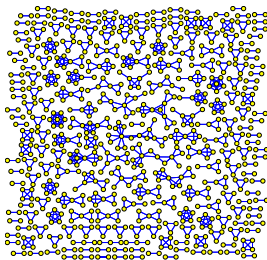


# New Approaches to Loopy Random Graph Ensembles

University of Warwick, 5th May 2014

ACC Coolen

King's College London  
and  
London Institute for Mathematical Sciences



## 1 Motivation

- Stochastic processes on networks
- Tailoring random graphs
- Graphs with many short loops

## 2 The Strauss ensemble

- Harmonic oscillator of loopy graphs
- Early results
- Generalisations

## 3 Analysis of generalised Strauss ensembles

- Calculation road map
- Conversion into a replica formulation
- Derivation of order parameter eqns
- Replica symmetry
- Phase transition formulae

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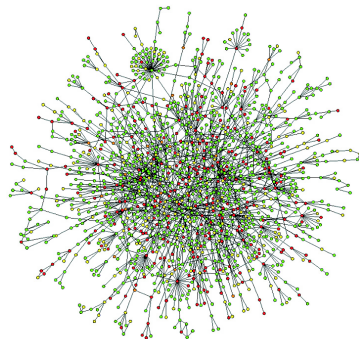
dynamics: chemical reaction eqns

nodes: proteins  $i, j = 1 \dots N$

links:  $c_{ij} = c_{ji} = 1$  if  $i$  can bind to  $j$   
 $c_{ij} = c_{ji} = 0$  otherwise

nondirected graphs,

$N \sim 10^4$ , links/node  $\sim 7$



- **gene regulation networks**

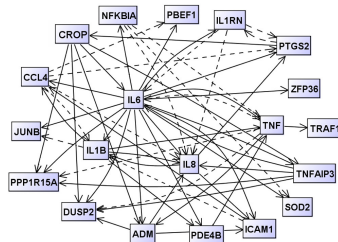
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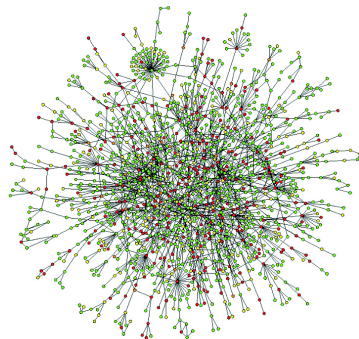
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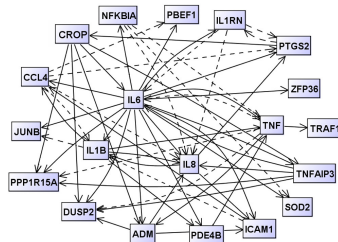
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use **random graph  $\mathbf{c}$**  as proxy

- tailored random graph ensemble  $\Omega_L$ :

maximum entropy ensemble, constrained by  
values of  $\omega(\mathbf{c}) = \{\omega_1(\mathbf{c}), \dots, \omega_L(\mathbf{c})\}$

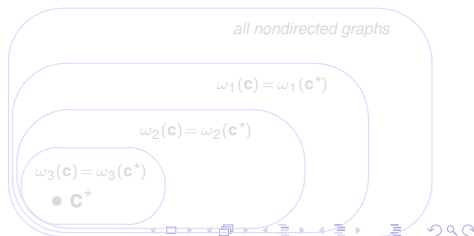
$$\Omega_L^{\text{hard}} : \quad p(\mathbf{c}) \propto \prod_{\ell \leq L} \delta_{\omega_\ell(\mathbf{c}), \omega_\ell(\mathbf{c}^*)}$$

$$\Omega_L^{\text{soft}} : \quad p(\mathbf{c}) \propto e^{\sum_{\ell=1}^L \hat{\omega}_\ell \omega_\ell(\mathbf{c})}, \quad \sum_{\mathbf{c}} p(\mathbf{c}) \omega_\ell(\mathbf{c}) = \omega_\ell(\mathbf{c}^*) \quad \forall \ell$$

- approx model solution:

average generating functions  
of process over  $\mathbf{c}$  in  $\Omega_L$

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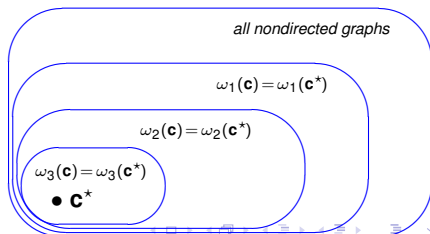
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## How to choose observables $\omega(\mathbf{c}) = \{\omega_1(\mathbf{c}), \dots, \omega_L(\mathbf{c})\}$ to carry over from $\mathbf{c}^*$ to the ensemble?

e.g. spin models  $H(\sigma) = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j$

- statics: replica method

$$\overline{e^{-\beta \sum_{\alpha=1}^n H(\sigma^\alpha)}} = \frac{\sum_{\mathbf{c}} \delta_{\omega, \omega(\mathbf{c})} e^{\sum_{i<j} c_{ij} A_{ij}}}{\sum_{\mathbf{c}} \delta_{\omega, \omega(\mathbf{c})}}, \quad A_{ij} = \beta J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha$$

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in both cases  
to be done *analytically*:

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$$p(k|\mathbf{c}) = \frac{1}{N} \sum_i \delta_{k, \sum_j c_{ij}}, \quad W(k, k'|\mathbf{c}) = \frac{1}{\bar{k}N} \sum_{ij} c_{ij} \delta_{k, \sum_r c_{ir}} \delta_{k', \sum_r c_{jr}}$$

