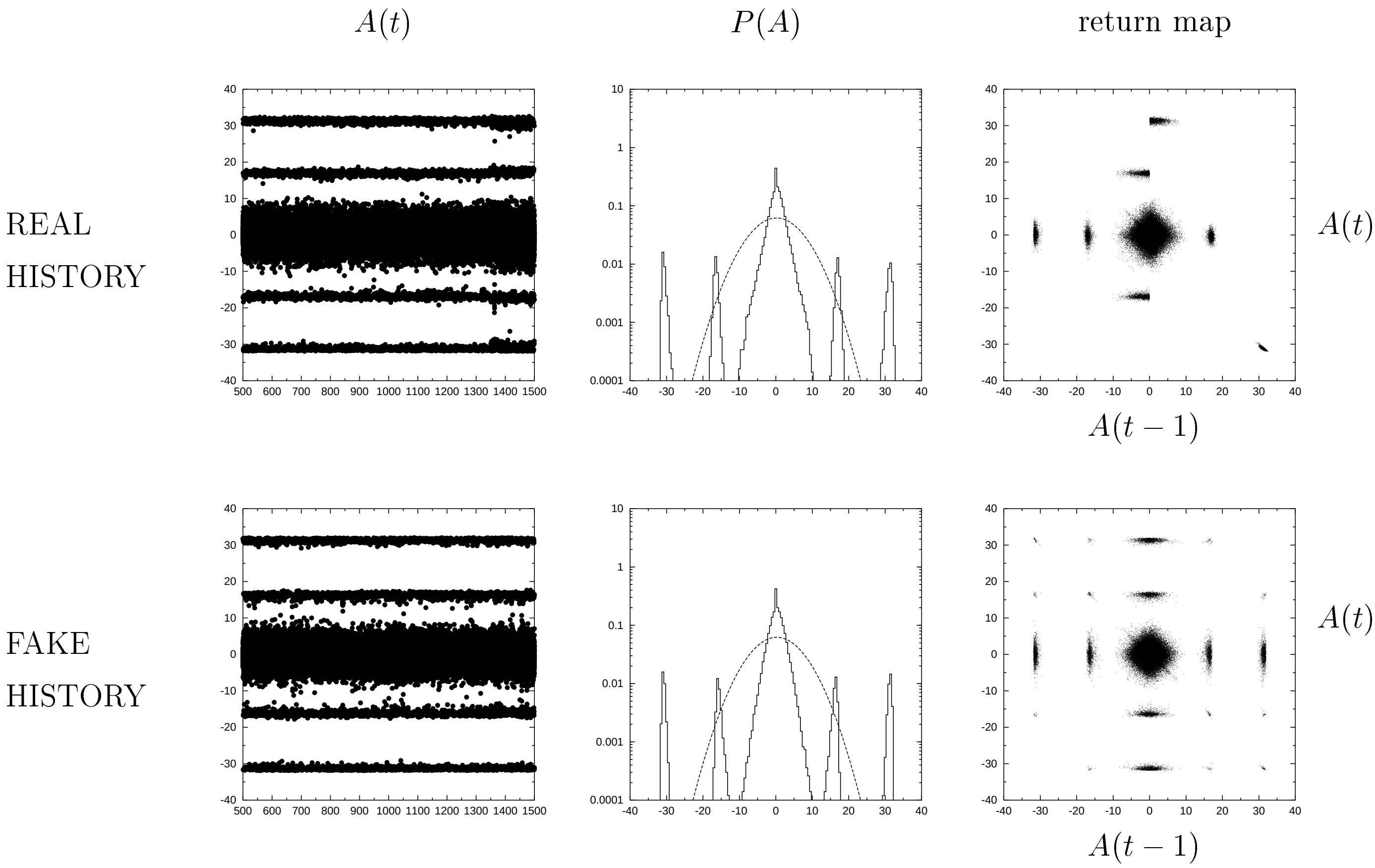
non-Markovian, memory depth $= \mathcal{O}(\log N)$

