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Divisibility by 11, 111, 1111, etc., and the birthday G_8 number

This article proposes an alternative rule for the divisibility of arbitrary natural numbers by the factor 11. The proposal leads to a general equation that generates a rule for the divisibility of arbitrary natural numbers of order M: $N_M \equiv [a_{M-1}a_{M-2}\cdots a_2a_1a_0]$ by $\underbrace{[111\cdots 111]}_{M}$.

The result is applied to 8-digit numbers that correspond to all 44,925 birthdays (dd-mm-yyyy) between 01-01-1900 and 31-12-2022. To that end the birthday-specified G_8 number is introduced and some special cases are discussed.

Divisibility by 11

One of the well-known rules for the divisibility of natural numbers by 11 is that when adding the odd-numbered digits (reading from left to right) and subtracting the even-numbered digits of the number, the total should add up to zero, or be divisible by 11. For example: 132 gives 1+2-3=0, and therefore is divisible by 11 (12x11). Similarly, 12573 gives 1+5+3-2-7=0, and it can be written as 1143 x 11.

Here we describe an alternative way to obtain divisibility by 11 from **arbitrary numbers.** The following rule holds for all integer numbers with absolute value |N| ≥ 10:

Any number N_M, for which the number of digits M≥2, becomes divisible by 11 when adding (M even) or subtracting (M odd) its reversed copy.

The *reversed copy* of a natural number, $N_{M,rev}$, is obtained by writing all its digits in reversed order (e.g., N_4 = 1234 becomes $N_{4,rev}$ = 4321).

M=2

Every two-digit number N_2 between 10 and 99 has the following property: if summed with its reversed copy, the result is divisible by 11.

For example, N_2 = 41 gives 41+14 = 55.

It can be readily shown that this property indeed holds for all numbers with M=2, written as $N_2 = [a_1a_0]$, which represents in decimal notation $N_2 = a_1\cdot 10^1+a_0\cdot 10^0$:

$S_2 = [a_1a_0] + [a_0a_1] = (a_0+a_1)\cdot(10^1+10^0) = (a_0+a_1)\cdot 11$ q.e.d.

In summary, the property reads: **mod ($[a_1a_0]$ + $[a_0a_1]$, 11) = 0**.

M>2

This property can be extended to arbitrarily large numbers of order M, written as $N_M = [a_{M-1}a_{M-2} ....a_2a_1a_0] = a_{M-1}\cdot10^{M-1}+ a_{M-2}\cdot10^{M-2}+......+ a_2\cdot10^2+ a_1\cdot10^1+a_0\cdot10^0$

Then, the following holds:

for **M even**: $S_{M+} = [a_{M-1}a_{M-2} ....a_2a_1a_0] + [a_0a_1a_2 ....a_{M-2}a_{M-1}]$ is divisible by 11.

For example: 8,647,332,126 + 6,212,337,468 = 14,859,669,594 = 1,350,879,054 x 11

for **M odd**: $S_{M-} = [a_{M-1}a_{M-2} ....a_2a_1a_0] - [a_0a_1a_2 ....a_{M-2}a_{M-1}]$ is divisible by 11.

For example: 18,647,332,126 - 62,123,374,681 = -43,476,042,555 = -3,952,367,505 x 11

Both properties can be proven by induction.

Proof:

M even: set M=2n. Express the numbers in their decimal expansion, and group digits with the same decimal:

$S_{M+} = N_M+N_{M,rev} = [a_{2n-1}a_{2n-2} ....a_2a_1a_0] + [a_0a_1a_2 ....a_{2n-2}a_{2n-1}] = \Sigma_k (a_{2n-k} + a_{k-1})\cdot(10^{2n-k} + 10^{k-1})$ for n=1,2,3,.... and $1\le k\le n$

It has to be shown that $KN = (10^{2n-k} + 10^{k-1}) = 10^{k-1}\cdot(10^{2(n-k)+1} + 1)$ is divisible by 11 for all n=1,2,3,.... and $1\le k\le n$

One therefore only has to look at the requirement: $\mathrm{mod}(10^{2(n-k)+1} + 1,11) = \mathrm{mod}(K,11) = 0$.

