Master Project Physics/Mathematics (12 months, 42 ec)

Calculating Acoustic Interactions in the Ear by Applying Stretched Coordinates From the Super Formula

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Acoustic waves interact with the human body, head and ears, before reaching the tympanic membranes in the ear canals. In this project we will develop an analytical method to calculate how these acoustic interactions occur. This is a highly nontrivial problem, because the acoustic wave equation (or, for harmonic waves, the Helmholtz equation) has to be solved on the complex geometry of the boundaries. Especially the human outer ear (pinna) poses a challenge in this respect, and so far there are no good analytical methods to deal with this problem.

The pinna has a complex geometrical shape (Fig 1), and it is well established that this shape serves to enable the human auditory system to localize sounds in the vertical and front/back (elevation) direction (i.e. in the medial plane of the head). It thus serves as a natural antenna for sound.

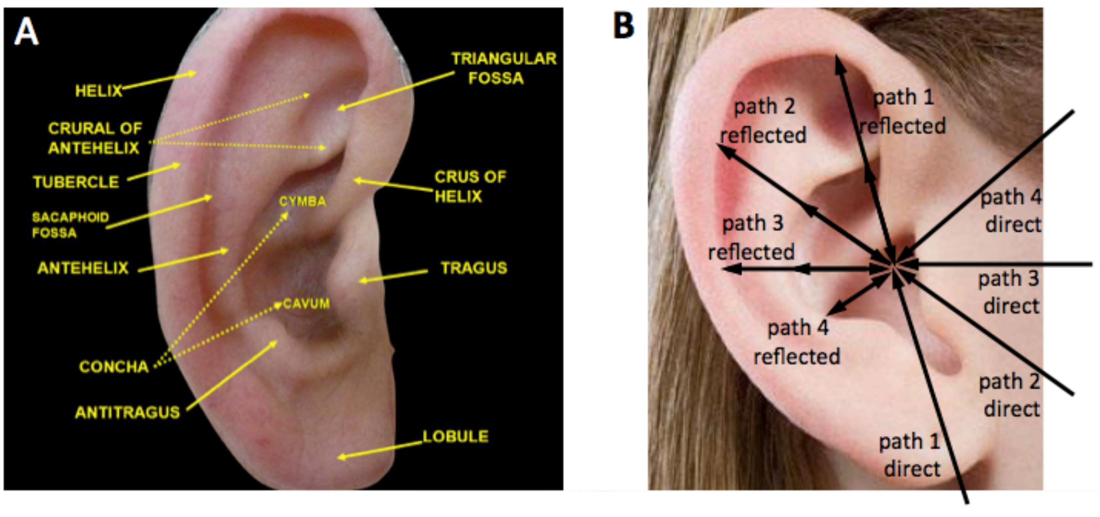

Fig. 1 *The human pinna acts as a direction-dependent, asymmetric acoustic aperture.* **A** *Anatomical features.* **B** *Acoustic waves from a particular direction reach the ear canal directly, and via reflections at the rims of the cavities, here simplified by a single reflection from the helix. Note that the reflections have different path lengths.*

The pinna is known to perform a direction-dependent (elevation angle) transformation on the acoustic input that is described in the literature by the so-called head-related transfer functions, or HRTFs (Fig. 2).

The simplest possible model that leads to a very basic understanding of these HRTFs is a simple reflection-delay model (Fig. 1B) in which the ear canal is stimulated by a direct wave as well as by a reflected wave(s). As a result of interference and path length differences (and associated time delays), frequency-dependent peaks (amplifications) and notches (attenuations) arise as a function of the incident angle of the sound.