Real vs Fake Histories



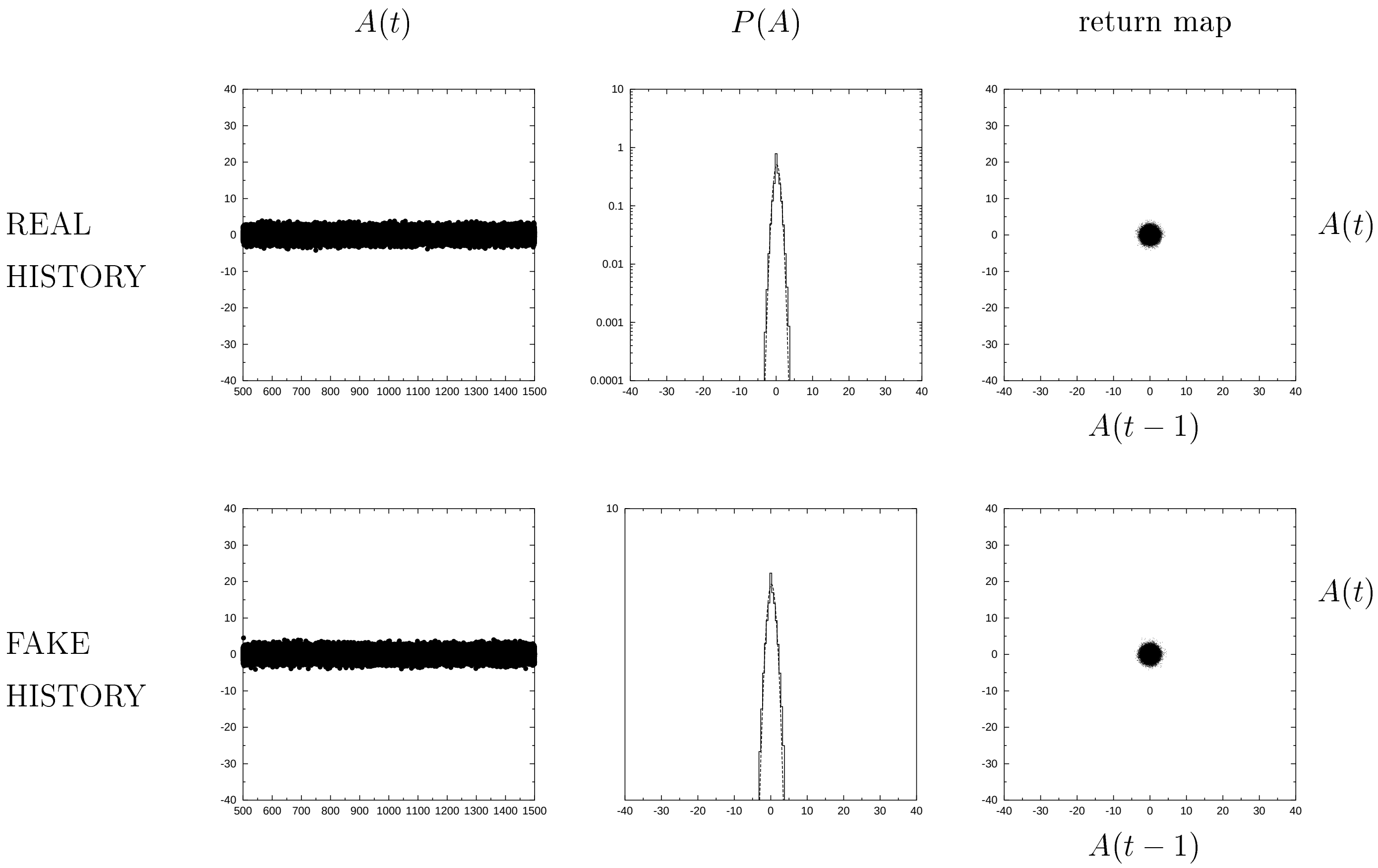
- Cavagna 1999:
  hardly any difference in volatitilies of standard MG
- Johnson et al 1999:
  big differences in MGs with other valuation update rules, or when agents do not see history strings of same length
- Challet & Marsili 2000:
  differences also in standard MG, approximate replica analysis, by 'fitting' a curve to observed history freq distribution
- Lee 2001:
  simulation study of bid periodicities due to real histories
- all other theoretical papers: fake histories only ...