Note that the requirement is alway true for any n when k = n: in that case, K = (10 + 1) = 1x11.

For k=n-1 one obtains: $K = (10^{2(n-n+1)+1} + 1) = 1001 = 91\text{x}11$.

Then for k=n-2: $K = (10^{2(n-n+2)+1} +1) = 100001 = 9091\text{x}11$, etc., down to k=1:

For k=1: $K = (10^{2n+1} +1)$ is divisible by 11 for all $n\ge2$.

This statement directly follows from induction (and from the well-known odd-even digit rule: 1-1=0).

q.e.d.

M odd: set M=2n+1. Write the numbers in their decimal expansion, and group digits with the same decimal (a_c is the central digit):

$S_{M-} = N_M-N_{M,rev} = [a_{2n}a_{2n-1} ....a_c.....a_2a_1a_0] - [a_0a_1a_2 ....a_c....a_{2n-1}a_{2n}] = \Sigma_k (a_{2n-k} - a_{k-1})\cdot(10^{2n-k} + 10^{k-1})$ for n=1,2,3,.... and $1\le k\le n$, see above.

q.e.d.

From the divisibility generator of 11, we now proceed to a rule that generates divisibility by 111 for N_3 numbers.

Divisibility by 111 for N_3 numbers (N ∈ 100 - 999)

To obtain divisibility by 111 for all N_3 numbers, the following rule holds:

Any number $N_3 = [a_2a_1a_0]$ becomes divisible by $10^2+10^1+1 = 111$ when taking the sum of all unique permutations of its digits.

Note that this rule is an extension of the divisibility-by-11 rule for N_2 numbers, which gave: $S_2 = [a_1a_0] + [a_0a_1]$.

Example: N_3=467 gives S_3 = 467+476+647+674+746+764 = 3774 (= 34 x 111)

Proof:

If all digits differ, the total sum of the six possible permutations gives:

$$S_3 = [a_2a_1a_0] + [a_2a_0a_1] + [a_1a_2a_0] + [a_1a_0a_2] + [a_0a_1a_2] + [a_0a_2a_1] = 2\cdot(a_0+a_1+a_2)\cdot 111$$

If one of the digits repeats (say, $a_0=a_1$) the total sum of unique permutations gives:

$$S_3 = [a_2a_1a_1] + [a_1a_2a_1] + [a_1a_1a_2] = (2a_1+a_2)\cdot 111$$

If all digits are identical $S_3 = [a_2a_2a_2] = a_2\cdot 111$

All three cases are indeed divisible by 111.

q.e.d.

From this result, it is straightforward to extend the rule to arbitrarily large numbers.

Divisibility of [111 ..... 111] by 11, 111, 1111, etc.

The following property holds for the divisibility of $\underbrace{[111\cdots 111]}_{2M}$ (an even number of ones) by 11:

$$\sum_{m=0}^{2M-1} 10^m = 11 \cdot \sum_{m=0}^{M-1} 10^{2m}$$

Examples: M = 1: 11 = 11 x 1 M = 2: 1,111 = 11 x 101 M = 3: 111,111 = 11 x 10,101 M = 4: 11,111,111 = 11 x 1,010,101 etc.

In the same way, it can be readily seen that for divisibility by 111 the following must hold for $\underbrace{[111\cdots 111]}_{3M}$ (number of digits divisible by 3):

$$\sum_{m=0}^{3M-1} 10^m = 111 \cdot \sum_{m=0}^{M-1} 10^{3m}$$

Examples: M = 1: 111 = 111 x 1 M = 2: 111,111 = 111 x 1001 M = 3: 111,111,111 = 111 x 1,001,001 M = 4: 111,111,111,111 = 111 x 1,001,001,001 etc.,

from which one can formulate for the general case (multiples of 1, 11, 111, 1,111, 11,111, etc.):

$$\sum_{m=0}^{K\cdot M-1} 10^m = \sum_{k=0}^{K-1} 10^k \cdot \sum_{m=0}^{M-1} 10^{K\cdot m} \ \ \textbf{for} \ \ M = 1, 2, 3, \cdots \ \ \textbf{and} \ \ K = 1, 2, 3, \cdots (\leq M)$$

Some examples:

M=2, K=2: 1,111 = 11 x 101

M=3, K=3: 111,111,111 = 111 x 1,001,001

M=6, K=4: 111,111,111,111,111,111,111,111 = 11,111 x 100,001,000,010,000,100,001

M=7, K=5: 11,111,111,111,111,111,111,111,111,111,111,111 = 111,111 x 1,000,001,000,001,000,001,000,001,000,001

etc.

