Generating Functional Analysis of Minority Games with Inner Product Strategy Definitions
(work in progress)

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• Brief intro to Minority Games
• Lookup-table versus inner product MGs
• Generalization of GFA for MGs with real histories
• Analysis of macroscopic laws
  – fake history limit
  – true history: random matrix theory
N agents, \( i = 1, \ldots, N \):

- At each round \( \ell \):
  - all get info \( I(\ell) \in \{I_1, \ldots, I_p\} \)
  - each places a bid \( b_i(\ell) \in \mathbb{R} \)

- Those in the **minority** group win
  \[ \sum_j b_j(\ell) > 0 : \quad b_i(\ell) < 0 \text{ wins} \]
  \[ \sum_j b_j(\ell) < 0 : \quad b_i(\ell) > 0 \text{ wins} \]

- Each agent \( i \) has \( S \) strategies converting \( I(\ell) \) into bid \( b_i(\ell) \)

- dynamics:
  - choice of strategy \( a \in \{1, \ldots, S\} \),
  - as a function of time, for all agents,
  - based on strategies' performance
**Lookup table MGs** (Challet, Marsili, Zhang, ...):

- **info**: $m$ step history, $\lambda(\ell) = (\text{sgn}[A(\ell-1)], \ldots, s$
- **strat**: lookup table, $R_{ia} = (R_{ia}^1, \ldots, R_{ia}^p)$
- **bid**: strategy $a$, info $\lambda$, $b_i(\ell) = R_{ia}^\lambda$

**history window**: $\Delta t = \Delta \ell/N = O(N^{-1} \log N)$

**Inner product MGs** (Cavagna, Sherrington, Garrahan):

- **info**: $p$ step history, $\lambda(\ell) = (f[A(\ell-1)], \ldots, f$
- **strat**: linear functional, $R_{ia} = (R_{ia}^1, \ldots, R_{ia}^p)$
- **bid**: strategy $a$, info $\lambda$, $b_i(\ell) = p^{-1} \sum_{\mu \leq p} R_{ia}^{\mu} \lambda_{\mu}(\ell)$

**history window**: $\Delta t = \Delta \ell/N = O(1)$
• Inner product MGs closer to how real agents predict time series (generalized linear models, ARMA, ARIMA, etc)

• Lookup-table MGs analyzed extensively using replica & GFA techniques (including real history version)

• Not true for inner product MGs

• Lookup table: history window $O(N^{-1}\log N) \to 0$
  Inner product: history window $O(N^0)$
history strings:
\[ \lambda(\ell) = (\text{sgn}[A(\ell - 1)], \ldots, \text{sgn}[A(\ell - m)]) \]

\[ \pi_\lambda = \lim_{L \to \infty} \frac{1}{L} \sum_{\ell \leq L} \delta_{\lambda, \lambda(\ell)} \]

\[ \varrho(f) = \lim_{N \to \infty} 2^{-m} \sum_{\lambda} \delta_{\lambda(f)} \]

\[ \alpha < \alpha_c \quad \alpha > \alpha_c \]

• Inner product MGs with real market history
no such thing as \( \varrho(f) \), history string is of length \( O(\)
Generalization of the GFA theory for MGs with real inner product and lookup table MGs as special cases (always $S = 2$)

- decision noise:

\[ \text{sgn}[q_i(\ell)] \rightarrow \sigma[q_i(\ell), z_i(\ell)] \]

\[ \text{additive: } \sigma[q, z] = \text{sgn} \]
\[ \text{multipl: } \sigma[q, z] = \text{sgn} \]

- real vs fake history:

\[ A(\ell - \mu) \rightarrow (1 - \zeta)A(\ell - \mu) + \zeta Z(\ell, \mu) \]

\[ Z(\ell, \mu) : \text{ Gaussian, } \langle Z \rangle = 0 \]
\[ \text{consistent: } Z(\ell, \mu) = Z(\ell - \mu), \langle Z(\ell)Z(\ell') \rangle = \xi \]
\[ \text{inconsistent: } Z(\ell, \mu) \text{ all indep, } \langle Z(\ell, \mu)Z(\ell', \mu') \rangle \]
strategy: \( \mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p) \) \quad 2^m = p

bid: \( b_i(\ell) = R_{ia}^\lambda \)

Inner product MGs:

info: \( \lambda(\ell) = (f[A(\ell-1)], \ldots, f[A(\ell-2)]) \)

strategy: \( \mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p) \)

bid: \( b_i(\ell) = p^{-\frac{1}{2}} \sum_{\mu \leq p} R_{ia}^\mu \lambda_\mu(\ell) \)