Simulations of the standard MG in the non-ergodic regime $\alpha < \alpha_c$



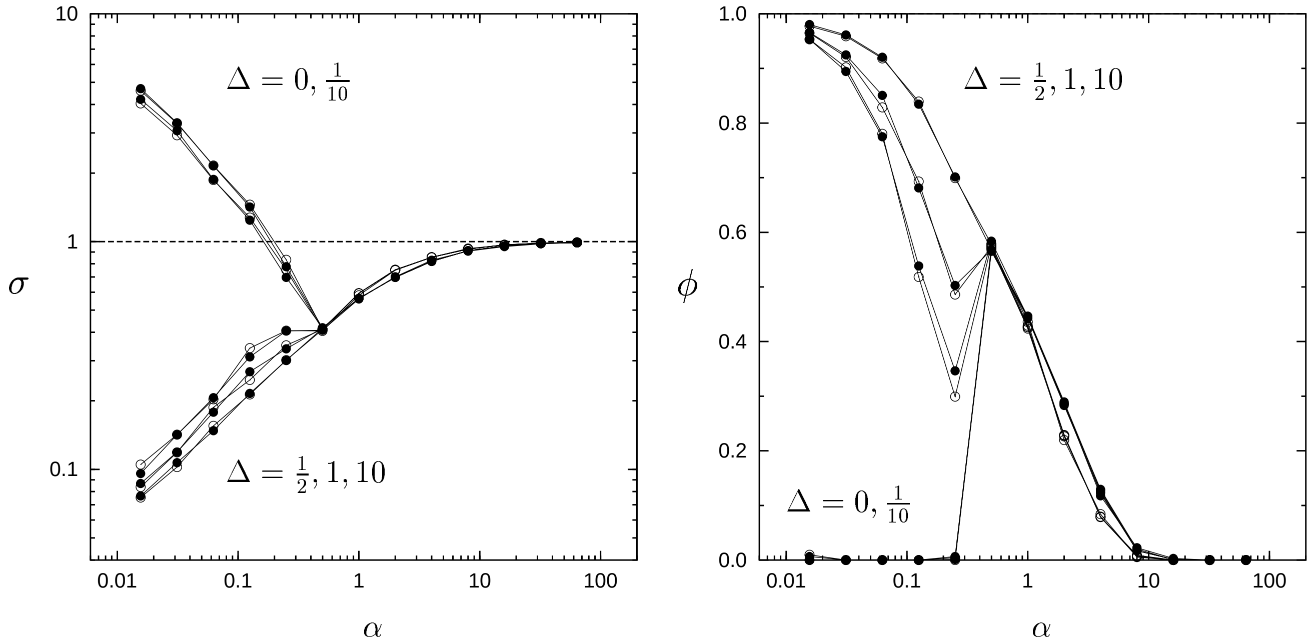
- two strategies per agent
- $M = 5$ (history depth), $N = 4097$ (nr of agents)
- binary individual bids: $b_i(t) \in \{-1, 1\}$
- overall bid: $A(t) = N^{-\frac{1}{2}} \sum_i b_i(t)$

Simulations of the standard MG in the ergodic regime $\alpha > \alpha_c$



- two strategies per agent
- $M = 16$ (history depth), $N = 4097$ (nr of agents)
- binary individual bids: $b_i(t) \in \{-1, 1\}$
- overall bid: $A(t) = N^{-\frac{1}{2}} \sum_i b_i(t)$

volatility σ and fraction ϕ of ‘frozen’ agents:



full circles: real memory
open circles: fake memory

initial conditions: $\Delta = \frac{1}{2}[p_{i1}(0) - p_{i2}(0)]$
$p_{ia}(t)$: valuation of strategy a of agent i at time t

