Statistical physics of tailored random graphs: entropies, processes, and generation

Lecture I. Common tools and tricks

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The δ-distribution

- **intuitive definition** of $\delta(x)$:

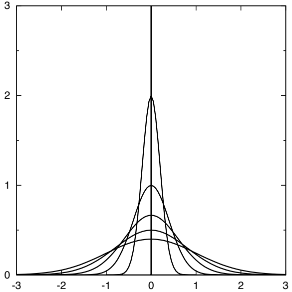
  prob distribution for a 'random' variable x
  that is always zero

  $$\langle f\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\delta(x) = f(0) \qquad \textit{for any } f$$

  for instance

  $$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{2\pi}}\ e^{-x^2/2\sigma^2}$$

  not a function: $\delta(x\neq 0)=0,\ \ \delta(0)=\infty$



- status of $\delta(x)$:

  $\delta(x)$ only has a meaning when appearing *inside an integration*,
  one takes the limit $\sigma\downarrow 0$ *after* performing the integration

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\delta(x) = \lim_{\sigma\downarrow 0} \int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} = \lim_{\sigma\downarrow 0} \int_{-\infty}^{\infty} \mathrm{d}x\ f(x\sigma)\frac{e^{-x^2/2}}{\sqrt{2\pi}} = f(0)$$

- **differentiation** of $\delta(x)$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; f(x)\delta'(x) = \int_{-\infty}^{\infty} \mathrm{d}x \left\{ \frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\delta(x)\Big) - f'(x)\delta(x) \right\}$$

$$= \lim_{\sigma\downarrow 0} \left[f(x) \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \right]_{x=-\infty}^{x=\infty} - f'(0) = -f'(0)$$

generalization:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; f(x) \frac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x) = (-1)^n \lim_{x\to 0} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \qquad (n = 0, 1, 2, \ldots)$$

- **integration** of $\delta(x)$:

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) \qquad \begin{array}{l} \theta(x<0) = 0 \\ \theta(x>0) = 1 \end{array}$$

Proof: both sides have same effect in integrals

$$\int \mathrm{d}x \left\{ \delta(x) - \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) \right\} f(x) = f(0) - \lim_{\epsilon\downarrow 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}x \left\{ \frac{\mathrm{d}}{\mathrm{d}x}\Big(\theta(x)f(x)\Big) - f'(x)\theta(x) \right\}$$

$$= f(0) - \lim_{\epsilon\downarrow 0} [f(\epsilon) - 0] + \lim_{\epsilon\downarrow 0} \int_0^{\epsilon} \mathrm{d}x\; f'(x) = 0$$

- **generalization** to vector arguments:

$$\mathbf{x} \in \mathrm{I\!R}^N: \qquad \delta(\mathbf{x}) = \prod_{i=1}^{N} \delta(x_i)$$

- **Integral representation** of $\delta(x)$

use defns of Fourier transforms and their inverse:

$$\hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\ e^{-2\pi \mathrm{i}kx} f(x) \qquad f(x) = \int_{-\infty}^{\infty} \mathrm{d}k\ e^{2\pi \mathrm{i}kx} \hat{f}(k) \quad \Rightarrow \quad f(x) = \int_{-\infty}^{\infty} \mathrm{d}k\ e^{2\pi \mathrm{i}kx} \int_{-\infty}^{\infty} \mathrm{d}y\ e^{-2\pi \mathrm{i}ky} f(y)$$

apply to $\delta(x)$: $\delta(x) = \int_{-\infty}^{\infty} \mathrm{d}k\ e^{2\pi \mathrm{i}kx} = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\ e^{\mathrm{i}kx}$

- invertible **functions of** x as arguments: $\delta\left[g(x) - g(a)\right] = \frac{\delta(x-a)}{|g'(a)|}$