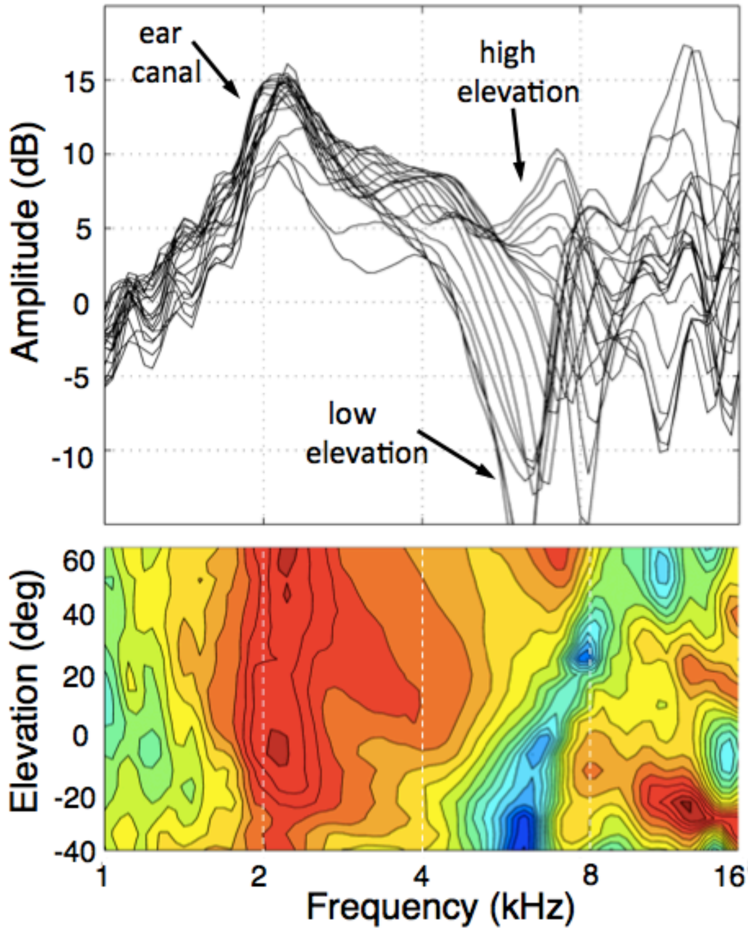

Fig. 2 *Measured head-related transfer functions of a male adult human subject. Data are shown on log-log scale. Top: Amplitude spectra (in dB re. subject's head absent) for GWN stimuli presented in the midsagittal plane at elevations between -40 to +60 deg in 5 deg steps. Up to about 3 kHz the curves all coincide, and show no direction dependence. The amplification around 2-2.5 kHz is due to the first resonance in the ear canal. Curves start to separate at about 4 kHz. Bottom: same data, plotted in color scale. Note the typical elevation-dependent notch (dark blue), running from about 6 kHz at -40 deg, to about 12 kHz at +60 deg.*

In a more realistic description of the ear acoustics, the peaks and notches of the human HRTF have to be understood from both diffraction and reflection, as well as specific resonances of sound waves within the complex shape of the pinna cavities and their boundaries (Taneka et al., 2012).

It has been a long-standing problem in the auditory research field to understand and apply the physical principles that allow one to predict the HRTFs from a given (3D) pinna shape.

Sound propagation through the air is described by the standard homogeneous wave equation:

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad (1)$$

and with harmonic solutions to this linear problem:

$$p(\vec{r},t) = p(\vec{r}) \cdot \exp(-j\omega t) \qquad (2)$$

this equation transforms into the well-known Helmholtz equation:

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \cdot p(\vec{r}) = 0 \qquad (3)$$

The Helmholtz equation has to be solved for every point in space and time, by applying the appropriate boundary conditions. So far, only for simple shapes (free field, points, bars, plates, cylinders, spheres) this equation can be solved with analytical methods. Therefore, to calculate solutions for arbitrary shapes discrete methods are typically applied (boundary element, or finite element methods) that allow one to solve the Helmholtz equation on small elementary shapes (e.g. on the vertices of small triangles) that cover the entire shape. To increase precision, the minimum size of the elementary shapes is determined by the wavelength of the highest frequency in the signals for which solutions are needed. However, the smaller the elements, the longer the calculations will take, making the problem nearly intractable for practical use.
For high sound frequencies (say, > 5kHz, where the HRTFs start to diverge, Fig. 2) the smallest elements are already in the sub-mm range, leading to millions of vertices for the BEM calculations.

Only recently Takemoto et al. (2012) have been able to apply the finite-difference time-domain (FDTD) method that allowed them to circumvent the extremely time-consuming calculations of more classical boundary element methods, especially for the high frequencies, even up to 24 kHz (Fig. 3).