History statistics:

History strings:

$$\boldsymbol{\lambda}(t) = (\mathrm{sgn}[A(t-1)], \ldots, \mathrm{sgn}[A(t-M)])$$

History frequency:

$$\pi_{\boldsymbol{\lambda}} = \lim_{L\to\infty} \frac{1}{L} \sum_{t=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(t)}$$

History frequency distribution:
($2^M = \alpha N$, α fixed)

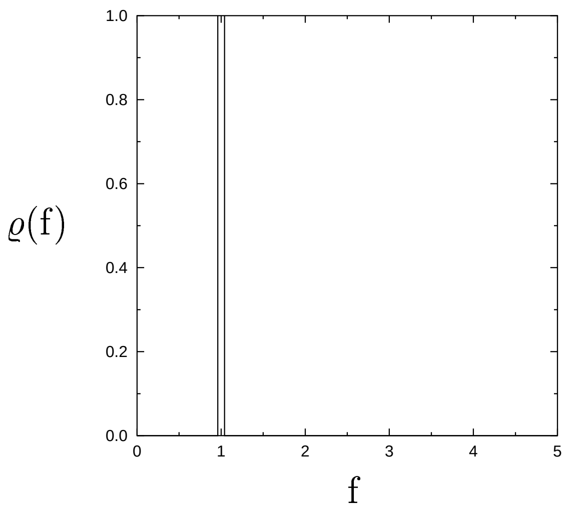
$$\varrho(\mathrm{f}) = \lim_{N\to\infty} 2^{-M} \sum_{\boldsymbol{\lambda}} \delta[\mathrm{f} - 2^M \pi_{\boldsymbol{\lambda}}]$$

fake hist:

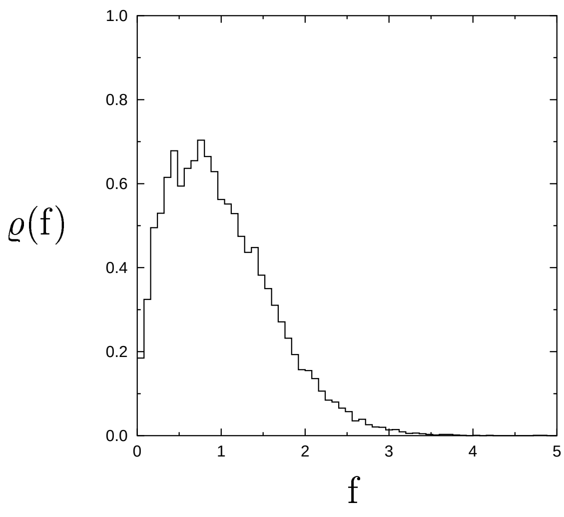
$$\pi_{\boldsymbol{\lambda}} = 2^{-M}, \qquad \varrho(\mathrm{f}) = \delta[\mathrm{f}-1]$$

real hist:

$\alpha < \alpha_c$



$\alpha > \alpha_c$



Pseudo-equilibrium Replica Analysis of MGs with Fake Histories

- neglect fluctuations in microscopic process:
  Lyapunov function H
- approximate fluctuations by effective Gaussian ones, added to gradient descent process on H
- use equilibrium statistical mechanics,
  carry out disorder average using replica theory

For:

first systematic theory for MGs
correct results for phase diagram

Against:

not exact
no dynamics
not applicable in non-ergodic regime of MG

Generating Functional Analysis of MGs with Fake Histories

1. generating functional:

$$\overline{Z[\boldsymbol{\psi}]} = \overline{\langle e^{-i\sum_i\sum_t \ \psi_i(t)\mathrm{sgn}[q_i(t)]}\rangle}$$

   average over process (i.e. over paths): $\langle \ldots \rangle$
   average over random strategies: $\overline{\ldots}$

$$\begin{aligned} C(t,t') &= \frac{1}{N}\sum_i \overline{\mathrm{sgn}[q_i(t)]\mathrm{sgn}[q_i(t')]} = -\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\partial^2 \overline{Z[\boldsymbol{\psi}]}}{\partial\psi_i(t)\partial\psi_i(t')} \\ G(t,t') &= \frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t')}\overline{\mathrm{sgn}[q_i(t)]} = -i\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\partial^2 \overline{Z[\boldsymbol{\psi}]}}{\partial\psi_i(t)\partial\theta_i(t')} \end{aligned}$$

