Eigenvalue spectra of graph adjacency matrices

Definition and basic properties

- spectrum $\varrho(\mu)$ of N-node graph $\mathbf{c} = \{c_{ij}\}$:

$$\mu x_i = \sum_j c_{ij} x_j, \qquad \varrho(\mu) = \frac{1}{N} \sum_{k=1}^{N} \delta[\mu - \mu_k]$$

- bounds:

$$-k_{\max} \leq \mu_{\min} \leq \langle k \rangle \leq \mu_{\max} \leq k_{\max}$$

- moments:

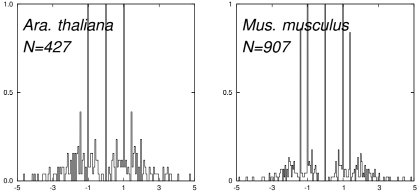
$$\int \mathrm{d}\mu\ \mu \varrho(\mu) = 0, \qquad \int \mathrm{d}_\mu\ \mu^2 \varrho(\mu) = \langle k \rangle, \qquad \int \mathrm{d}\mu\ \mu^{\ell>2} \varrho(\mu) = \frac{1}{N} \mathrm{Tr}(\mathbf{c}^\ell)$$

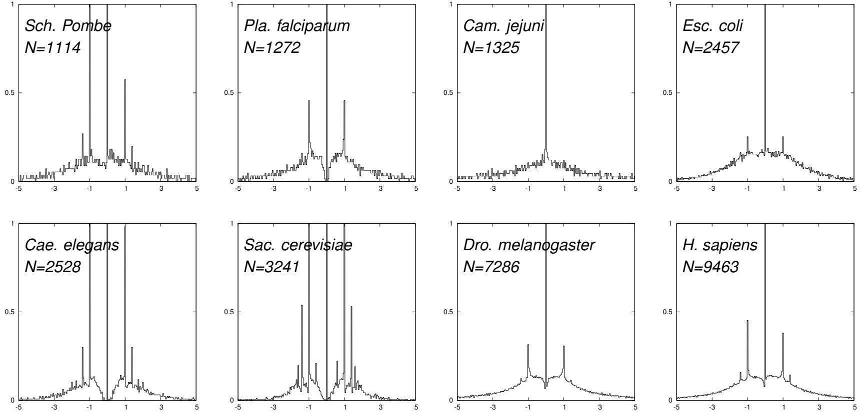
$\mathrm{Tr}(\mathbf{c}^\ell)$ = nr of closed paths of length ℓ

- q-regular graphs:

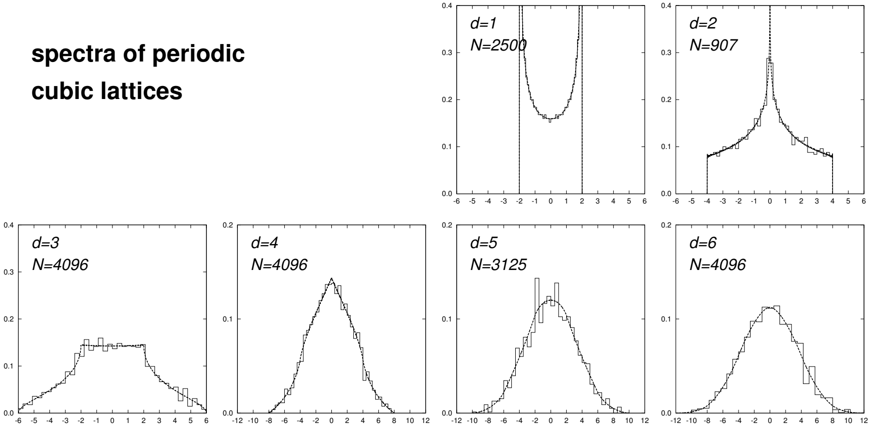
$$-q \leq \mu_{\min} \leq \mu_{\max} = q, \qquad \int \mathrm{d}\mu\ \mu^\ell \varrho(\mu) \leq q^{\ell-1}$$

spectra of protein interaction networks





spectra of periodic cubic lattices



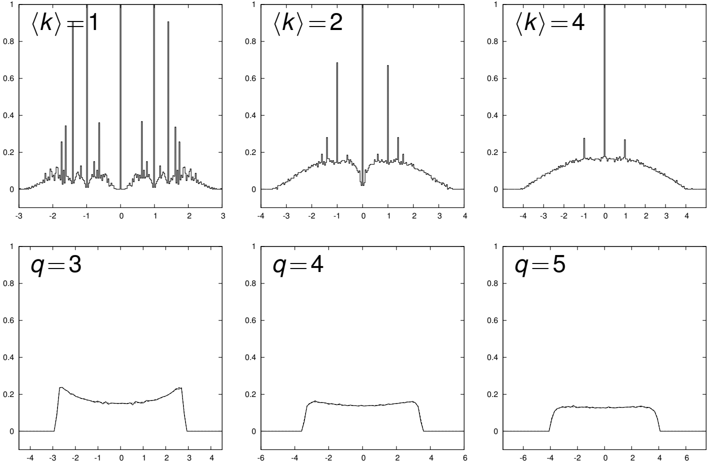
$$N \to \infty: \qquad \varrho_{d+1}(\mu) = \int_0^1 \mathrm{d}x \; \varrho_d(\mu - 2\cos(\pi x)), \qquad \varrho_1(\mu) = \frac{\theta(2-|\mu|)}{\pi\sqrt{4-\mu^2}}$$