- add bid perturbation term:

\[
A(t) = \frac{1}{\sqrt{N}} \sum_i b_i(t) + A_e(t)
\]
generating functional
\[
\overline{Z[\psi]} = \left\langle e^{-i \sum \psi_i(\ell) \sigma[q_i(\ell), z_i(\ell)]} \right\rangle
\]

manipulations:
\[
\overline{Z[\psi]} = \int \mathcal{D}C \mathcal{D}\hat{C} \mathcal{D}G \mathcal{D}\hat{G} \ldots e^{N\Psi[C, \hat{C}, G, \hat{G}, \ldots]}
\]

saddle-point equations
for dynamic order parameters
\[
C(\ell, \ell') = \lim_{N \to \infty} \frac{1}{N} \sum_i \left\langle \sigma[q_i(\ell), z_i(\ell)] \sigma[q_i(\ell'), z_i(\ell')] \sigma[q_i(\ell), z_i(\ell')] \right\rangle
\]
\[
G(\ell, \ell') = \lim_{N \to \infty} \frac{1}{N} \sum_i \frac{\partial}{\partial \theta_i(\ell')} \left\langle \sigma[q_i(\ell), z_i(\ell)] \right\rangle
\]
• Effective single agent process:
\[
\frac{d}{dt} q(t) = \theta(t) - \alpha \int_0^t dt' R(t, t') \sigma[q(t')] + \sqrt{\alpha} \eta(t)
\]

Gaussian \( \eta(t), \langle \eta(t) \rangle = 0, \langle \eta(t) \eta(t') \rangle = 0 \), \( \sigma[q] = \int Dz \sigma[q, z] \)

• Closed eqns for order parameters:
\[
C(t, t') = \langle \sigma[q(t)] \sigma[q(t')] \rangle_\star \quad G(t, t') = \frac{\delta}{\delta \theta(t')} \langle \sigma \rangle
\]

• All complications: in relation between \( \{ R, \Sigma \} \) and involves effective overall bid process
\[ C(t, t') = C(t - t'), \quad G(t, t') = G(t - t') \]
\[ R(t, t') = R(t - t'), \quad \Sigma(t, t') = \Sigma(t - t') \]

static order pars:
\[ \chi = \int_0^\infty dt \ G(t), \quad \chi_R = \int_0^\infty dt \ R(t) \]
\[ c = \lim_{t \to \infty} C(t), \quad \phi : \text{fraction of ‘frozen’ ag} \]
given by
\[ \phi = 1 - \text{Erf}[u] \]
\[ c = \sigma^2[\infty] \left\{ 1 - \text{Erf}[u] + \frac{1}{2u^2} \text{Erf}[u] - \frac{1}{u\sqrt{\pi}} e^{-u^2} \right\} \]
\[ \chi = \text{Erf}[u] / \alpha \chi_R \]
\[ u = \sqrt{\alpha \chi_R \sigma[\infty]} / S_0 \sqrt{2}, \quad S_0^2 = \Sigma(\infty) \]
• $S_0^2 = \Sigma(\infty)$ and $\chi_R = \int_0^\infty dt \; R(t)$

\[
\chi_R = \lim_{\delta \to 0} \left\{ \overline{W}[0, 0; \{A, Z\}] + \sum_{\ell=1}^{\infty} \frac{\partial}{\partial A(0)} \langle \langle \overline{W}[\ell, 0; \{A, Z\}] \rangle \rangle \right\}
\]

\[
S_0^2 = \lim_{\delta \to 0} \lim_{L \to \infty} \frac{\tilde{\eta}}{L^2 \delta} \sum_{\ell, \ell' = 1}^{L} \langle \langle \overline{W}[\ell, \ell'; \{A, Z\}] A(\ell) A(\ell') \rangle \rangle \]

• Differences between models: $\overline{W}[\ell, \ell'; \{A, Z\}]$
  
  similarity between histories observed at times $\ell$ and $\ell'$
  
  lookup table: $\overline{W}[\ell, \ell'; \{A, Z\}] = \delta_{\lambda(\ell,A,Z),\lambda(\ell',A,Z)}$
  
  inner product: $\overline{W}[\ell, \ell'; \{A, Z\}] = \frac{1}{\alpha N} \sum_{\lambda \leq p} f \left[(1-\zeta)A(\ell-\lambda) + \zeta Z(\ell, \lambda)\right] f \left[(1-\zeta)A(\ell'
Effective overall bid process:

\[ A(\ell) = A_e(\ell) + \varphi \ell - \frac{1}{2} \sum_{\ell' < \ell} G(\ell, \ell') W_{\ell, \ell'} \]