2. after standard manipulations:

$$\overline{Z[\boldsymbol{\psi}]} = \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}G\mathcal{D}\hat{G}\ldots \ e^{N\Psi[C,\hat{C},G,\hat{G},\ldots]}$$

3. $N\to\infty$, steepest descent:

   exact closed eqns for C and G
   defined in terms of effective 'single agent' process

4. solve/analyze effective single agent process:

   phase diagrams, short-time dynamics, ...

Examples of 'effective single agent' processes, for fake histories:

- standard 'batch' MG:

$$q(t+1) = q(t) + \theta(t) - \alpha \sum_{t' \leq t} R(t,t')\mathrm{sgn}[q(t')] + \sqrt{\alpha}\ \eta(t)$$

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \Sigma(t,t')$$

$$R = R(C,G), \qquad \Sigma = \Sigma(C,G)$$

- standard 'on-line' MG with decision noise:

$$\frac{d}{dt} q(t) = \theta(t) - \alpha \int_0^t dt'\ R(t,t')\ \langle \sigma[q(t'),z] \rangle_z + \sqrt{\alpha}\ \eta(t)$$

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \Sigma(t,t')$$

$$R = R(C,G), \qquad \Sigma = \Sigma(C,G)$$

- 'batch' MG with decision noise and 'trend-followers':

$$q(t+1) = q(t) - \varepsilon\tilde{\theta}(t) + \alpha\varepsilon \sum_{t' \leq t} R(t,t')\sigma[q(t'),z(t')|T] + \sqrt{\alpha}\,\eta(t)$$

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \Sigma(t,t')$$

$$R = R(C,G), \qquad \Sigma = \Sigma(C,G)$$

Generating Functional Analysis of MGs with Real Histories

mathematical complications:

- non-Markovian microscopic laws
- no 'batch' versions possible
  (batch = average over all histories at each step ...)
- on-line: no temporal regularization possible
  (messes up timing, disaster for non-Markovian models ...)

Define generating functional,
for un-regularized on-line process with real histories

$$\overline{Z[\boldsymbol{\psi}]} = \overline{\left\langle e^{-i\sum_i \sum_t \; \psi_i(t)\mathrm{sgn}[q_i(t)]}\right\rangle}$$

average over process: $\langle \ldots \rangle$
average over random strategies: $\overline{\ldots}$

add overall bid perturbation term:

$$A(t) = \frac{1}{\sqrt{N}} \sum_i b_i(t) + A_e(t)$$

after *non-standard* manipulations:

$$\overline{Z[\boldsymbol{\psi}]} = \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}G\mathcal{D}\hat{G} \ldots \; e^{N\Psi[C,\hat{C},G,\hat{G},\ldots]}$$

$C(t,t')$: two-time correlation function
$G(t,t')$: two-time response function

Effective single agent process

similarities between real and fake history MGs:

- both cases: effective single agent equation of the form

$$\frac{d}{dt}q(t) = \theta(t) - \alpha \int_0^t dt'\ R(t,t')\ \sigma[q(t')] + \sqrt{\alpha}\ \eta(t)$$

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \Sigma(t,t')$$

  from which $\{C, G\}$ are to be solved self-consistently

- scaling with N of characteristic times are identical, provided we avoid highly biased global bid initializations *differences between real and fake history MGs:*