Proof: both sides have same effect in integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x) \left\{ \delta\left[g(x) - g(a)\right] - \frac{\delta(x-a)}{|g'(a)|} \right\} = \int_{-\infty}^{\infty} \mathrm{d}x\ g'(x) \frac{f(x)}{g'(x)} \delta\left[g(x) - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= \int_{g(-\infty)}^{g(\infty)} \mathrm{d}k\ \frac{f(g^{\mathrm{inv}}(k))}{g'(g^{\mathrm{inv}}(k))} \delta\left[k - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= sgn[g'(a)] \int_{-\infty}^{\infty} \mathrm{d}k\ \frac{f(g^{\mathrm{inv}}(k))}{g'(g^{\mathrm{inv}}(k))} \delta\left[k - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= sgn[g'(a)] \frac{f(a)}{g'(a)} - \frac{f(a)}{|g'(a)|} = 0$$

Gaussian integrals

- one-dimensional:

$$\int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2/\sigma^2} = 1, \qquad \int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\, x\, e^{-\frac{1}{2}x^2/\sigma^2} = 0, \qquad \int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\, x^2 e^{-\frac{1}{2}x^2/\sigma^2} = \sigma^2$$

$$\int \frac{\mathrm{d}x}{\sqrt{2\pi}}\, e^{kx-\frac{1}{2}x^2} = e^{\frac{1}{2}k^2} \quad (k \in \mathbb{C})$$

- N-dimensional:

$$\int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\, e^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = 1, \qquad \int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\, x_i e^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = 0,$$

$$\int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\, x_i x_j e^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = C_{ij}$$

- multivariate Gaussian distribution:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \det\mathbf{C}}}\, e^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}}$$

$$\int \mathrm{d}\mathbf{x}\; p(\mathbf{x}) x_i x_j = C_{ij}, \qquad \int \mathrm{d}\mathbf{x}\; p(\mathbf{x}) e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{C}\mathbf{k}}$$

Steepest descent integration

Objective of steepest descent
(or 'saddle-point') integration:
large N behavior of integrals of the type

$$I_N = \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x})\; e^{-Nf(\mathbf{x})}$$

- $f(\mathbf{x})$ real-valued, smooth, bounded from below, and with unique minimum at $\mathbf{x}^\star$

  expand f around minimum:

$$f(\mathbf{x}) = f(\mathbf{x}^\star) + \frac{1}{2}\sum_{ij=1}^{p} A_{ij}(x_i - x_i^\star)(x_j - x_j^\star) + \mathcal{O}(|\mathbf{x}-\mathbf{x}^\star|^3) \qquad A_{ij} = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}^\star}$$

  Insert into integral,
  transform $\mathbf{x} = \mathbf{x}^\star + \mathbf{y}/\sqrt{N}$:

$$\begin{aligned} I_N &= e^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x}) e^{-\frac{1}{2}N\sum_{ij}(x_i - x_i^\star)A_{ij}(x_j - x_j^\star) + \mathcal{O}(N|\mathbf{x}-\mathbf{x}^\star|^3)} \\ &= N^{-\frac{p}{2}} e^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g\Big(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}}\Big)\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})} \end{aligned}$$

$$\int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x})\; e^{-Nf(\mathbf{x})} = N^{-\frac{p}{2}} e^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g\Big(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}}\Big)\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}$$

- first result, for $p \ll N/\log N$:

$$-\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; e^{-Nf(\mathbf{x})}$$

$$= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} - \frac{1}{N}\log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}\Big]$$

$$= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} - \frac{1}{2N}\log\Big(\frac{(2\pi)^p}{\det \mathbf{A}}\Big) - \frac{1}{N}\log\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)\Big]$$

$$= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} + \mathcal{O}(\frac{p}{N}) + \mathcal{O}(\frac{p^{3/2}}{N^{3/2}})\Big] = f(\mathbf{x}^\star)$$