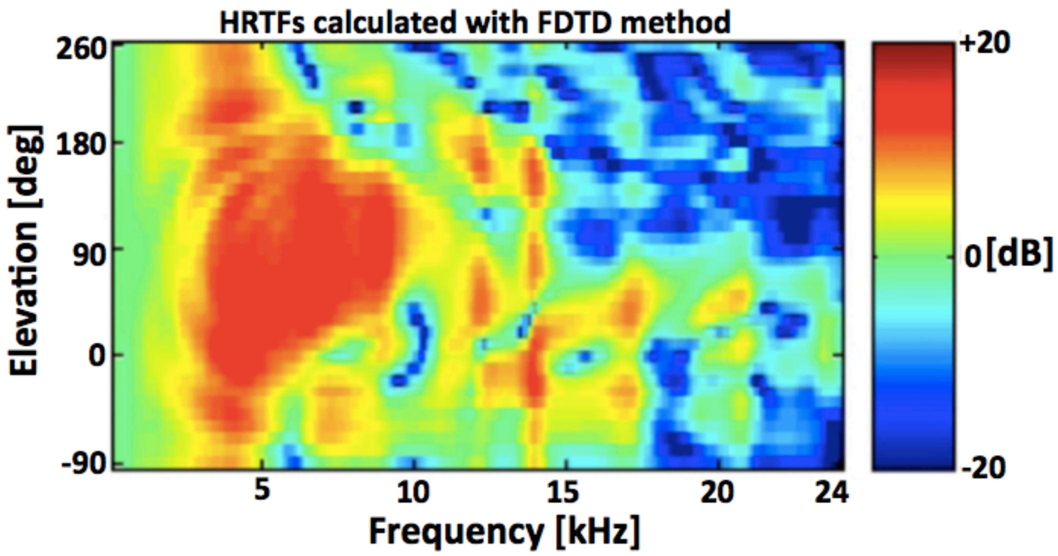

Fig. 3 *Modeled HRTFs of a human ear, calculated over the full elevation and frequency domain on a MRI-generated 3D volume of head and pinna with the FDTD method. After Takemoto et al., 2012.*

Towards an analytical treatment of more complex geometries:
Interestingly, Caratelli and colleagues (e.g. Caratelli et al., 2010) have recently developed a powerful analytical tool to solve the Helmholtz equation that can cope with much more complex shapes than simple spheres. The idea is based on a coordinate transformation from the unit sphere to the actual shape (e.g. a star-like shape) by so-called '*stretched polar coordinates*', which can then be directly applied to the original Helmholtz equation. These stretched coordinates are based on the so-called Super Formula of Gielis (2003), which is a mathematical extension of the super-circle/ellipse equation:

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \qquad (4)$$

in the following way (described in 2D polar coordinates, (r, φ), where x=r cos (φ) and y=r sin (φ)):

$$r(\varphi) = \left[\left[\left(\left|\frac{1}{a}\cos\left(\varphi\frac{m}{4}\right)\right|\right)^{n_2} + \left(\left|\frac{1}{b}\sin\left(\varphi\frac{m}{4}\right)\right|\right)^{n_3}\right]\right]^{-1/n_1} \qquad (5)$$

with free parameters n_1, n_2, n_3, a, b and m. Many complex natural shapes (leaves, snow flocks, etc.) can be described by this formula. For example, n_i=2 and m=4 yields the ellipse. When also a=b it becomes a circle. The parameter *m* divides the plane into m equal sectors, which can now be arbitrarily numerous (instead of the standard four quadrants for m=4), and asymmetries can be introduced by selecting different exponents for the different sectors. The formula can be readily extended to higher dimensions (Fig. 4).