- real history: $\{R,\Sigma\}$ are to be solved from an effective equation for the stochastically evolving global bid:

$$R(t,t') = \frac{\delta}{\delta A_e(t')} \lim_{\delta\to 0} \left\langle A(\ell)\delta_{\boldsymbol{\lambda}(\ell,A),\boldsymbol{\lambda}(\ell',A)}\right\rangle_{\{A\}}\Big|_{\ell=t/\delta,\ell'=t'/\delta}$$
$$\Sigma(t,t') = \eta \lim_{\delta\to 0} \frac{1}{\delta}\left\langle A(\ell)A(\ell')\delta_{\boldsymbol{\lambda}(\ell,A),\boldsymbol{\lambda}(\ell',A)}\right\rangle_{\{A\}}\Big|_{\ell=t/\delta,\ell'=t'/\delta}$$

  Here:
  $\lambda(\ell,A)$ = history string at time ℓ, given overall bid 'path' A

- effective global bid process:

$$A(\ell) = A_e(\ell)+\phi_\ell-\frac{1}{2}\eta\sum_{\ell'<\ell}G(\ell,\ell')\delta_{\boldsymbol{\lambda}(\ell,A),\boldsymbol{\lambda}(\ell',A)}\,A(\ell')$$

  zero-average Gaussian random fields $\{\phi\}$:

$$\langle\phi_\ell\phi_{\ell'}\rangle_{\{\phi|A\}} = \frac{1}{2}[1+C(\ell,\ell')]\,\delta_{\boldsymbol{\lambda}(\ell,A),\boldsymbol{\lambda}(\ell',A)}$$

- The overall bid process is in itself independent of the effective single trader process; they are linked only via the (time dependent) order parameters occurring in their definitions

- $A(\ell)$ is coupled directly only to those bids in the past at times ℓ' with *identical realization of the M-bit history string*

Role and Calculation of History Statistics

History coincidence kernels:

$$\Delta_k(\ell_1,\ldots,\ell_k) \;=\; p^{k-1}\sum_{\boldsymbol{\lambda}}\langle\;\prod_{i=1}^{k}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A)}\;\rangle_{\{A\}}$$

probability of finding *identical* histories
at k prescribed non-identical times $\{\ell_1,\ldots,\ell_k\}$
divided by the probability for this to happen
for randomly drawn fake histories

One can express $\{R,\Sigma\}$ in terms of these kernels:

$$R(t,t') \;=\; \delta(t-t')+\lim_{\delta\to 0}\Big\{\sum_{r>0}(-\delta)^{r-1}\sum_{\ell_1\ldots\ell_{r-1}}G(\ell_0,\ell_1)\ldots G(\ell_{r-1},\ell_r) \times\,\Delta_{r+1}(\ell_0,\ldots,\ell_r)\Big\}\,\Big|_{\ell_0=\frac{t}{\delta},\ell_r=\frac{t'}{\delta}}$$

$$\Sigma(t,t') \;=\; \lim_{\delta\to 0}\Big\{\sum_{r,r'\geq 0}(-\delta)^{r+r'}\sum_{\ell_1\ldots\ell_r}G(\ell_0,\ell_1)\ldots G(\ell_{r-1},\ell_r) \times\sum_{\ell'_1\ldots\ell'_r}G(\ell'_0,\ell'_1)\ldots G(\ell'_{r-1},\ell'_r)\,[1+C(\ell_r,\ell'_{r'})] \times\,\Delta_{r+r'+2}(\ell_0,\ldots,\ell_r,\ell'_0,\ldots,\ell'_{r'})\Big\}\,\Big|_{\ell_0=\frac{t}{\delta},\ell'_0=\frac{t'}{\delta}}$$

Time translation invariant states with short history correlation times L_h

$$\frac{1}{2L}\sum_{\ell'=\ell-L}^{\ell+L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell',A)} = \pi_{\boldsymbol{\lambda}}(A)\;[1+\mathcal{O}((L_h/L)^{\frac{1}{2}})]$$