- second result, for $p \ll \sqrt{N}$:

$$\lim_{N\to\infty} \frac{\int \mathrm{d}\mathbf{x}\; g(\mathbf{x}) e^{-Nf(\mathbf{x})}}{\int \mathrm{d}\mathbf{x}\; e^{-Nf(\mathbf{x})}} = \lim_{N\to\infty} \Bigg[\frac{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g(\mathbf{x}^\star + \mathbf{y}/\sqrt{N})\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}}{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}}\Bigg]$$

$$= \frac{g(\mathbf{x}^\star)\Big(1 + \mathcal{O}(\frac{p^2}{N})\Big)\sqrt{\frac{(2\pi)^p}{\det \mathbf{A}}}\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)}{\sqrt{\frac{(2\pi)^p}{\det \mathbf{A}}}\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)} = g(\mathbf{x}^\star)$$

- $f(\mathbf{x})$ complex-valued:

  – deform integration path in complex plane, using Cauchy's theorem, such that along deformed path the imaginary part of $f(\mathbf{x})$ is constant, and preferably zero

  – proceed using Laplace's argument, and find the leading order in N by extremization of the real part of $f(\mathbf{x})$

  similar fomulae,
  but with (possibly complex) extrema
  that need no longer be maxima:

$$
-\lim_{N\to\infty}\frac{1}{N}\log\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\; e^{-Nf(\mathbf{x})} \;=\; \mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})
$$

$$
\lim_{N\to\infty}\frac{\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\; g(\mathbf{x})e^{-Nf(\mathbf{x})}}{\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\; e^{-Nf(\mathbf{x})}} = g\Big(\arg\,\mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})\Big)
$$

- stuff never mentioned in papers ...

  – in practice we can often not trace the contour deformation in detail
  – often we can *choose* the scaling with N of terms in the exponent, what to do? (check Curie-Weiss magnet, very instructive!)

Exponential distributions

Often we study stochastic processes for $\mathbf{x} \in X \subseteq {\rm I\!R}^N$,
that evolve to a stationary state, with prob distribution $p(\mathbf{x})$
many are of the following form:

- stationary state is *minimally informative*, subject to a number of constraints

$$\sum_{\mathbf{x}\in X} p(\mathbf{x})\omega_1(\mathbf{x}) = \Omega_1 \quad \ldots\ldots \quad \sum_{\mathbf{x}\in X} p(\mathbf{x})\omega_L(\mathbf{x}) = \Omega_L$$

This is enough to calculate $p(\mathbf{x})$:

- information content of $\mathbf{x}$: Shannon entropy hence

$$\textit{maximize} \quad S = -\sum_{\mathbf{x}\in X} p(\mathbf{x}) \log p(\mathbf{x})$$

$$\textit{subject to}: \begin{cases} p(\mathbf{x}) \geq 0 \ \ \forall \mathbf{x}, \ \ \sum_{\mathbf{x}\in X} p(\mathbf{x}) = 1 \\ \sum_{\mathbf{x}\in X} p(\mathbf{x})\omega_\ell(\mathbf{x}) = \Omega_\ell \ \ \textit{for all } \ell = 1\ldots L \end{cases}$$

- solution using Lagrange's method:

$$\frac{\partial}{\partial p(\mathbf{x})}\Big\{\lambda_0\sum_{\mathbf{x}'\in X}p(\mathbf{x}')+\sum_{\ell=1}^{L}\lambda_\ell\sum_{\mathbf{x}'\in X}p(\mathbf{x}')\omega_\ell(\mathbf{x}')-\sum_{\mathbf{x}'\in X}p(\mathbf{x}')\log p(\mathbf{x}')\Big\}=0$$

$$\lambda_0+\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x})-1-\log p(\mathbf{x})=0 \quad\Rightarrow\quad p(\mathbf{x})=\mathrm{e}^{\lambda_0-1+\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x})}$$

($p(\mathbf{x})\geq 0$ *automatically satisfied*)