We now proceed with the question how to make arbitrary natural numbers, written as N_M, divisible by [111 ... 111] (M ones)

Divisibility of N_M by [111 ...... 111]

Any M-digit number N_M = [$a_{M-1}a_{M-2}$ ...... a_2 a_1 a_0] can be made divisible by $\sum_{m=0}^{M-1} 10^m = \underbrace{[111 \cdots 111]}_{M}$

after taking the sum of all its unique digit permutations.

This sum is obtained from the following expression:

$$S_M = \frac{(M-1)!}{\prod_{s=1}^{m} n_s!} \cdot \left(\sum_{k=0}^{M-1} a_k \right) \cdot \sum_{m=0}^{M-1} 10^m \quad \textbf{with} \quad \sum_{s=1}^{m} n_s \leq M$$

the m digits a_s that repeat n_s times in the number.

Here, the ratio on the left quantifies the total number of unique permutations.

If all digits are unique (which is possible for numbers with M≤10), the equation reduces to

$$S_M = (M-1)! \cdot \left(\sum_{k=0}^{M-1} a_k \right) \cdot \sum_{m=0}^{M-1} 10^m$$

Note that these equations also capture the N_2 and N_3 cases:

M=2 gives S_2 = 1!·(a_0+a_1)·11 (both digits unique), and S_2 = (1!/2!)·(a_1+a_1)·11 = a_1·11 (identical digits).

M=3 yields 2!·(a_0+a_1+a_2)·111 (unique), or (2!/2!)·(a_1+a_1+a_2)·111 = (2a_1+a_2)·111 (one repetition), or (2!/3!)·(a_2+a_2+a_2)·111 = a_2·111.

Examples: for N_6=563,892 (all six digits unique), one obtains S_6 = 5!·33·111,111 = 439,999,560 and for N_6 = 563,863 (2 repetitions for digits 6 and 3), this yields S_6 = 5!/(2!2!)·30·111,111 = 99,999,900

The maximum S_M(M)

For an arbitrary number N_M, the maximum value of S_M is obtained when (i) its digits are maximally different (leading to the largest number of permutations), and (ii) its digits are largest (giving the highest sum of its digits).

For example, for the maximum value of S_6 the number N_6 should contain the digits 9,8,7,6,5 and 4 (digit sum = 39). For any such number (there are 6! = 720 different ones), the sum of all 720 digit permutations yields S_6 = 5!·39·111,111 = 519,999,480, which is the largest possible value of S_6. For M>10 digit repetitions become unavoidable. Thus, for the largest possible S_M the number of repetitions for all 10 digits (0, 1, 2, ...,9) is n_{rep} = div(M,10), and then for the remaining mod(M,10) digits (starting at 9, and counting downwards) the number of repetitions is n_{rep} = div(M,10)+1.

For example, the largest possible S_{34} is generated by adding all permutations in N_{34} containing 4 repetitions of {9,8,7,6} and 3 repetitions of {5,4,3,2,1,0}; e.g. N_{34} = 9,999,888,877,776,666,555,444,333,222,111,000

For this number (of which there are $5.61\cdot10^{26}$ unique permutations), the total sum yields S_{34} = $33!/(4!^4 3!^6)\cdot165\cdot\sum_{m=0}^{33} 10^m \approx 9.2559\cdot10^{28}\cdot\sum_{m=0}^{33} 10^m \approx 1.0284\cdot10^{62}$.

