The Auditory System and Human Sound-Localization Behavior

Exercises Chapter 4

Problem 4-1

(a) For the third-order nonlinear system of Fig. 4-1, calculate the output spectrum when the input is given by: $x(t) = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$, where $f_2 > f_1$.

(b) Write a Matlab script (or use your preferred programming environment) to calculate and plot the amplitude spectrum for the output of a finite-order (up to n=5) polynomial nonlinearity in response to the stimulus of (a) for arbitrary frequencies.

Problem 4-2

In analogy to the Taylor expansion for the exponential function, give the Volterra kernels for the following nonlinear functions:

(a) $y(t) = \log(\|x(t)\| + C)$ for C=1.

(b) $y(t) = \exp(x(t))$ preceded by a linear system with impulse response, g(τ)

(c) $y(t) = \cos(x(t))$

Problem 4-3

The Wiener functionals are orthogonal to each other. As a consequence, if the series is terminated at a certain order, say *N*, it is the *best possible* approximation of the system up to that order in the least-squared error sense. We are going to show this by the following example. Assume that signal *x(t)* (defined on the interval *[0, T]*) is represented by an infinite series of orthogonal signals:

$$x(t) = \sum_{k=0}^{\infty} a_k \cdot f_k(t)$$

with *{f_k(t)}* basis functions that are orthonormal with respect to a given weighting function, *w(t)*, on the interval *[0,T]*. This means that the following inner-product condition holds:

$$\int_0^T f_k(t) \cdot f_m(t) \cdot w(t) dt = \delta_{km} \text{ with } \delta_{km} = \begin{matrix} 1 \text{ for } k = m \\ 0 \text{ for } k \neq m \end{matrix}$$

For simplicity, we take $w(t) = 1$ (think of the Fourier representation, as an example).

(a) Check that the coefficients a_k are determined by:

$$a_k = \int_0^T x(t) \cdot f_k(t) dt$$

Now suppose that we approximate $x(t)$ by taking the first N terms of the series:

$$\tilde{x}(t) \approx \sum_{k=0}^{N} a_k \cdot f_k(t)$$

As a measure for the error in this approximation, we take the mean-squared-error (mse), ε:

$$\varepsilon = \int_0^T [x(t) - \tilde{x}(t)]^2 dt$$

(b) Show that the a_k for which the error reaches its minimum are given by your answer in (a). In other words, the orthogonal approximation is the most efficient.

Problem 4-4

Use the autocorrelation properties of GWN (Eqn. 4.16) and Gramm-Schmidt orthogonalization to determine the second-order Wiener functional, by making the inhomogeneous functional

$$G_2[x(t), P] = k_{02} + \int_0^\infty k_{12}(\tau) \cdot x(t-\tau) \cdot d\tau + \int_0^\infty \int_0^\infty k_{22}(\tau_1, \tau_2) \cdot x(t-\tau_1) \cdot x(t-\tau_2) d\tau_1 d\tau_2$$

orthogonal to the zeroth- and first-order Wiener functionals of Eqn. 4.17 (see Eqn. 4.18 for the answer).

Problem 4-5

Verify that the third-order Wiener functional is indeed orthogonal to the first-order Wiener functional (see Eqn. 4.18 for their formulations).

Problem 4-6

Apply the cross-correlation method of Lee and Schetzen to determine the Wiener kernels of the system in Fig. 4.5, and verify that they are indeed identical to the Volterra kernels of the system.

Problem 4-7

Use the Hermite polynomials to write down the full expression for the fifth Wiener functional, $G_4[h_4, x(t); P]$.

Problem 4-8

Determine the second-order Wiener kernel for the model of Fig. 4.11 (BM to IHC response).

Problem 4-9

In the model of Fig. 4.13 the linear filter is described by the following function:

$$g(t) = \frac{a}{m} \exp\left(-kt\right) \cdot \sin\left(mt\right)$$

with a=2, m=0.3, and k=0.08.

Calculate the Volterra kernels for this system.

***Problem 4-10**

In this Matlab computer exercise you perform the Lee-Schetzen (1965) cross-correlation technique to identify a nonlinear system on the basis of a single pair of input-output data. All you know is that the system either consists of a L – NL, or a NL – L cascade, in which NL is a static second-order nonlinearity:

$$y(t) = a \cdot x(t) + b \cdot x(t)^2$$

and the linear filter has the following appearance:

$$y(t) = m \cdot \exp\left(-kt\right) \cdot \sin(nt)$$

You may obtain the input and output noise signals from the model system (sampled at 1 ms temporal resolution, with a total signal duration is 4.0 seconds) from the book's Website, where you will also find some relevant tips, tricks and scripts to implement cross-correlation functions and linear convolutions.

Estimate the parameter values of the system, [*a, b, m, k* and *n*], as well as the order in which the subsystems were concatenated.

Problem 4-11

When Gaussian White Noise is filtered by a first-order low-pass *linear* system with finite bandwidth and impulse response *k(τ),* the output is also band-limited. Yet, it still has Gaussian statistics, and hence is a Gaussian process. Show this, by determining the expectation values $\langle y \rangle$, $\langle y^2 \rangle$, $\langle y^3 \rangle$ and $\langle y^n \rangle$ of the system's output, *y(t)*, for independent output samples, and use the properties of GWN autocorrelations as expressed in Eqn. 4.16. Distinguish *n*=odd and *n*=even.
If the moments of independent output samples are the same as for GWN, then they represent the same stochastic process.