Gaussian random fields:

\[ \langle \phi_{\ell} \rangle_{\phi | A, Z} = 0, \quad \langle \phi_{\ell} \phi_{\ell'} \rangle_{\phi | A, Z} = \frac{1}{2} [1 + C(\ell, \ell')] W \]
• Put $\zeta \to 1$, calculate kernels (grammatically), define $\kappa = \int Dz \ f^2[Sz]$

• Inner product and lookup table MGs differ only in characteristic amplitude, that can be transformed away:

$$G_{IP}(t, t') = \kappa^{-1}G_{LU}(t, t'), \quad \theta_{IP}(t) = \kappa\theta_{LU}(t), \quad \sigma_{IF}^2$$
The real problem: generalized MGs with true market

- define random matrix $\mathbf{B}(\ell)$:

$$B_{\lambda\lambda'}(\ell) = \mathcal{F}_\lambda[\ell, A, Z] \mathcal{F}_{\lambda'}[\ell, A, Z]$$

LU:

$$\mathcal{F}_\lambda[\ell, A, Z] = \sqrt{\alpha N} \delta_{\lambda,\lambda}(\ell, A, Z) \lambda$$

IP:

$$\mathcal{F}_\lambda[\ell, A, Z] = f[(1 - \zeta)A(\ell - \lambda) + \zeta Z(\ell, \lambda)] \lambda$$

- $\{R, \Sigma\}$ can be written in terms of

$$\Delta_{r+1}(\ell_0, \ldots, \ell_r) = \frac{1}{p} \text{Tr} \left\langle \left\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r) \right\rangle \right\rangle_{\{A}$$

$$\tilde{\Delta}_{r+r'+2}(\ell_0, \ldots, \ell_r; \ell'_0, \ldots, \ell'_{r'}) =$$

$$\frac{1}{p} \text{Tr} \left\langle \left\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1) \right\rangle \right\rangle_{\{A}$$

$$\tilde{\Delta}_{r+r'+1}(\ell_0, \ell_1 \ldots, \ell_r; \ell_0, \ell'_1, \ldots, \ell'_{r'}) =$$

$$\frac{1}{p} \text{Tr} \left\langle \left\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1) \right\rangle \right\rangle_{\{A}.$$
\[ \prod_{i=1}^{r} B(\ell_i) \to B^r(A, Z) \quad B(A, Z) = \lim_{L \to \infty} \frac{1}{L} \sum_{\ell \leq L} \mu_i(A, Z) e_v o f ~ B(A, Z) : \quad \varrho(\mu) = \langle \langle \frac{1}{p} \sum_{i=1}^{p} \delta[\mu - \mu_i(A, Z)] \rangle \rangle \]

\[ \Delta_{r+1}(\ell_0, \ldots, \ell_r) \to \int_0^{\infty} d\mu \ \varrho(\mu) \ \mu^{r+} \]

\[ \tilde{\Delta}_{r+r'+2}(\ell_0, \ldots, \ell_r; \ell'_0, \ldots, \ell'_{r'}) \to \int_0^{\infty} d\mu \ \varrho(\mu) \ \mu^{r'+} \]

\[ \tilde{\Delta}_{r+r'+1}(\ell_0, \ell_1 \ldots, \ell_r; \ell'_0, \ell'_1, \ldots, \ell'_{r'}) \to \int_0^{\infty} d\mu \ \varrho(\mu) \ \mu^{r'+} \]

\[ \chi_R = \int_0^{\infty} d\mu \ \varrho(\mu) \frac{\mu}{1 + \mu \chi} \quad S_0^2 = (1 + c) \int_0^{\infty} d\mu \ \varrho(\mu). \]
eigenvalue distribution \( \varrho(\mu) \) of random matrix \( B(A \) takes over role of history frequency distribution \( \varrho(f \) in lookup table MGs

- link with lookup table MG theory:

\[
B_{\lambda \lambda'}(A, Z) = p\delta_{\lambda \lambda'}\pi_\lambda(A, Z)
\]

\[
\varrho(\mu) = \lim_{p \to \infty} \frac{1}{p} \sum_{\lambda} \langle \langle \delta [\mu - p\pi_\lambda(A, Z)] \rangle \rangle_{\{A, Z\}}
\]

(recover the history frequency distr)
• Generation functional analysis of MGs can be generalized to include lookup table & inner product as special cases.

• Exact macroscopic theory in terms of effective single and (in case of real histories) an effective overall bid plays a crucial role.

• Fake histories:
  - Inner product and lookup table MGs differ only in simple rescaling of observables, phase diagrams identical.

• Real histories:
  - Short history correlation time ansatz can be generalized.
  - Role of history frequency distribution in lookup table taken over by eigenvalue spectrum of a random matrix (randomness: generated by the effective bid process).

• To be finished:
  - Calculation of random matrix eigenvalue spectrum