Figures 1 and 2 show that S_M(M) grows approximately exponentially fast with M (Fig. 1; calculated for M≤100). The growth-rate, dS_M/dM, however, is not constant (Fig. 2): at every transition where a new 9 is repeated (i.e., after M=m·10, with m=1,2,3,....) the growth rate jumps downward. Note that for large M the downward jumps become smaller, and $dS_M/dM \approx 100$.

Application to birthdays: M=8 and calculating the G_8 number

One can calculate the S_8 value for a date (e.g., a birthday, given by dd-mm-yyyy). For dates, however, the possible N_8 numbers are subjected to some restrictions, since dd ≤ 31, mm ≤ 12, and if one only considers the possible birthdays of all people living (near) today, we may restrict the years to 1900 ≤ yyyy ≤ 2022. This restriction leaves 44,925 possible dates between 01-01-1900 and 31-12-2022, including leap-days.

Birthdays may consist of 8 unique digits, e.g. 26-03-1957, but it is more common that there will be digit repetitions, e.g. in 01-01-1900, and in 11-11-1958. Since for all birthdays the common factor is 11,111,111, we can normalise the S_8 by this factor, and call it the **G_8 number** (after R.J. Goderie):

$$G_8 = \frac{7!}{\prod_{s=1}^{m} n_s!}\cdot\left(\sum_{k=0}^{7} a_k\right) \quad \text{with} \quad \sum_{s=1}^{m} n_s \leq 8$$ the m (≤ 4) digits a_s that repeat n_s times.

The following can be noted:

- the number of unique permutations, given by the left-hand factor, can only attain 19 different values, for 21 different possible repetition sequences. They are all listed in Table 1, and may be observed as the more or less isolated clusters in Fig. 3.

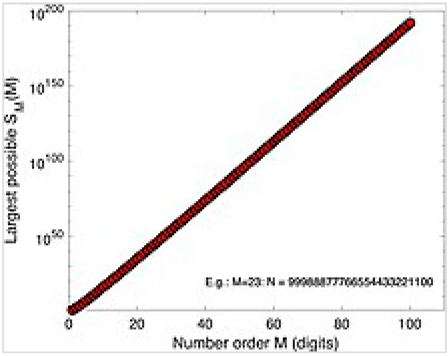

Fig. 1: Increase of $S_M(M)$ as function of M on logarithmic scale. The line suggests and exponential increase, but Fig. 2 shows that the slope of the line is not constant. Maximum numbers for general M are obtained by adding the maximum digit after div(M,10) repetitions, see N_{23} as an example. Note that 10^{190} is vastly much larger than the estimated number of Planck volumes in the observable universe (~10^{185}).

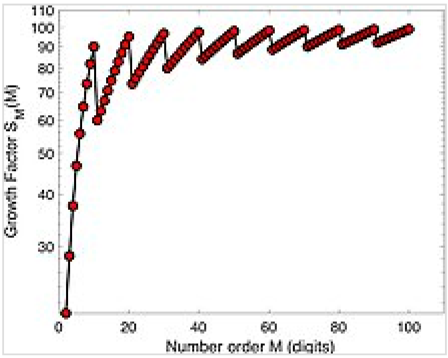

Fig. 2: Growth rate of the $S_M(M)$ function (local slope of the line in Fig. 1) with M. Note that the growth rate approximates the value of 100 for large M. The downward jumps at M=11,21, 31, etc. result from adding the extra repetition of digit 9 (e.g. for M=10: $N_{10,max}$ = 9876543210, and for M=11: $N_{11,max}$ = 99876543210).

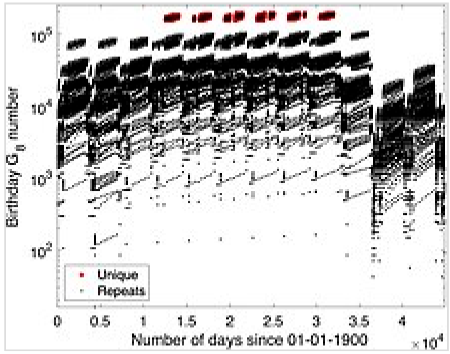

Fig. 3: All 273 possible G8 numbers, generated for all 44,925 dates since 01-01-1900 on logarithmic scale. The red dots correspond to the 360 unique-digit dates, and yield only 9 different G_8 numbers. The different clusters correspond to the 19 different permutation patterns of Table 1. The sudden downward jump on the right is the millennium change from 31-12-1999 to 01-01-2000.