If $L_h \ll N$:
stationary state order parameter equations in terms of history frequency distribution $\varrho(\mathrm{f})$ only

$$\begin{aligned}
u &= \frac{\sqrt{\alpha}\chi_R}{S_0\sqrt{2}} \qquad \chi = \frac{1-\phi}{\alpha\chi_R} \qquad \phi = 1-\mathrm{Erf}[u] \\
c &= 1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}e^{-u^2} \\
\chi_R &= \int_0^\infty d\mathrm{f}\;\varrho(\mathrm{f})\;\frac{\mathrm{f}}{1+\chi\mathrm{f}} \\
S_0^2 &= (1+c)\int_0^\infty d\mathrm{f}\;\varrho(\mathrm{f})\;\frac{\mathrm{f}^2}{(1+\chi\mathrm{f})^2}
\end{aligned}$$

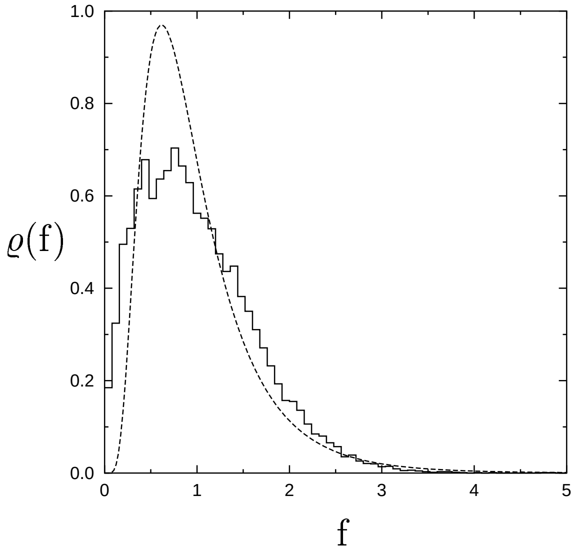
with

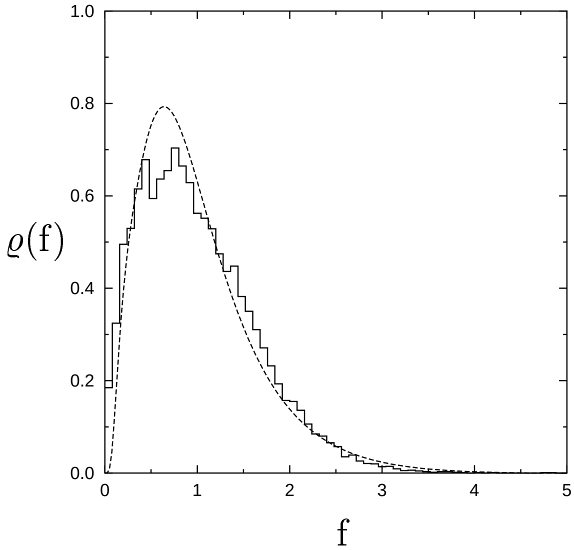
$$\chi = \int_0^\infty dt\,G(t) \qquad \chi_R = \int_0^\infty dt\,R(t) \qquad S_0^2 = \Sigma(\infty) \qquad c = C(\infty)$$

Calculation of history frequency distribution $\varrho(\mathrm{f})$ as expansion for small width

If Fourier transform of $\varrho(\mathrm{f})$ well-defined: moment expansion

$$\begin{aligned}
\mu_k &= \int_0^\infty df\ \varrho(f) f^k \\
\varrho(f) &= \int \frac{d\omega}{2\pi} e^{i\omega f} \sum_{k\geq 0} \frac{\mu_k}{k!}(-i\omega)^k
\end{aligned}$$