## Shannon entropy

$$S = N^{-1} \sum_{\mathbf{c}} p(\mathbf{c}) \log p(\mathbf{c})$$

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$$\lim_{N \rightarrow \infty} \epsilon_N = 0$$

$$\pi(k) = e^{-\langle k \rangle} \langle k \rangle^k / k!$$

similar for directed graphs,  
formulae in terms of

$$\vec{k} = (k_{\text{in}}, k_{\text{out}}), \quad p(\vec{k}) \quad \text{and} \quad W(\vec{k}, \vec{k}')$$

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## Tailoring random graphs further ...

all nondirected graphs  $\mathbf{c}$

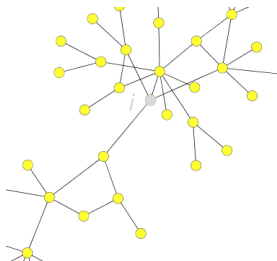
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next box?

•  $\mathbf{c}^*$

obvious candidates:  
*generalised degrees,*  
*node neighbourhoods,*  
...



$$k_i = \sum_j c_{ij} = 4$$

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$$n_i = (k_i; \{\xi_i^s\}) = (4; 3, 4, 6, 7)$$

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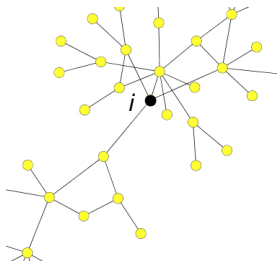
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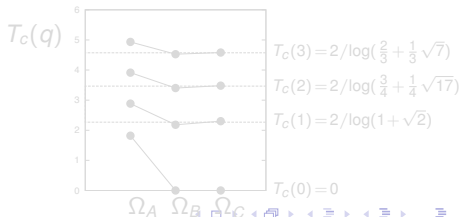
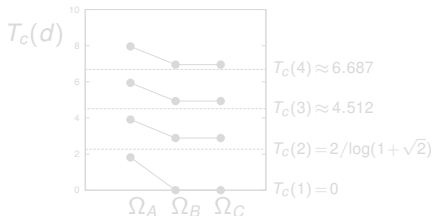
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 $p(k) = \delta_{k,2d}$



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 $p(k \geq 2) = e^{-q} q^{k-2} / (k-2)!$



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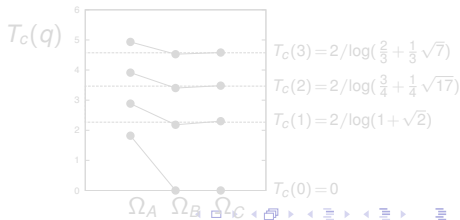
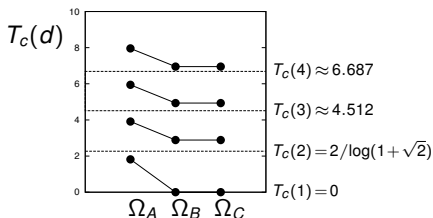
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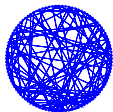
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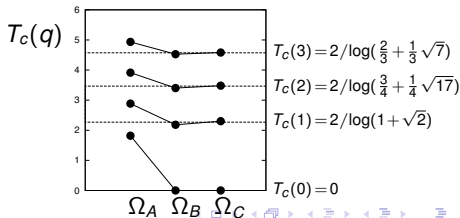
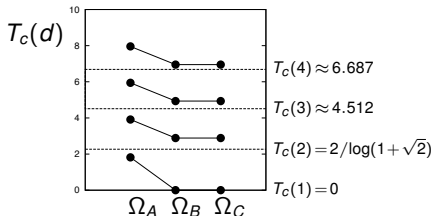
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## most informative next observable $\omega(\mathbf{c})$ ?

- random graphs with prescribed  $p(k)$  and  $W(k, k')$ :  
*locally tree-like ...*

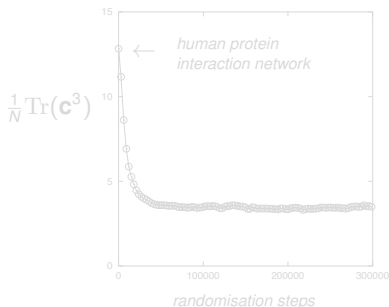
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lattice-like networks  $\mathbf{c}^*$ :  
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but most of our analysis methods  
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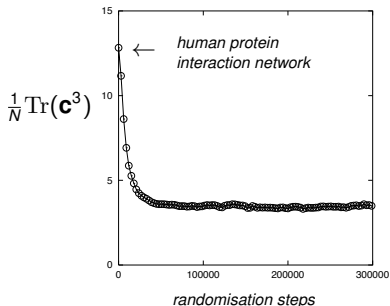
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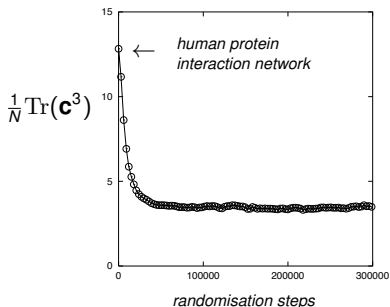
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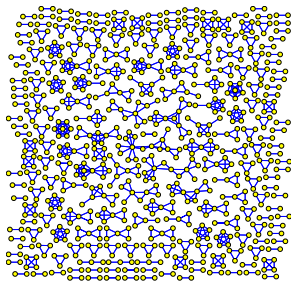
## Immune model of Agliari and Barra (2013)

interaction between B-cells and T-cells

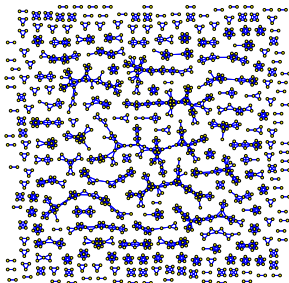
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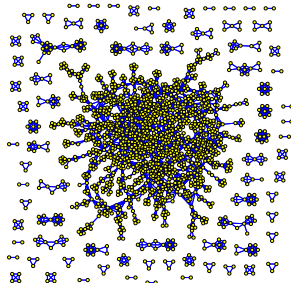
$\alpha C^2 < 1$