- 'exponential distribution':

$$p(\mathbf{x})=\frac{\mathrm{e}^{\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x})}}{Z(\boldsymbol{\lambda})},\qquad Z(\boldsymbol{\lambda})=\sum_{\mathbf{x}\in X}\mathrm{e}^{\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x})}$$

$$\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_L):\quad \textit{solved from}\quad \sum_{\mathbf{x}\in X}p(\mathbf{x})\omega_\ell(\mathbf{x})=\Omega_\ell\quad(\ell=1\ldots L)$$

example:
physical systems in thermal equilibrium
$L=1$, $\omega(\mathbf{x})=E(\mathbf{x})$ (energy), $\lambda=-1/k_BT$

$$p(\mathbf{x})=\frac{\mathrm{e}^{-E(\mathbf{x})/k_BT}}{Z(T)},\qquad Z(T)=\sum_{\mathbf{x}\in X}\mathrm{e}^{-E(\mathbf{x})/k_BT}$$

Generating functions

$$p(\mathbf{x}) = \frac{e^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\mathbf{x})}}{Z(\boldsymbol{\lambda})}, \qquad Z(\boldsymbol{\lambda}) = \sum_{\mathbf{x}\in X} e^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\mathbf{x})}, \qquad \langle f \rangle = \sum_{\mathbf{x}\in X} p(\mathbf{x}) f(\mathbf{x})$$

Idea behind generating functions:
reduce nr of state averages to be calculated ...

- define

  $F(\boldsymbol{\lambda}) = \log Z(\boldsymbol{\lambda})$

$$\frac{\partial F(\boldsymbol{\lambda})}{\partial \lambda_k} = \frac{\sum_{\mathbf{x}\in X} \omega_k(\mathbf{x}) e^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\mathbf{x})}}{\sum_{\mathbf{x}\in X} e^{\sum_{\ell=1}^{L} \lambda_\ell \omega_\ell(\mathbf{x})}} = \langle \omega_k(\mathbf{x}) \rangle$$

- how to calculate
  arbitrary state average $\langle \psi \rangle$?

$$F(\boldsymbol{\lambda}, \mu) = \log \Big[\sum_{\mathbf{x}\in X} e^{\mu\psi(\mathbf{x}) + \sum_\ell \lambda_\ell \omega_\ell(\mathbf{x})} \Big]$$

$$\langle \psi \rangle = \lim_{\mu\to 0} \frac{\partial F(\boldsymbol{\lambda}, \mu)}{\partial \mu}, \qquad \langle \omega_\ell \rangle = \lim_{\mu\to 0} \frac{\partial F(\boldsymbol{\lambda}, \mu)}{\partial \lambda_\ell}$$

The replica method

replica method

A clever trick that enables the analytical calculation of averages that are normally impossible to do, except numerically.

is particularly useful for

Complex heterogeneous systems composed of *many* interacting variables, and with *many* parameters on which we have only statistical information. (too large for numerical averages to be computationally feasible)

gives us

Analytical predictions for the behaviour of *macroscopic* quantities in *typical* realisations of the systems under study.

first appearance: Marc Kac 1968
first in physics: Sherrington & Kirkpatrick 1975
first in biology: Amit, Gutfreund & Sompolinksy 1985

- Consider processes with many fixed (pseudo-)random parameters $\boldsymbol{\xi}$, distributed according to $\mathcal{P}(\boldsymbol{\xi})$

$$p(\mathbf{x}|\boldsymbol{\xi}) = \frac{\mathrm{e}^{\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}}{Z(\boldsymbol{\lambda},\boldsymbol{\xi})}, \qquad Z(\boldsymbol{\lambda},\boldsymbol{\xi}) = \sum_{\mathbf{x}\in X}\mathrm{e}^{\sum_{\ell=1}^{L}\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}$$

  – calculating state averages $\langle f\rangle_{\boldsymbol{\xi}}$ for each realisation of $\boldsymbol{\xi}$ is usually impossible