Table 1: All 21 possible permutations for dates between 01-01-1900 and 31-12-2022

7! = 5040	**7!/2! = 2520**	**7!/(2!2!) = 1260**	**7!/(2!2!2!) = 630**	**7!/(2!2!2!2!) = 315**	**7!/3! = 840**
7!/(3!2!) = 420	**7!/(3!2!2!) = 210**	**7!/(3!3!) = 140**	**7!/(3!3!2!) = 70**	**7!/(4!) = 210**	**7!/(4!2!) = 105**
7!/(4!2!2!) = 52.5	**7!/(4!3!) = 35**	**7!/(4!4!) = 8.75**	**7!/5! = 42**	**7!/(5!2!) = 21**	**7!/6! = 7**
7!/(5!3!) = 7	**7!/(6!2!) = 3.5**	**7!/7! = 1**			

- The total number of different G_8 numbers in these 123 years (i.e., 44,925 dates) is only **$N(G_8)$ = 273.**
- The birthday with the largest number of digit repetitions (n_{rep} = 7) is 11-11-1911, leading to the **smallest G_8 number** of all dates: **$G_{8,MIN}$ = 16**.
- The largest G_8 numbers are found for dates with 8 unique digits. There are 360 such dates (only 0.8% of the total data set), with digit sums between 30 and 38 (Fig. 3, red dots). This means that these 360 dates can only attain **9 different G_8 values**. Only 24 of these dates have the highest digit sum of 38, e.g., 28-07-1956, which has **$G_{8,MAX}$ = 191,520**, which is 11,970 times as large as the smallest G_8. Note also that these 360 dates are clustered over a restricted time window within the 123 years: between 26-05-1934 (digit sum: 30, day 12,564) and 25-06-1987 (digit sum: 38, day 31,952).
- Clearly, not every birthday yields a unique G_8: for example, 13-12-1958, with G_8 = 25,200 gives the same result as 31-12-1958, and 371 more dates, such as 24-03-1901, 30-04-1921, 11-08-1937, 27-11-1945, etc. (in this case, these dates have either a single triple repetition and a total digit sum of 30, like in 13-12-1958, or two double repetitions with a digit sum of 20, like in 30-04-1921). This also holds for the four dates with the **lowest digit sum of 4**: 01-01-2000, 10-01-2000, 01-10-2000 and 10-10-2000, which all yield G_8 = 84.

- For the entire period of 123 years (which covers all ages of the total world population), only 13 birthdays have a **unique G_8 number**. Table 2 lists them all.
- The frequency distribution of all G_8 numbers is shown in Fig. 4. The 13 unique dates are seen at the bottom of the graph. Although distributed over the entire period of 123 years (Table 2), they all have relatively low G_8 numbers.
- The G_8 number that occurs **most frequently** in these 123 years is **G_8 = 32,760**. It is generated by **838 different birthdays** that share the following properties: eiher the date contains two double repetitions and a digit sum of 26 (this occurs 806 times; some examples are 28-05-1901, 28-04-1902, 16-07-2019, 25-07-2019, etc.), or the date contains one triple repetition with a digit sum of 39 (this occurs only 32 times, like in 26-03-1999, and 25-04-1999).

Table 2: The 13 unique birthdays and their G_8

Date	G_8 number
11-11-1911*	16
11-11-2011	56
22-12-2022	91
22-09-2022	399
22-12-1922	441
13-11-1933	770
07-07-2007	805
14-11-1944	875
19-09-1999	987
09-09-2009	1015
27-07-2007	1750
29-09-1999**	2016
29-09-2009	2170

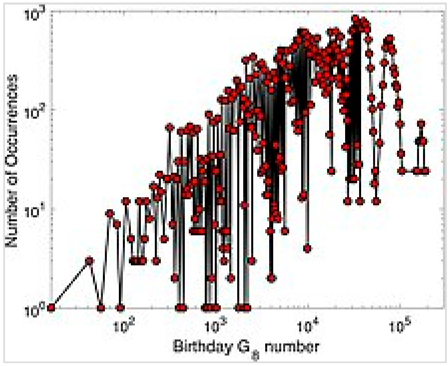

Fig. 4: Frequency of occurrence of all 273 birthday G_8 numbers. Only 13 numbers are uniquely coupled to a single date (at the bottom of the graph; Table 2). All others are generated by multiple birthdays, with the most popular number G_8 = 32,760 (838 times; peak of the graph).