$\alpha C^2 = 1$



$\alpha C^2 > 1$



◀ ◻ (see also Newman, 2003) ◻ ▶

## Immune versus neural network models

mathematically very similar ...

both store and recall information ...

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 $c_{ij}$ : finitely connected tree-like graph

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*recall of one  $N$ -bit pattern at a time*

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## Model exactly solvable

using replica techniques,  
in spite of the many short loops ...

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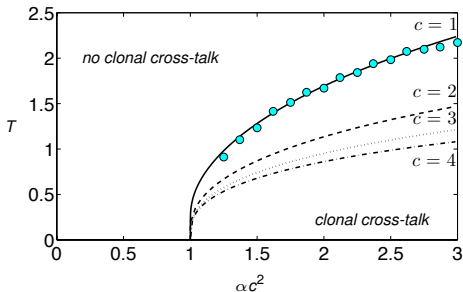
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$\xi$ : sparse  $p \times N$  matrix with indep distributed entries

Hubbard-Stratonovich type mapping  
to model with spins + Gaussian fields,  
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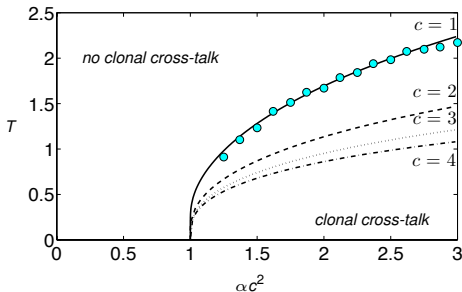
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# Harmonic oscillator of loopy graphs

## Simplest nontrivial graph ensemble

controlled average connectivity,  
controlled nr of **triangles**  
(Strauss, 1986)

$$p(\mathbf{c}) \propto e^{u \sum_{ij} c_{ij} + v \sum_{ijk} c_{ij} c_{jk} c_{ki}}$$

- quantities of interest:

$$\langle k \rangle = \left\langle \frac{1}{N} \sum_{ij} c_{ij} \right\rangle, \quad \langle m \rangle = \left\langle \frac{1}{N} \sum_{ijk} c_{ij} c_{jk} c_{ki} \right\rangle, \quad S = -\frac{1}{N} \sum_{\mathbf{c}} p(\mathbf{c}) \log p(\mathbf{c})$$

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# Early results

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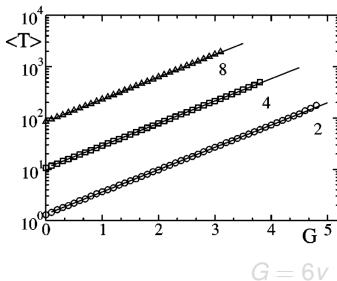
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- **Burda et al (2004)**

- $u = -\frac{1}{2} \log(N/\bar{k} - 1)$ ,  $v = \mathcal{O}(1)$   
if  $v=0$ : ER ensemble with  $\langle k \rangle = \bar{k}$
- diagrammatic perturbation theory in  $v$ ,  
formula for nr of triangles:  
 $\lim_{N \rightarrow \infty} \langle T \rangle = \bar{k}^3 e^v$
- regular regime, 'clumped' regime
- unresolved subtleties in expansion,  
expect series to explode for  $v \sim \log N$   
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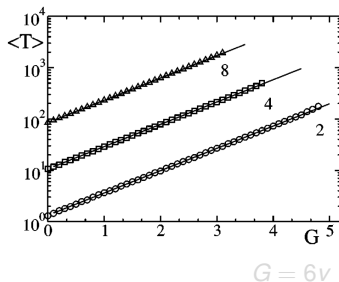
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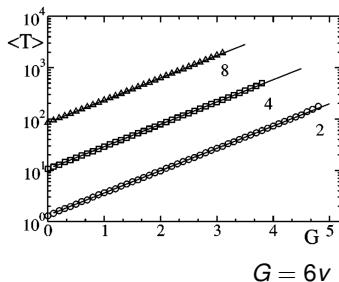
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## intuition behind Burda's results

- exact identities:

$$\phi = \frac{1}{2}(N-1) \log(e^{2u} + 1) + \frac{1}{N} \log \sum_{r \geq 0} p(r|u) e^{vr}$$

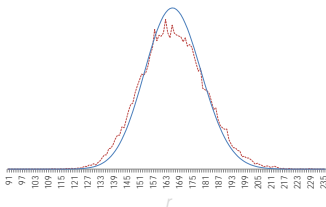
$$p(r|u) = \sum_{\mathbf{c}} p_{\text{ER}}(\mathbf{c}|u) \delta_{r, \sum_{ijk} c_{ij} c_{jk} c_{ki}} \quad (\text{ER triangle distr})$$

$$\bar{r}(u) = \frac{N(N-1)(N-2)}{(1 + e^{-2u})^3}$$

- assume  $p(r|u)$  is Poissonian:

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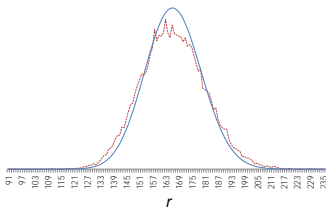
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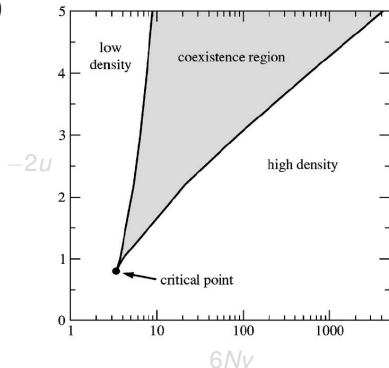
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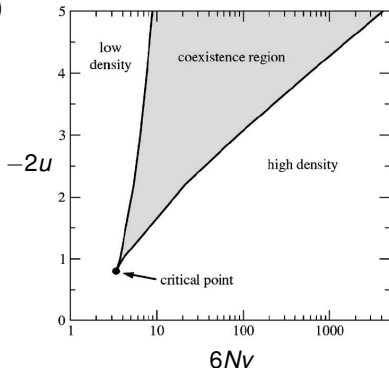
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# Generalisations

- control closed paths of all lengths  $\ell \leq L$

(Strauss:  $L = 3$ )

generating function:  
(use  $\sum_{ij} c_{ij} = \sum_{ij} c_{ij} c_{ji}$ )

$$p(\mathbf{c}) \propto e^{u \sum_{ij} c_{ij} + \sum_{\ell=3}^L v_{\ell} \sum_{i_1 \dots i_{\ell}} c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_{\ell} i_1}}$$

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$$(\text{above: } \hat{\varrho}(\mu) = \sum_{\ell} v_{\ell} \mu^{\ell})$$

generating function:  $\phi = \frac{1}{N} \log \sum_{\mathbf{c}} e^{N \int d\mu \hat{\varrho}(\mu) \varrho(\mu | \mathbf{c})}$

$$\varrho(\mu) = \frac{\delta \phi}{\delta \hat{\varrho}(\mu)}, \quad S = \phi - \int d\mu \hat{\varrho}(\mu) \varrho(\mu)$$

# Generalisations

- control closed paths of all lengths  $\ell \leq L$

(Strauss:  $L = 3$ )

generating function:  
(use  $\sum_{ij} c_{ij} = \sum_{ij} c_{ij} c_{ji}$ )

$$p(\mathbf{c}) \propto e^{u \sum_{ij} c_{ij} + \sum_{\ell=3}^L v_{\ell} \sum_{i_1 \dots i_{\ell}} c_{i_1 i_2} c_{i_2 i_3} \dots c_{i_{\ell} i_1}}$$

$$\phi = \frac{1}{N} \log \sum_{\mathbf{c}} e^{u \text{Tr}(\mathbf{c}^2) + \sum_{\ell=3}^L v_{\ell} \text{Tr}(\mathbf{c}^{\ell})}$$

$$\langle k \rangle = \frac{\partial \phi}{\partial u}, \quad \langle m_{\ell} \rangle = \frac{1}{N} \langle \text{Tr}(\mathbf{c}^{\ell}) \rangle = \frac{\partial \phi}{\partial v_{\ell}}, \quad \mathcal{S} = \phi - u \langle k \rangle - \sum_{\ell=3}^L v_{\ell} \langle m_{\ell} \rangle$$

- control closed paths of all lengths, since  $\text{Tr}(\mathbf{c}^{\ell}) = N \int d\mu \mu^{\ell} \varrho(\mu | \mathbf{c})$ :

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- Stochastic processes on networks
- Tailoring random graphs
- Graphs with many short loops

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- Harmonic oscillator of loopy graphs
- Early results
- Generalisations

## 3 Analysis of generalised Strauss ensembles

- **Calculation road map**
- Conversion into a replica formulation
- Derivation of order parameter eqns
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- Phase transition formulae

## 4 Summary

# Calculation road map

target:  $\phi = \frac{1}{N} \log \sum_{\mathbf{c}} e^{u \sum_{ij} c_{ij} + N \int d\mu \hat{\rho}(\mu) \rho(\mu|\mathbf{c})}$       Strauss :  $\hat{\rho}(\mu) = v\mu^3$

Burda's scaling regime :  $u = \frac{1}{2} \log(\bar{k}/N)$ ,  $N \rightarrow \infty$

Shannon entropy? observables? phase transitions?

- derive expression in which sum over graphs can be done:

$$\phi = \lim_{\epsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{u \sum_{ij} c_{ij}} \prod_{\mu} \left[ Z(\mu + i\epsilon|\mathbf{c})^{i\Delta\lambda(\mu)} \overline{Z(\mu + i\epsilon|\mathbf{c})}^{-i\Delta\lambda(\mu)} \right]$$

$$Z(\mu|\mathbf{c}) = \int d\phi e^{-\frac{1}{2}i\phi \cdot [\mathbf{c} - \mu \mathbf{1}] \phi}, \quad \lambda(\mu) = \frac{1}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu)$$

- replica analysis, saddle-point eqns for  $N \rightarrow \infty$ , analytical continuation to *imaginary* dimension, limits  $\epsilon \downarrow 0$  and  $\Delta \downarrow 0$
- replica symmetry, bifurcation analysis, phase transitions and entropy