  – we are mostly interested in *typical* values of state averages

  – for $N\to\infty$ macroscopic averages will not depend on $\boldsymbol{\xi}$, only on $\mathcal{P}(\boldsymbol{\xi})$, 'self-averaging': $\lim_{N\to\infty}\langle f\rangle_{\boldsymbol{\xi}}$ indep of $\boldsymbol{\xi}$

  so focus on

$$\overline{\langle f\rangle_{\boldsymbol{\xi}}} \;=\; \sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\langle f\rangle_{\boldsymbol{\xi}} = \sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\Big\{\sum_{\mathbf{x}\in X}p(\mathbf{x}|\boldsymbol{\xi})f(\mathbf{x},\boldsymbol{\xi})\Big\}$$

- new generating function:

$$\overline{F}(\boldsymbol{\lambda},\mu)=\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\log Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi}),\qquad Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi})=\sum_{\mathbf{x}\in X}\mathrm{e}^{\mu\psi(\mathbf{x},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}$$

$$\lim_{\mu\to0}\frac{\partial}{\partial\mu}\overline{F}(\boldsymbol{\lambda},\mu)=\lim_{\mu\to0}\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\left\{\frac{\sum_{\mathbf{x}\in X}\psi(\mathbf{x},\boldsymbol{\xi})\mathrm{e}^{\mu\psi(\mathbf{x},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}}{\sum_{\mathbf{x}\in X}\mathrm{e}^{\mu\psi(\mathbf{x},\boldsymbol{\xi})+\sum_\ell\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}}\right\}$$

$$=\sum_{\boldsymbol{\xi}}\mathcal{P}(\boldsymbol{\xi})\left\{\frac{\sum_{\mathbf{x}\in X}\psi(\mathbf{x},\boldsymbol{\xi})\mathrm{e}^{\sum_\ell\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}}{\sum_{\mathbf{x}\in X}\mathrm{e}^{\sum_\ell\lambda_\ell\omega_\ell(\mathbf{x},\boldsymbol{\xi})}}\right\}=\overline{\langle\psi\rangle_{\boldsymbol{\xi}}}$$

- main obstacle in calculating $\overline{F}$:
  the logarithm ...

  replica identity : $\overline{\log Z}=\lim_{n\to0}\frac{1}{n}\log\overline{Z^n}$

  proof:

$$\lim_{n\to0}\frac{1}{n}\log\overline{Z^n}=\lim_{n\to0}\frac{1}{n}\log\overline{[\mathrm{e}^{n\log Z}]}=\lim_{n\to0}\frac{1}{n}\log\overline{[1+n\log Z+\mathcal{O}(n^2)]}$$

$$=\lim_{n\to0}\frac{1}{n}\log[1+n\overline{\log Z}+\mathcal{O}(n^2)]=\overline{\log Z}$$

- apply $\overline{\log Z} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n}$
  (simplest case $L = 1$)

$$
\begin{aligned}
\overline{F}(\lambda) &= \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \log \Big[\sum_{\mathbf{x}\in X} \mathrm{e}^{\lambda \omega(\mathbf{x},\boldsymbol{\xi})} \Big] = \lim_{n\to 0} \frac{1}{n} \log \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \Big[\sum_{\mathbf{x}\in X} \mathrm{e}^{\lambda \omega(\mathbf{x},\boldsymbol{\xi})} \Big]^n \\
&= \lim_{n\to 0} \frac{1}{n} \log \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \Big[\sum_{\mathbf{x}^1\in X} \ldots \sum_{\mathbf{x}^n\in X} \mathrm{e}^{\lambda \sum_{\alpha=1}^{n} \omega(\mathbf{x}^\alpha,\boldsymbol{\xi})} \Big] \\
&= \lim_{n\to 0} \frac{1}{n} \log \Big[\sum_{\mathbf{x}^1\in X} \ldots \sum_{\mathbf{x}^n\in X} \sum_{\boldsymbol{\xi}} \mathcal{P}(\boldsymbol{\xi}) \mathrm{e}^{\lambda \sum_{\alpha=1}^{n} \omega(\mathbf{x}^\alpha,\boldsymbol{\xi})} \Big]
\end{aligned}
$$

- notes:
  - impossible $\boldsymbol{\xi}$-average converted into simpler one ...
  - calculation involves n ‘replicas’ $\mathbf{x}^\alpha$ of original system
  - but $n \to 0$ at the end ... ?
  - penultimate step true only for *integer* n,
    so limit requires *analytical continuation* ...

since then: alternative (more tedious) routes,
these confirmed correctness of the replica method!

