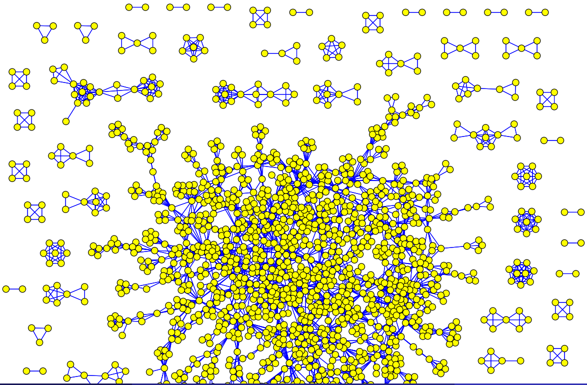
Statistical physics of tailored random graphs: entropies, processes, and generation

Lecture V. New replica methods for loopy graphs

ACC Coolen, King's College London



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Tailoring random graph ensembles

Motivation:

stat mechanics of process on complex network $\mathbf{c}^\star$,
use *random graph* $\mathbf{c}$ as proxy

- max entropy ensemble Ω_L, constrained by values of $\omega_1(\mathbf{c})\ldots\omega_L(\mathbf{c})$

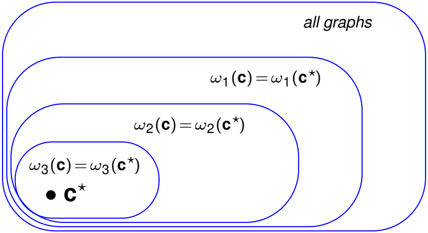
hard constraints: $$p(\mathbf{c}) \propto \prod_{\ell\leq L} \delta_{\omega_\ell(\mathbf{c}),\omega_\ell(\mathbf{c}^\star)}$$

soft constraints: $$p(\mathbf{c}) \propto \mathrm{e}^{\sum_{\ell=1}^{L}\hat{\omega}_\ell\omega_\ell(\mathbf{c})}, \quad \langle\omega_\ell(\mathbf{c})\rangle = \omega_\ell(\mathbf{c}^\star)\ \ \forall\ell$$

- approximate process on $\mathbf{c}^\star$:

  average generating function of process over graphs in Ω_L

  larger $L \to$ better approxim



which observables

$\boldsymbol{\omega}(\mathbf{c}) = \{\omega_1(\mathbf{c}), \ldots, \omega_L(\mathbf{c})\}$

should our graphs inherit from $\mathbf{c}^\star$?

e.g. spin system on nodes of graph $\mathbf{c}$,
Hamiltonian $H(\boldsymbol{\sigma}) = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j$

- statics, replica method:

$$\overline{\mathrm{e}^{-\beta \sum_{\alpha=1}^{n} H(\boldsymbol{\sigma}^\alpha)}} = \frac{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})} \mathrm{e}^{\sum_{i<j} c_{ij} K_{ij}}}{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}, \qquad K_{ij} = \beta J_{ij} \sum_{\alpha=1}^{n} \sigma_i^\alpha \sigma_j^\alpha$$

- dynamics, GFA:

$$\overline{\mathrm{e}^{-\mathrm{i} \sum_{it} \hat{h}_i(t) \sum_j c_{ij} J_{ij} \sigma_j(t)}} = \frac{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})} \mathrm{e}^{\sum_{i<j} c_{ij} K_{ij}}}{\sum_{\mathbf{c}} \delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}, \qquad K_{ij} = -\mathrm{i} J_{ij} \sum_t [\hat{h}_i(t)\sigma_j(t) + \hat{h}_j(t)\sigma_i(t)]$$

in both cases
to do *analytically*:

$$\sum_{\mathbf{c}} \underbrace{\delta_{\boldsymbol{\omega},\boldsymbol{\omega}(\mathbf{c})}}_{hard} \underbrace{\mathrm{e}^{\sum_{i<j} c_{ij} K_{ij}}}_{easy}$$

*boils down to: can we
calculate ensemble entropy?*

Shannon entropy per node

- constraint: $\langle k \rangle$

$$S = \frac{1}{2}\langle k\rangle[1+\log(\frac{N}{\langle k\rangle})] + \ldots$$

(Erdös-Rényi)