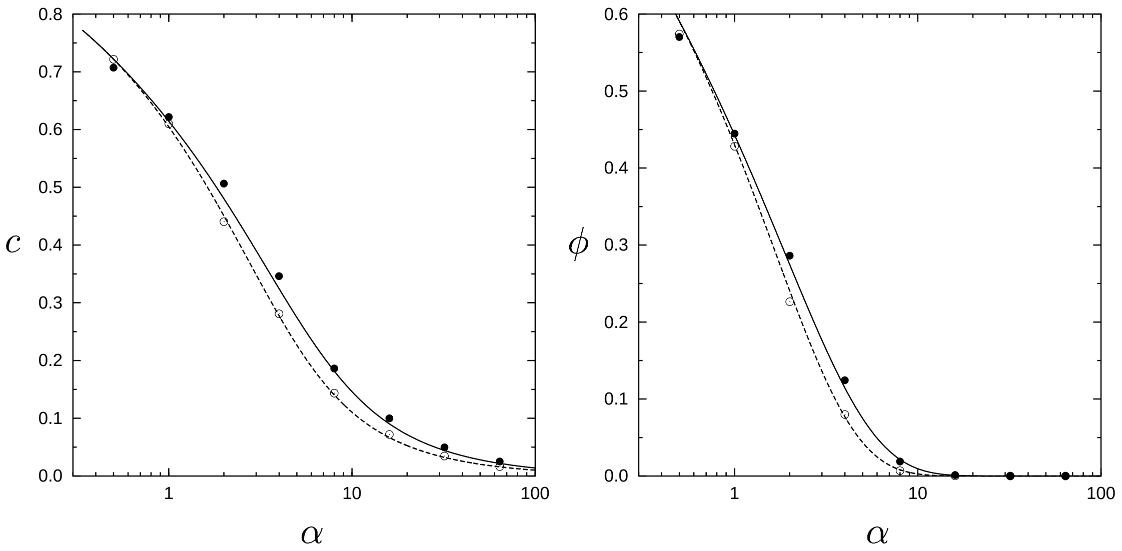


$$\lim_{M\to\infty} \log(\mu_k) = \frac{1}{2}\Omega k(k-1) - \frac{1}{12}\Omega^2 k(k-1)(2k-3) + \mathcal{O}(\Omega^3)$$

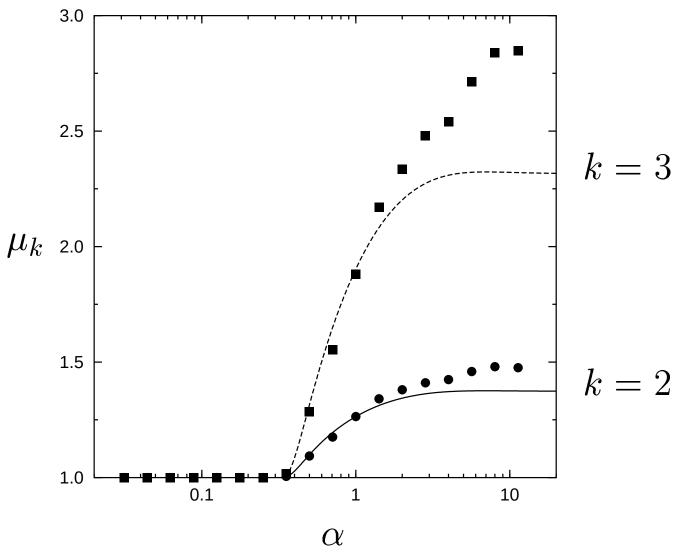
$$\Omega = \frac{4}{\pi}\int_0^\infty df\ \varrho(f) f\ \ \arctan\left[1 + \frac{2(1+c)}{(1+\chi f)^2(1-c)}\right]^{\frac{1}{2}} - 1$$

Final Theory versus Simulations

persistent correlations c and fraction ϕ of frozen agents:



moments μ_k of history frequency distribution:



Conclusions

- The generating functional methods of De Dominicis can be applied successfully to non-Markovian disordered microscopic stochastic processes
- We now have the mathematical tools to study the more subtle and more realistic MG versions with *real* market histories
- The present method to solve the effective bid process is an expansion for small width of the history frequency distribution; it would be helpful to develop alternatives

Further Reading

- details: `cond-mat/0410335`
- *Minority games – interacting agents in financial markets*
  Oxford Univ Press, Nov 2004
  D Challet, M Marsili and YC Zhang
- *The mathematical theory of minority games – statistical mechanics of interacting agents*
  Oxford Univ Press, Feb 2005
  ACC Coolen