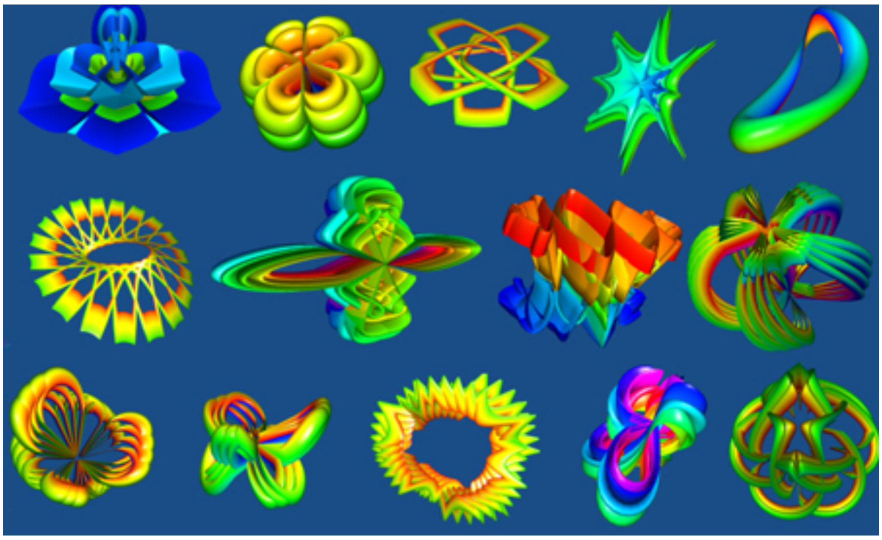

***Fig. 4**. 3D shapes generated with (an extended form of) the superformula.*

After the coordinate transformation, the Helmholtz equation can be solved with standard Fourier methods. (Caratelli et al., 2010)

In this project we will study whether and how we can apply this method to the acoustics of the human head and ears.

Project
(1) As a first attempt we will apply the method to a simple human head model, still without the complex geometries of the pinnae (Fig. 5).

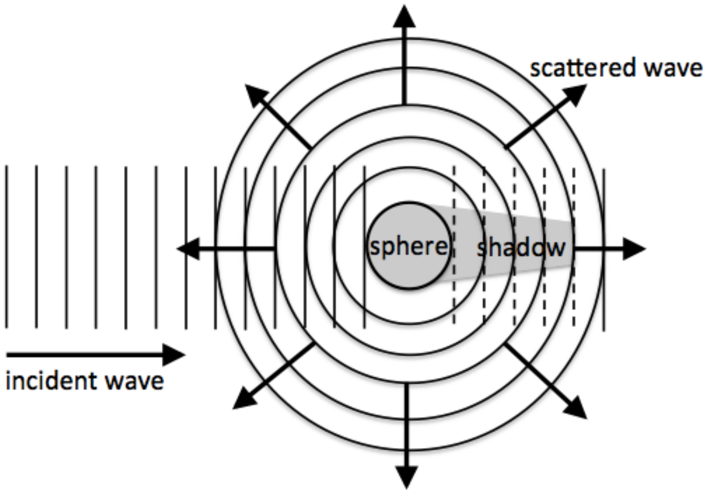

Fig. 5 A simple spherical shape in an acoustic field casts an acoustic shadow, which depends on the sound frequency.

First, the head will be a perfect sphere (radius R), where the ears will be modeled by two (cylindrical) ear canals, diametrically opposed to each other with a certain depth, D~0.2R) and diameter (d), and closed by a circular tympanic membrane that is elastic, e.g. with a given, frequency independent acoustic impedance (this impedance depends on the system that is attached to the tympanic membrane, and consists of the middle-ear bones and the water-filled cochlea; as a start we may make this impedance infinite, i.e. a perfect reflector).

We aim to calculate the air pressure at the two tympanic membranes for planar incident sound waves of different frequencies and incident angles in both the horizontal (azimuth) and vertical (elevation) directions.

We should be able to see the Head-Shadow-Effect (see Van Opstal, Ch 7) as function of frequency and incident azimuth (horizontal) angle for this simple system.

(2) Then, we will change the shape of the obstruction to a slightly more realistic head shape (ellipsoid), and also include the effect of model shoulders and torso. Now we expect some direction sensitivity in the elevation direction too.

(3) Finally, we will develop a suitable morphing coordinate transformation in order to approximate an elementary pinna shape, which is asymmetric with respect to the ear canal (e.g., López-Poveda and Meddis, 1996, approximated the pinna by a parabolic plate (the concha; Fig. 1A) and a reflector helix in Fig. 1A).

Reference material

Chapter 7 of AJ van Opstal's
"*Human Audition and Sound-Localization Behavior*"
(attached as pdf)
(for an elementary background on the human sound-localization cues)

Caratelli D, Natalini P and Ricci PE (2010): Fourier solution for the wave equation for a star-like-shaped vibrating membrane. *Computers and Mathematics with Applications* 59: 176-184

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Takemoto H., Mokhtari P., Kato H. and Nishimura R. (2012): Mechanism for generating peaks and notches of head-related transfer functions in the median plane. *Journal of the Acoustical Society of America* 132: 3832-3841

Lopez-Poveda EA and Meddis R (1996): A physical model of sound diffraction and reflections in the human concha. *Journal of the Acoustical Society of America* 100, 3248-3259 (1996).