locally tree-like random graphs

$p(\mathbf{c}) \propto \delta_{\mathbf{k},\mathbf{k}(\mathbf{c})}$

$$p(k) = \mathrm{e}^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

$$p(k) = \delta_{kq}$$



regular graphs (*McKay* '81) : $\varrho(\mu) = \theta\left[2\sqrt{q-1} - |\mu|\right] \, \dfrac{q\sqrt{4(q-1)-\mu^2}}{2\pi(q^2-\mu^2)}$

Formulae for spectra

formula A : $$\varrho(\mu) = \frac{1}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\,\mathrm{Tr}\left(\mathbf{c} - (\mu+\mathrm{i}\varepsilon)\mathbb{1}\right)^{-1}$$

formula B : $$\varrho(\mu) = \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\frac{\partial}{\partial\mu} \log Z(\mu+\mathrm{i}\varepsilon), \qquad Z(\mu) = \int \mathrm{d}\boldsymbol{\phi}\; \mathrm{e}^{-\frac{1}{2}i\boldsymbol{\phi}\cdot(\mathbf{c}-\mu\mathbb{1})\boldsymbol{\phi}}$$

use the following representation of $\delta(z)$:

$$\delta(z) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Re}\left\{ \frac{\int \mathrm{d}x\; x^2 \mathrm{e}^{-\frac{1}{2}(\varepsilon+\mathrm{i}z)x^2}}{\int \mathrm{d}x\; \mathrm{e}^{-\frac{1}{2}(\varepsilon+\mathrm{i}z)x^2}} \right\}$$

proof of ***B***

$$\frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\frac{\partial}{\partial\mu} \log Z(\mu+\mathrm{i}\varepsilon) = \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\frac{\partial}{\partial\mu} \log \int \mathrm{d}\boldsymbol{\phi}\; \mathrm{e}^{-\frac{1}{2}i\boldsymbol{\phi}\cdot[\mathbf{c}-(\mu+\mathrm{i}\varepsilon)\mathbb{1}]\boldsymbol{\phi}}$$

$$= \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\frac{\partial}{\partial\mu} \log \prod_i \int \mathrm{d}x\; \mathrm{e}^{-\frac{1}{2}\mathrm{i}x^2(\mu_i-\mu-\mathrm{i}\varepsilon)}$$

$$= \frac{1}{N\pi} \sum_i \lim_{\varepsilon\downarrow 0} \mathrm{Im}\left\{\mathrm{i}\frac{\int \mathrm{d}x\; x^2 \mathrm{e}^{-\frac{1}{2}x^2(\varepsilon+\mathrm{i}(\mu_i-\mu))}}{\int \mathrm{d}x\; \mathrm{e}^{-\frac{1}{2}x^2(\varepsilon+\mathrm{i}(\mu_i-\mu))}}\right\} = \frac{1}{N}\sum_i \delta(\mu_i-\mu) = \varrho(\mu)$$

next we use

$$\log\det\mathbf{A} = \mathrm{Tr}\log\mathbf{A}, \qquad \frac{\partial}{\partial\mu}\log(\mathbf{A}+\mu\mathbb{1}) = (\mathbf{A}+\mu\mathbb{1})^{-1}$$

proof of *A*

$$\begin{aligned}
\frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log Z(\mu+\mathrm{i}\varepsilon) &= \frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log\int \mathrm{d}\boldsymbol{\phi}\;\mathrm{e}^{-\frac{1}{2}i\boldsymbol{\phi}\cdot[\mathbf{c}-(\mu+\mathrm{i}\varepsilon)\mathbb{1}]\boldsymbol{\phi}} \\
&= \frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log\left\{\frac{(2\pi)^{N/2}}{\sqrt{\det\big(\varepsilon\mathbb{1}+\mathrm{i}(\mathbf{c}-\mu\mathbb{1})\big)}}\right\} \\
&= -\frac{1}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log\det\big(\varepsilon\mathbb{1}+\mathrm{i}(\mathbf{c}-\mu\mathbb{1})\big) \\
&= -\frac{1}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\;\mathrm{Tr}\frac{\partial}{\partial\mu}\log\big(\varepsilon\mathbb{1}+\mathrm{i}(\mathbf{c}-\mu\mathbb{1})\big) \\
&= \frac{1}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\left\{\mathrm{i}\mathrm{Tr}\big(\varepsilon\mathbb{1}+\mathrm{i}(\mathbf{c}-\mu\mathbb{1})\big)^{-1}\right\} \\
&= \frac{1}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\;\mathrm{Tr}\big(\mathbf{c}-(\mu+\mathrm{i}\varepsilon)\mathbb{1}\big)^{-1}
\end{aligned}$$