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## 4 Summary

# Conversion into a replica formulation

ensemble constraints written in terms of spectrum,  
use Edwards-Jones formula (1976):

$$\varrho(\mu|\mathbf{c}) = \frac{2}{N\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \frac{\partial}{\partial \mu} \log Z(\mu + i\varepsilon|\mathbf{c}), \quad Z(\mu|\mathbf{c}) = \int d\phi e^{-\frac{1}{2}i\phi \cdot [\mathbf{c} - \mu \mathbf{1}] \phi}$$

insert into  $\phi$ ,  
integrate by parts,  
discretise  $\mu$ -integral:

$$\begin{aligned} \phi &= \frac{1}{N} \log \sum_{\mathbf{c}} e^{U \sum_{ij} c_{ij} + N \int d\mu \hat{\rho}(\mu) \frac{2}{N\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im} \frac{\partial}{\partial \mu} \log Z(\mu + i\varepsilon|\mathbf{c})} \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{U \sum_{ij} c_{ij} - \frac{2}{\pi} \int d\mu \operatorname{Im} \log Z(\mu + i\varepsilon|\mathbf{c})} \frac{d}{d\mu} \hat{\rho}(\mu) \\ &= \lim_{\varepsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{U \sum_{ij} c_{ij} - \frac{2\Delta}{\pi} \sum_{\mu} \operatorname{Im} \log Z(\mu + i\varepsilon|\mathbf{c})} \frac{d}{d\mu} \hat{\rho}(\mu) \\ &= \lim_{\varepsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{U \sum_{ij} c_{ij}} \prod_{\mu} e^{-2\operatorname{Im} \log Z(\mu + i\varepsilon|\mathbf{c})} \cdot \frac{\Delta}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu) \end{aligned}$$

$$e^{-2 \operatorname{Im} \log z} = z^i \bar{z}^{-i}$$

$$\phi = \lim_{\varepsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{U \sum_{ij} c_{ij}} \prod_{\mu} \left[ Z(\mu + i\varepsilon|\mathbf{c})^i \overline{Z(\mu + i\varepsilon|\mathbf{c})}^{-i} \right]^{\frac{\Delta}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu)}$$

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## Flavours of the replica method

the replica dimension  $n$  ...

- $n=0$ : Kac (1968), Sherrington, Kirkpatrick (1975), Parisi (1979)  
stat mech of disordered spin systems

$$\log Z = \lim_{n \rightarrow 0} \frac{1}{n} (Z^n - 1), \quad \overline{\log Z} = \lim_{n \rightarrow 0} \frac{1}{n} \log \overline{Z^n}$$

- $n \in \mathbb{R}, > 0$ : Sherrington (1980), Coolen, Penney, Sherrington (1993)  
'slow' dynamics of parameters in 'fast' spin system  
(partial annealing,  $n = T/T'$ )
- $n \in \mathbb{R}, < 0$ : Dotsenko, Franz, Mezard (1994)  
slow dynamics evolves to *maximise* free energy of fast system

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# Derivation of order parameter eqns

replica dimensions:

$$n_\mu = -m_\mu = \frac{i\Delta}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu)$$

- prepare summation over graphs:

$$\phi = \lim_{\epsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{u \sum_{ij} c_{ij}} \prod_{\mu} \left[ Z(\mu + i\epsilon | \mathbf{c})^{n_\mu} \overline{Z(\mu + i\epsilon | \mathbf{c})}^{m_\mu} \right]$$

use  $u = -\frac{1}{2} \log(N/\bar{k})$ , and

$$p_{\text{ER}}(\mathbf{c} | \bar{k}) = \prod_{i < j} \left[ \frac{\bar{k}}{N} \delta_{c_{ij}, 1} + \left(1 - \frac{\bar{k}}{N}\right) \delta_{c_{ij}, 0} \right]$$

to get

$$\phi = \frac{1}{2} \bar{k} + \lim_{\epsilon, \Delta \downarrow 0} \frac{1}{N} \log \left\langle \prod_{\mu} \left[ Z(\mu + i\epsilon | \mathbf{c})^{n_\mu} \overline{Z(\mu + i\epsilon | \mathbf{c})}^{m_\mu} \right] \right\rangle_{\text{ER}} + \mathcal{O}\left(\frac{1}{N}\right)$$

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- evaluate  $Z(\mu+i\epsilon|\mathbf{c})^{n_\mu}$  and  $\overline{Z(\mu+i\epsilon|\mathbf{c})}^{m_\mu}$   
for integer  $n_\mu$  and  $m_\mu$ ,  
and do average over graphs

$$\left\langle \prod_{\mu} \left\{ \left[ \prod_{\alpha_{\mu}=1}^{n_{\mu}} \int_{\mathbb{R}^N} d\phi e^{-\frac{1}{2}\epsilon\phi^2 - \frac{1}{2}i\phi \cdot (\mathbf{c} - \mu \mathbf{1})\phi} \right] \left[ \prod_{\beta_{\mu}=1}^{m_{\mu}} \int_{\mathbb{R}^N} d\psi e^{-\frac{1}{2}\epsilon\psi^2 + \frac{1}{2}i\psi \cdot (\mathbf{c} - \mu \mathbf{1})\psi} \right] \right\} \right\rangle_{\text{ER}}$$

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$$\phi^i = \{\phi_{\mu, \alpha_{\mu} \leq n_{\mu}}^i\}, \quad \psi^i = \{\psi_{\mu, \beta_{\mu} \leq m_{\mu}}^i\}$$