*Date with the largest number of digit repetitions (n_{rep}=7), only to be beaten by 11-11-1111.

**This date yields the highest possible digit sum (48) of all >1,000,000 dates between 01-01-0001 and the end of the 28th century (2899)!

All dates between 01-01-0001 and 31-12-2099

One can perform the same analysis on all dates since January 1 of the Year 1. Up until the last day of this century (31-12-2099) this yields **766,644 days** in total. Interestingly, this vast expansion of the number of possible dates has some, albeit relatively small, influence on the numbers mentioned above. For example:

- The total number of days containing only unique digits grows from 360 to **N = 2520**, but this is still only a tiny fraction of **only 0.3%.** The very first day for which this occurs is **27-06-1345** (digit sum: 28), and the very last day for which it occurs remains **25-06-1987** (digit sum: 38). These unique dates are confined to a relatively tight time window across the 21 centuries, covering only 12,564 days = 1.6%. Fig. 5 shows these dates on linear scale to highlight that these 2520 dates generate only **11 different G_8 numbers** (only 2 more than over the last 123 years).

- The total number of different possible G_8 numbers increases slightly from 273 to (still only) **$N(G_8)$ = 302**, because one new permutation is added: 7!/8! = 0.125 (8 identical digits: 11-11-1111), in combination with a larger range of digit sums (the minimum is now 3, for 01-01-0001, and 10-10-1000, etc.).
- There are only **5 dates** with a **unique G_8** number! They are listed in Table 3. The other 12 unique birthdays of Table 2 are therefore confined to the 123 years for the current world population.

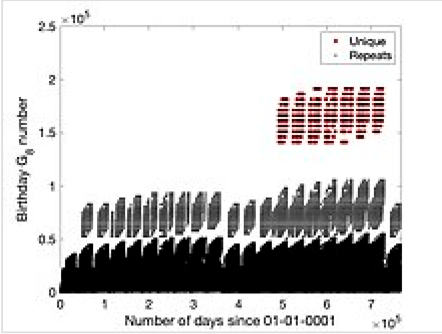

Fig. 5: All 302 G_8 numbers for all 766,644 possible dates between 01-01-0001 and 31-12-2099 on linear scale. Each vertical cluster of points belongs to one century. The red dots (7 clusters) show the dates for which all 8 digits are different, and span the period between 27-06-1345 and 25-06-1987. Note that these 2520 dates only generate 11 different G_8 numbers. Note also the deep dips at the two millennium transitions, at day numbers 364,878 (01-01-1000) and day 730,120 (01-01-2000), respectively.

Table 3: Unique G8 numbers over 2100 years

Date	G_8 number
11-11-1111	1
26-06-1666	1386
27-07-1777	1596
28-08-1888	1806
29-09-1999**	2016

- The most popular G_8 number of all times is **G_8 = 8,400**, which occurs **14,208 times**. It consists mostly of dates with a single triple repetition and one double repetition (14,074 times), for which the digit sum is 20, starting in 29-12-0006, and ending at 10-10-2097, but also including dates like 26-10-1316. For 134 of these popular G_8 dates the digit sum is 40, and it contains one triple repetition and two doubles, the first occurrence being 19-09-0669, and the last one is found at 27-12-1999.

- In summary, two dates stick out in human history since the birth of Christ: **11-11-1111** and **29-09-1999**. The former contains 8 repetitive digits and the absolute lowest possible G_8=1, whereas the latter generate the highest possible digit sum.

What happens to the G8 distributions if we adopt a different calendar??