Alternative forms of the replica identity

suppose we need averages, but for
a $p(\mathbf{x}|\boldsymbol{\xi})$ that is not of an exponential form?

or we need to average quantities that we
don't want in the exponent of $Z(\lambda\boldsymbol{\xi})$?

$$p(\mathbf{x}|\boldsymbol{\xi}) = \frac{W(\mathbf{x},\boldsymbol{\xi})}{\sum_{\mathbf{x}'\in X} W(\mathbf{x}',\boldsymbol{\xi})}, \qquad \overline{\langle f\rangle_{\boldsymbol{\xi}}} = \overline{\sum_{\mathbf{x}\in X} p(\mathbf{x}|\boldsymbol{\xi}) f(\mathbf{x},\boldsymbol{\xi})}$$

- main obstacle here:
  the fraction ...

$$\begin{aligned}
\overline{\langle f\rangle_{\boldsymbol{\xi}}} &= \overline{\left[\frac{\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi}) f(\mathbf{x},\boldsymbol{\xi})}{\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi})}\right]} = \overline{\left[\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi}) f(\mathbf{x},\boldsymbol{\xi})\right]\left[\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi})\right]^{-1}} \\
&= \lim_{n\to 0} \overline{\left[\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi}) f(\mathbf{x},\boldsymbol{\xi})\right]\left[\sum_{\mathbf{x}\in X} W(\mathbf{x},\boldsymbol{\xi})\right]^{n-1}} \\
&= \lim_{n\to 0} \sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \overline{f(\mathbf{x}^1,\boldsymbol{\xi}) W(\mathbf{x}^1,\boldsymbol{\xi})\ldots W(\mathbf{x}^n,\boldsymbol{\xi})}
\end{aligned}$$

(again: used integer n, but $n\to 0$ ...)

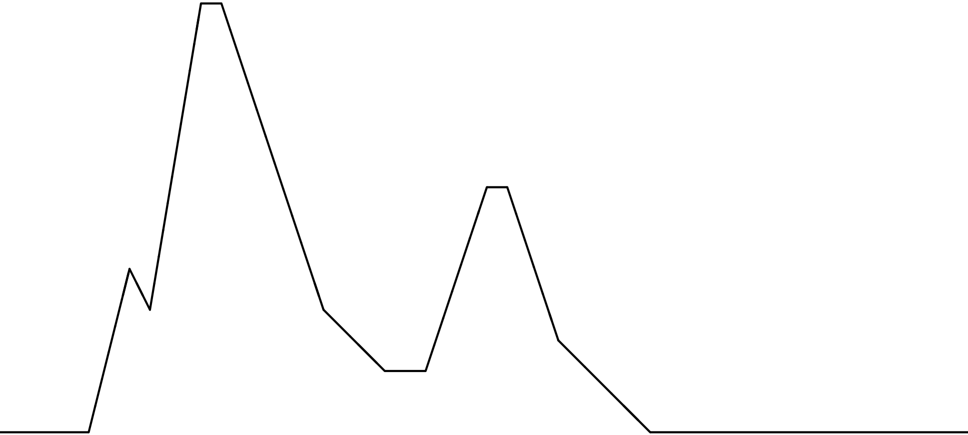
- equivalence between two forms of replica identity, if

$$W(\mathbf{x},\boldsymbol{\xi}) = \mathrm{e}^{\sum_\ell \lambda_\ell \phi_\ell(\mathbf{x},\boldsymbol{\xi})}$$

proof:

$$\begin{aligned}
\overline{\langle f\rangle_{\boldsymbol{\xi}}} &= \lim_{n\to 0} \sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \overline{f(\mathbf{x}^1,\boldsymbol{\xi})W(\mathbf{x}^1,\boldsymbol{\xi})\ldots W(\mathbf{x}^n,\boldsymbol{\xi})} \\
&= \lim_{n\to 0} \sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \overline{f(\mathbf{x}^1,\boldsymbol{\xi})\; \mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\mathbf{x}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{n\to 0} \frac{1}{n} \sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \overline{\Big[\sum_{\alpha=1}^n f(\mathbf{x}^\alpha,\boldsymbol{\xi})\Big]\; \mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\mathbf{x}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{n\to 0} \frac{1}{n} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \overline{\mathrm{e}^{\sum_{\alpha=1}^n \sum_\ell \lambda_\ell \phi_\ell(\mathbf{x}^\alpha,\boldsymbol{\xi}) + \mu\sum_{\alpha=1}^n f(\mathbf{x}^\alpha,\boldsymbol{\xi})}} \\
&= \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \lim_{n\to 0} \frac{1}{n} \overline{\sum_{\mathbf{x}^1\in X}\cdots\sum_{\mathbf{x}^n\in X} \mathrm{e}^{\sum_{\alpha=1}^n \left[\sum_\ell \lambda_\ell \phi_\ell(\mathbf{x}^\alpha,\boldsymbol{\xi}) + \mu f(\mathbf{x}^\alpha,\boldsymbol{\xi})\right]}} \\
&= \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \lim_{n\to 0} \frac{1}{n} \overline{Z^n(\boldsymbol{\lambda},\mu,\boldsymbol{\xi})}, \qquad Z(\boldsymbol{\lambda},\mu,\boldsymbol{\xi}) = \sum_{\mathbf{x}\in X} \mathrm{e}^{\sum_\ell \lambda_\ell \phi_\ell(\mathbf{x},\boldsymbol{\xi}) + \mu f(\mathbf{x},\boldsymbol{\xi})}
\end{aligned}$$

stat mech of complex systems



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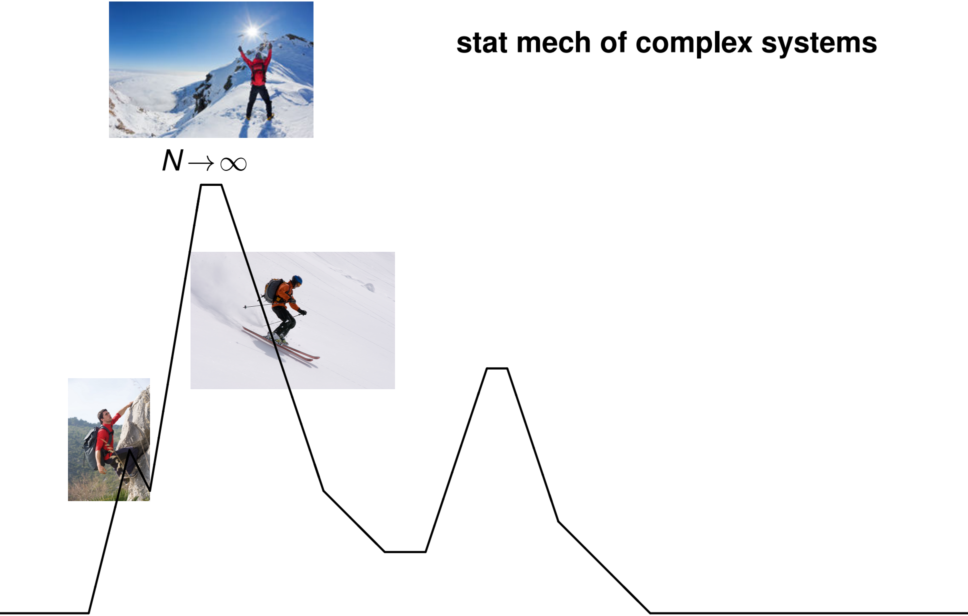

nothing $\leftarrow$ $\rightarrow$ *in business*

stat mech of complex systems