- introduce functional order parameter

$$\forall \phi = \{\phi_{\mu, \alpha_{\mu} \leq n_{\mu}}\} : \quad 1 = \int d\mathcal{P}(\phi, \psi) \delta \left[ \mathcal{P}(\phi, \psi) - \frac{1}{N} \sum_i \delta(\phi - \phi^i) \delta(\psi - \psi^i) \right]$$

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*theory in terms of  
 $\mathcal{P}(\phi, \psi)$  and  $\hat{\mathcal{P}}(\phi, \psi)$*

- evaluate  $Z(\mu+i\epsilon|\mathbf{c})^{n_\mu}$  and  $\overline{Z(\mu+i\epsilon|\mathbf{c})}^{m_\mu}$  for integer  $n_\mu$  and  $m_\mu$ , and do average over graphs

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*theory in terms of  $\mathcal{P}(\phi, \psi)$  and  $\hat{\mathcal{P}}(\phi, \psi)$*

- path integral form:

$$\phi = \lim_{N \rightarrow \infty} \lim_{\epsilon, \Delta \downarrow 0} \frac{1}{N} \log \int \{d\mathcal{P}d\hat{\mathcal{P}}\} e^{N\Psi[\{\mathcal{P}, \hat{\mathcal{P}}\}]} = \lim_{\epsilon, \Delta \downarrow 0} \text{extr}_{\{\mathcal{P}, \hat{\mathcal{P}}\}} \Psi[\{\mathcal{P}, \hat{\mathcal{P}}\}]$$

$$\begin{aligned} \Psi[\{\mathcal{P}, \hat{\mathcal{P}}\}] &= i \int d\phi d\psi \hat{\mathcal{P}}(\phi, \psi) \mathcal{P}(\phi, \psi) \\ &+ \frac{1}{2} \bar{k} \int d\phi d\psi d\phi' d\psi' \mathcal{P}(\phi, \psi) \mathcal{P}(\phi', \psi') e^{i(\psi \cdot \psi' - \phi \cdot \phi')} \\ &+ \log \int d\phi d\psi e^{-\frac{1}{2} \phi \cdot (\epsilon \mathbf{I} - i\mathbf{M}) \phi - \frac{1}{2} \psi \cdot (\epsilon \mathbf{I} + i\mathbf{M}) \psi - i\hat{\mathcal{P}}(\phi, \psi)} \end{aligned}$$

$$M_{\mu, \alpha; \mu', \alpha'} = \mu \delta_{\mu\mu'} \delta_{\alpha\alpha'}$$

$$\begin{aligned} \phi &= \{\phi_{\mu, \alpha_{\mu} \leq n_{\mu}}\}, & n_{\mu} &= \frac{i\Delta}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu) \\ \psi &= \{\psi_{\mu, \beta_{\mu} \leq m_{\mu}}\}, & m_{\mu} &= -\frac{i\Delta}{\pi} \frac{d}{d\mu} \hat{\rho}(\mu) \end{aligned}$$

- saddle-point eqns,  $\Omega = \exp[-i\hat{\mathcal{P}}]$ :

$$\Omega(\phi, \psi) = \exp \left[ \bar{k} \int d\phi' d\psi' \mathcal{P}(\phi', \psi') e^{i(\psi \cdot \psi' - \phi \cdot \phi')} \right]$$

$$\mathcal{P}(\phi, \psi) = \frac{\Omega(\phi, \psi) e^{-\frac{1}{2} \phi \cdot (\epsilon \mathbf{I} - i\mathbf{M}) \phi - \frac{1}{2} \psi \cdot (\epsilon \mathbf{I} + i\mathbf{M}) \psi}}{\int d\phi' d\psi' \Omega(\phi', \psi') e^{-\frac{1}{2} \phi' \cdot (\epsilon \mathbf{I} - i\mathbf{M}) \phi' - \frac{1}{2} \psi' \cdot (\epsilon \mathbf{I} + i\mathbf{M}) \psi'}}$$



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$$\mathcal{P}(\phi, \psi) = \frac{\mathcal{Q}(\phi, \psi) e^{-\frac{1}{2} \phi \cdot (\epsilon \mathbf{I} - i\mathbf{M}) \phi - \frac{1}{2} \psi \cdot (\epsilon \mathbf{I} + i\mathbf{M}) \psi}}{\int d\phi' d\psi' \mathcal{Q}(\phi', \psi') e^{-\frac{1}{2} \phi' \cdot (\epsilon \mathbf{I} - i\mathbf{M}) \phi' - \frac{1}{2} \psi' \cdot (\epsilon \mathbf{I} + i\mathbf{M}) \psi'}}$$

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- Derivation of order parameter eqns
- **Replica symmetry**
- Phase transition formulae

## 4 Summary

# Replica symmetric ansatz

- de Finetti's theorem, combined with  $\mathcal{P}(\psi, \phi) = \overline{\mathcal{P}(\phi, \psi)}$ :

$$\mathcal{P}(\phi, \psi) = \int \{d\pi\} \mathcal{W}[\{\pi\}] \left[ \prod_{\mu} \prod_{\alpha_{\mu}=1}^{n_{\mu}} \pi(\phi_{\mu, \alpha_{\mu}} | \mu) \right] \left[ \prod_{\mu} \prod_{\beta_{\mu}=1}^{m_{\mu}} \overline{\pi(\psi_{\mu, \beta_{\mu}} | \mu)} \right]$$

$$\int \{d\pi\} \mathcal{W}[\{\pi\}] = 1, \quad \int d\phi \pi(\phi | \mu) = 1 \text{ for all } \mu$$