Ok, so you say, what the heck? This is some quirky property of our peculiar Julian Calendar.... To check for this possibility, let's introduce a more regular calendar consisting of 13 months of 28 days (= 4 weeks), which leaves only one month with 29 days (e.g., month 13 = December). The new extra month we will call **Trajan** (after the Roman emperor Trajan, 98-117 AD), and is month 8.

The leap day (again, every 4 years) is also added to month 13; in that case, December will have 30 days.

So, let's call the months:

Table 4: The months of the New G8 Calendar with their number of days

1 January 28	2 February 28	3 March 28	4 April 28
5 May 28	6 June 28	7 July 28	**8 Trajan 28**
9 August 28	10 September 28	11 October 28	12 November 28
13 December 29/30			

In this way, the first 12 months of the year always start on the same week day, which is handy (isn't it?). The end of the year marks a transition: if in year N the months start on Monday, then in Year N+1 they will start on Tuesday (except when it's a leap year, in which case they will start on Wednesday).

- **Map dates from the current 12-month calendar to the new G8 calendar (or v.v.):**

Some examples:

```
e.g. Old:   March 26, 1957  =  day 31 +28 + 26 = 85 of the year
      New:      day 85 => 28 + 28 + 28 + 1 = April 1, 1957

      Old:  11-11-1958 = day 31+28+31+30+31+30+31+31+30+31+11 = 304
    New:      day 304 = 10 x 28 + 24 = October 24, 1958
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      Old:  10-05-1953 = day 31+28+31+30+10 = 130
    New:      day 130 = 4 x 28 + 18 = May 18, 1953
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      Old:  17-07-1956 = day 31+29+31+30+31+30+17 = 199
    New:      day 199 = 7 x 28 + 3 = Trajan 3, 1956
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The other way around:
                26-03-1957 (new) = day 2x28+26   =  82 => 23-03-1957 (old)
                11-11-1958 (new) = day 11x28+11 = 319 => 25-11-1958 (old)
```

- Next, do the same statistical analysis for all new G8 calendar dates, from 01-01-001 until 29-13-2099....

Figures 6 and 7 show the graphs that result from this analysis:

- Interestingly, we obtain exactly the same number of dates with unique digits (N=2520), but now they are distributed over 13 centuries. Yet, they still generate only 13 different G8 numbers.

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  The first one occurs on 27-13-0456
  and the last one occurs again on 25-06-1987 (!) So, June 25,
1987 is a truly special date!
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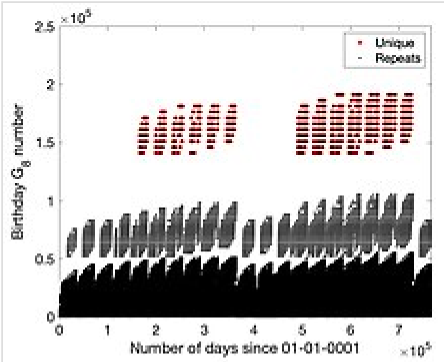

Fig. 6: Distribution of the G8 birthday numbers according to the new calendar.

- There now are 305 different G8 numbers (i.e., only 3 more than with the original calendar), and the most popular

one (G8 =8400) occurs 14,101 times.

There are only 6 days which generate a unique G8 number. These are same 5 as with the normal calendar (see above), PLUS one extra, which is 28-09-1999

- To summarise, the properties of G8 numbers that are derived from birthdays appear to have a peculiar statistic that is not critically dependent on the type of calendar that is being used.

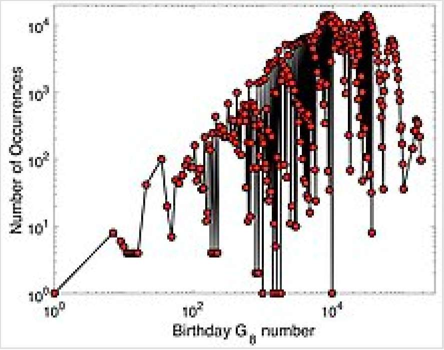

Fig. 7: Distribution of the unique G8 birthday numbers according to the new calendar.