- insert into saddle-point eqns,

$$\Delta \downarrow 0: \quad \Delta \sum_{\mu} \rightarrow \int d\mu$$

$$\mathcal{W}[\{\pi\}] = \frac{\sum_{\ell \geq 0} e^{-\bar{k} \frac{\bar{k}^{\ell}}{\ell!}} \int \left( \prod_{r \leq \ell} \{d\pi_r\} \mathcal{W}[\{\pi_r\}] \right) \mathcal{D}[\{\pi_1, \dots, \pi_{\ell}\}] \delta \left[ \pi - \mathcal{F}[\{\pi_1, \dots, \pi_{\ell}\}] \right]}{\sum_{\ell \geq 0} e^{-\bar{k} \frac{\bar{k}^{\ell}}{\ell!}} \int \left( \prod_{r \leq \ell} \{d\pi_r\} \mathcal{W}[\{\pi_r\}] \right) \mathcal{D}[\{\pi_1, \dots, \pi_{\ell}\}]}$$

$$\mathcal{F}(\phi | \mu; \pi_1, \dots, \pi_{\ell}) = \frac{e^{-\frac{1}{2}(\varepsilon - i\mu)\phi^2} \prod_{r \leq \ell} \hat{\pi}_r(\phi | \mu)}{\int d\phi' e^{-\frac{1}{2}(\varepsilon - i\mu)\phi'^2} \prod_{r \leq \ell} \hat{\pi}_r(\phi' | \mu)}, \quad \hat{\pi}(\phi | \mu) = \int dx \pi(x | \mu) e^{-ix\phi}$$

$$\mathcal{D}[\{\pi_1, \dots, \pi_{\ell}\}] = e^{-\frac{2}{\pi} \text{Im} \int d\mu \frac{d}{d\mu} \hat{\nu}(\mu) \log \int ds e^{-\frac{1}{2}(\varepsilon - i\mu)s^2} \prod_{r \leq \ell} \hat{\pi}_r(s | \mu)}, \quad \text{no loops: } \mathcal{D}[\dots] = 1$$

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- nature of RS solutions  $\mathcal{W}[\{\pi\}]$   
(similar to spectrum calculations)

$$\pi(\phi|\mu) = \frac{e^{-\frac{1}{2}\epsilon\phi^2 - \frac{1}{2}ix(\mu)\phi^2 + y(\mu)\phi}}{\int d\phi' e^{-\frac{1}{2}\epsilon\phi'^2 - \frac{1}{2}ix(\mu)\phi'^2 + y(\mu)\phi'}} : \quad \mathcal{W}[\{\pi\}] \rightarrow \mathcal{W}[\{x, y\}]$$

- insert into RS eqns,  
take  $\epsilon \downarrow 0$

$$W[\{x, y\}] = \frac{\sum_{\ell \geq 0} \rho_\ell \int (\prod_{r \leq \ell} \{dx_r dy_r\}) W[\{x_r, y_r\}] \tilde{\mathcal{D}}[\dots] \delta[x - F_x[\dots]] \delta[y - F_y[\dots]]}{\sum_{\ell \geq 0} \rho_\ell \int (\prod_{r \leq \ell} \{dx_r dy_r\}) W[\{x_r, y_r\}] \tilde{\mathcal{D}}[\dots]}$$

$$F_x[\mu|\{x_1, \dots, x_\ell\}] = -\mu - \sum_{r \leq \ell} \frac{1}{x_r(\mu)}$$

$$F_y[\mu|\{x_1, y_1, \dots, x_\ell, y_\ell\}] = - \sum_{r \leq \ell} \frac{y_r(\mu)}{x_r(\mu)} \quad \rho_\ell = e^{-\bar{k}} \bar{k}^\ell / \ell!$$

$$\tilde{\mathcal{D}}[\{x_1, y_1, \dots, x_\ell, y_\ell\}] = e^{\int d\mu \frac{d}{d\mu} \hat{\rho}(\mu) \left[ \frac{1}{2} \text{sgn}(F_x[\mu|x_1, \dots, x_\ell]) + \frac{1}{\pi} \frac{F_y^2[\mu|x_1, y_1, \dots, x_\ell, y_\ell]}{F_x[\mu|x_1, \dots, x_\ell]} \right]}$$

*everything now real-valued!*

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- alternative form

$$W[\{x, y\}] = \frac{\tilde{\mathcal{D}}[\{x, y\}] \sum_{\ell \geq 0} \rho_\ell \int (\prod_{r \leq \ell} \{dx_r dy_r\} W[\{x_r, y_r\}]) \delta[x - F_x[\dots]] \delta[y - F_y[\dots]]}{\sum_{\ell \geq 0} \rho_\ell \int (\prod_{r \leq \ell} \{dx_r dy_r\} W[\{x_r, y_r\}]) \tilde{\mathcal{D}}[F_x[\dots], F_y[\dots]]}$$

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## 4 Summary

# Phase transition formulae

- Remaining symmetry:

invariance under  $\mathcal{P}(\phi, \psi) \rightarrow \mathcal{P}(-\phi, -\psi)$

here:  $\{y\} \rightarrow \{-y\}$

*weakly symmetric solns* :  $W[\{x, -y\}] = W[\{x, y\}]$

*strongly symmetric solns* :  $W[\{x, y\}] = W[\{x\}] \delta[\{y\}]$

$$W[\{x\}] = \frac{\tilde{\mathcal{D}}[\{x, 0\}] \sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\dots]]}{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \tilde{\mathcal{D}}[F_x[\dots], 0]}$$

- symmetry-breaking:

*SG type* :  $W[\{x, y\}] = W[\{x\}] \delta[\{y\}] \rightarrow W[\{x, -y\}] = W[\{x, y\}]$

*F type* :  $W[\{x, y\}] = W[\{x\}] \delta[\{y\}] \rightarrow W[\{x, -y\}] \neq W[\{x, y\}]$

continuous transitions:

*functional  
moment  
expansion*

$$\begin{aligned} \psi(\mu_1, \dots, \mu_n | \{x\}) &= \int \{dy\} W[\{y|x\}] y(\mu_1) \dots y(\mu_n) \\ &= \mathcal{O}(\varepsilon^n) \quad \text{close to transition} \end{aligned}$$

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lowest orders:

- F-type bifurcation,  $\mathcal{O}(\epsilon)$ :

$$\psi(\mu|\{x\}) = \int \{dx'\} \mathcal{B}_1[\mu; \{x, x'\}] \psi(\mu|\{x'\}) + \mathcal{O}(\epsilon^2)$$

$$\mathcal{B}_1[\mu; \{x, x'\}] = -\bar{k} \frac{W[\{x'\}]}{x'(\mu)} \frac{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\{x_1, \dots, x_\ell, x'\}]]}{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\{x_1, \dots, x_\ell\}]]}$$

$$F_x[\{x_1, \dots, x_\ell, x'\}] = F_x[\{x_1, \dots, x_\ell\}] - \frac{1}{x'}:$$

$$\begin{aligned} \mathcal{B}_1[\mu; \{x, x'\}] &= -\bar{k} \frac{W[\{x'\}]}{x'(\mu)} \frac{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x + \frac{1}{x'} - F_x[\{x_1, \dots, x_\ell\}]]}{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\{x_1, \dots, x_\ell\}]]} \\ &= -\bar{k} \frac{W[\{x'\}]}{x'(\mu)} \frac{W[\{x + \frac{1}{x'}\}]}{\tilde{\mathcal{D}}[\{x + \frac{1}{x'}, 0\}]} \frac{\tilde{\mathcal{D}}[\{x, 0\}]}{W[\{x\}]} \end{aligned}$$

hence

$$\int \{dx'\} \mathcal{B}[\{x, x'\}] \zeta(\mu|\{x'\}) = -x(\mu) \zeta(\mu|\{x\})$$

$$\mathcal{B}[\{x, x'\}] = \bar{k} \frac{W[\{x + \frac{1}{x'}\}] \tilde{\mathcal{D}}[\{x', 0\}]}{\tilde{\mathcal{D}}[\{x + \frac{1}{x'}, 0\}]}$$

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$$\mathcal{B}_1[\mu; \{x, x'\}] = -\bar{k} \frac{W[\{x'\}]}{x'(\mu)} \frac{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\{x_1, \dots, x_\ell, x'\}]]}{\sum_{\ell \geq 0} p_\ell \int (\prod_{r \leq \ell} \{dx_r\} W[\{x_r\}]) \delta[x - F_x[\{x_1, \dots, x_\ell\}]]}$$

$$F_x[\{x_1, \dots, x_\ell, x'\}] = F_x[\{x_1, \dots, x_\ell\}] - \frac{1}{x'}:$$

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hence

$$\int \{dx'\} \mathcal{B}[\{x, x'\}] \zeta(\mu|\{x'\}) = -x(\mu) \zeta(\mu|\{x\})$$

$$\mathcal{B}[\{x, x'\}] = \bar{k} \frac{W[\{x + \frac{1}{x'}\}] \tilde{\mathcal{D}}[\{x', 0\}]}{\tilde{\mathcal{D}}[\{x + \frac{1}{x'}, 0\}]}$$

lowest orders:

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$$\psi(\mu, \mu' | \{x\}) = \int \{dx'\} \mathcal{B}_2(\mu, \mu'; \{x, x'\}) \psi(\mu, \mu' | \{x'\})$$

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## RS Entropy

follows directly from

$$\begin{aligned}\phi_{\text{RS}} = & \log \sum_{\ell \geq 0} e^{-\bar{k}} \frac{\bar{k}^\ell}{\ell!} \int \left( \prod_{r \leq \ell} \{d\pi_r\} \mathcal{W}[\{\pi_r\}] \right) \mathcal{D}[\{\pi_1, \dots, \pi_\ell\}] \\ & + \bar{k} - \frac{1}{2} \bar{k} \int \{d\pi d\pi'\} \mathcal{W}[\{\pi\}] \mathcal{W}[\{\pi'\}] \mathcal{E}[\{\pi, \pi'\}]\end{aligned}$$

with

$$\mathcal{E}[\{\pi, \pi'\}] = e^{\frac{2}{\pi} \text{Im} \int d\mu \hat{\varrho}(\mu) \frac{d}{d\mu} \log \int dx dx' e^{-ixx'} \pi(x|\mu) \pi'(x'|\mu)}$$

Extremisation:

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# Summary

- new analytical approach to loopy random graph ensembles, developed for generalised Strauss model
- based on replica form for generating function, in which sum over graphs can be done

$$\phi = \lim_{\epsilon, \Delta \downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} e^{u \sum_{ij} c_{ij}} \prod_{\mu} \left[ Z(\mu + i\epsilon | \mathbf{c})^{i\Delta\lambda(\mu)} \overline{Z(\mu + i\epsilon | \mathbf{c})}^{-i\Delta\lambda(\mu)} \right]$$

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