- constraints: $p(k) = \langle \frac{1}{N}\sum_i \delta_{k,k_i(\mathbf{c})}\rangle$

$$S = \frac{1}{2}\langle k\rangle[1+\log(\frac{N}{\langle k\rangle})] - \sum_k p(k)\log[\frac{p(k)}{\tilde{p}(k)}] + \ldots$$

$$\tilde{p}(k) = \mathrm{e}^{-\langle k\rangle}\langle k\rangle^k/k!$$

- constraints: $\mathbf{k}(\mathbf{c}) = \mathbf{k}$

$$S = \frac{1}{2}\langle k\rangle[1+\log(\frac{N}{\langle k\rangle})] - \sum_k p(k)\log[\frac{p(k)}{\tilde{p}(k)}] + \sum_k p(k)\log p(k) + \ldots$$

- constraints: $\mathbf{k}(\mathbf{c}) = \mathbf{k}$, $W(k,k') = \frac{1}{\langle k\rangle N}\sum_{ij} c_{ij}\delta_{k,k_i(\mathbf{c})}\delta_{k',k_j(\mathbf{c})}$

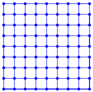
$$S = \frac{1}{2}\langle k\rangle[1+\log(\frac{N}{\langle k\rangle})] + \sum_k p(k)\log\tilde{p}(k) - \frac{1}{2}\langle k\rangle\sum_{k,k'} W(k,k')\log\left[\frac{W(k,k')}{W(k)W(k')}\right] + \ldots$$

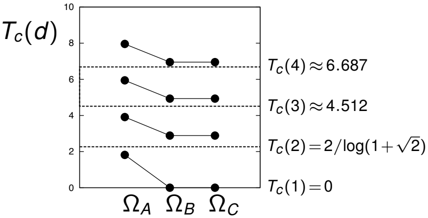
Ising spin models on tailored random graphs

yardstick: transition temperature T_c

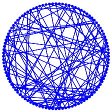
Ω_A: correct $\langle k \rangle$
Ω_B: correct $p(k)$
Ω_C: correct $p(k)$ and $W(k, k')$

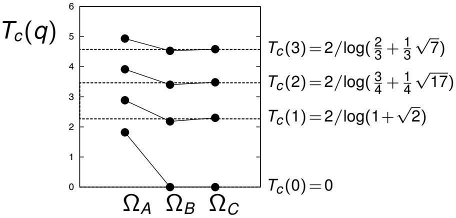
- $\mathbf{c}^\star$ = d-dim cubic lattice
  $p(k) = \delta_{k,2^d}$





- $\mathbf{c}^\star$ = 'small world' lattice
  $p(k \geq 2) = \mathrm{e}^{-q} q^{k-2}/(k-2)!$



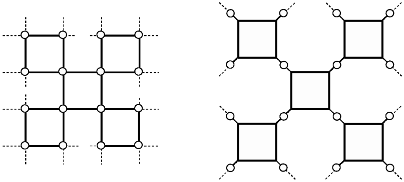


transition temperatures T_c

	degrees	4-loops	$d=1$	$d=2$	$d=3$	$d=4$
random, $\langle k \rangle = 2d$			1.820	3.915	5.944	7.958
random, $p(k) = \delta_{k,2d}$	✓		0	2.885	4.933	6.952
hypercubic Bethe	✓	✓	0	2.771	4.839	6.879
true cubic lattice	✓	✓	0	2.269	4.511	6.680

hypercubic Bethe lattice:
'tree of hypercubes'

– *correct local degrees*
– *geometric (non-random)*
– *finite nr of short loops per site*



The problem

- biological networks,
  physical lattices,
  communication networks,
  distribution networks,
  socio-ecomomic networks, ...

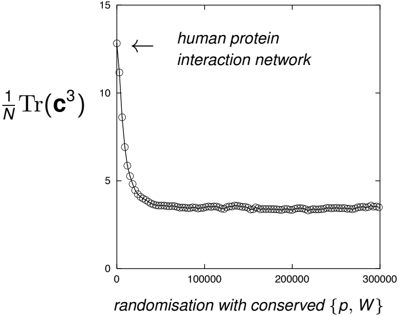
  $\rightarrow$ *sparse graphs,*
  $\rightarrow$ *many short loops*

- max entropy graph ensembles with prescribed $p(k), W(k, k')$:

  $\rightarrow$ *sparse graphs,*
  $\rightarrow$ *locally tree-like*

- realistic tailoring of graphs requires adding $\omega(\mathbf{c})$ that enforces short loops

- available analysis methods,
  e.g. replicas, GFA, cavity, belief propagation ...
  work only for locally tree-like graphs



exceptions:

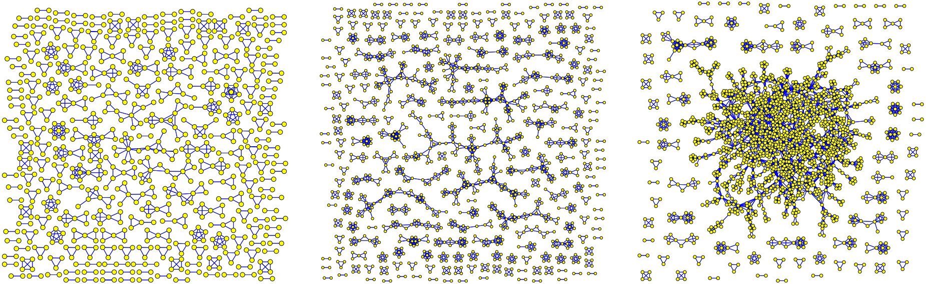
cubic lattices $d < 3$
spherical models
recent immune networks

Immune model
of Agliari and Barra

B-clones $\{b_\mu\}$, T-clones $\{\sigma_i\}$ and cytokines $\{\xi_i^\mu\}$
map to model with effective T-T interactions

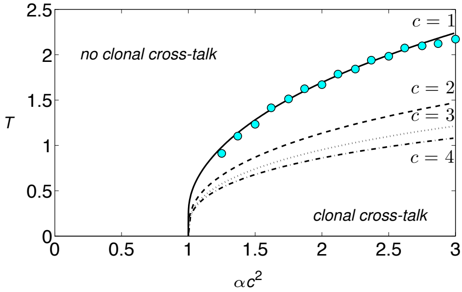
$$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j, \qquad J_{ij} = \sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu, \qquad p(\xi_i^\mu) = \frac{c}{2N}\left[\delta_{\xi_i^\mu,1} + \delta_{\xi_i^\mu,-1}\right] + (1-\frac{c}{N})\delta_{\xi_i^\mu,0}$$

$\alpha c^2 < 1$ $\qquad$ $\alpha c^2 = 1$ $\qquad$ $\alpha c^2 > 1$



Exactly solvable
in spite of short loops ...

[Agliari, Annibale, Barra, ACCC, Tantari, 2013]



here: $\mathbf{J} = \boldsymbol{\xi}^\dagger\boldsymbol{\xi}$
$\boldsymbol{\xi}$: sparse $p\times N$ matrix with iid entries

map to model with spins + Gaussian fields,
on tree-like bipartite graph $\boldsymbol{\xi}$

$$\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta\sum_{i<j} J_{ij}\sigma_i\sigma_j} = \int\frac{\mathrm{d}\mathbf{z}}{(2\pi)^{p/2}}\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\sqrt{\beta}\sum_{\mu i} z_\mu \xi_{\mu i}\sigma_i - \frac{1}{2}\sum_\mu z_\mu^2}$$

is a special case!

Loopy random graph ensembles

Simplest loopy ensemble

control average degree $\langle k\rangle$
and density of triangles $\langle m\rangle$
(Strauss '86, Jonasson '99)

$$p(\mathbf{c}) \propto e^{u\sum_{ij} c_{ij}+v\sum_{ijk} c_{ij}c_{jk}c_{ki}}$$

- to calculate:

$$\langle k\rangle = \Big\langle \frac{1}{N}\sum_{ij} c_{ij}\Big\rangle, \qquad \langle m\rangle = \Big\langle \frac{1}{N}\sum_{ijk} c_{ij}c_{jk}c_{ki}\Big\rangle, \qquad S = -\frac{1}{N}\sum_{\mathbf{c}} p(\mathbf{c})\log p(\mathbf{c})$$

- generating function:

$$\phi(u,v) = \frac{1}{N}\log\sum_{\mathbf{c}} e^{u\sum_{ij} c_{ij}+v\sum_{ijk} c_{ij}c_{jk}c_{ki}}$$

$$\langle k\rangle = \partial\phi/\partial u$$
$$\langle m\rangle = \partial\phi/\partial v$$
$$S = \phi - u\langle k\rangle - v\langle m\rangle$$

challenge:
sum over graphs ...

Early results

ensemble:

$$p(\mathbf{c}) \propto \mathrm{e}^{u\sum_{ij} c_{ij} + v \sum_{ijk} c_{ij}c_{jk}c_{ki}}$$

- Strauss '86

  — simulations
  — triangles 'clump together'

- Jonasson '99

  — $u = -\frac{1}{2}\alpha \log N + \ldots$
  — phase transition, $v_c = \frac{\alpha}{2N} \log N + \ldots$

- Burda et al '04

  — $u = -\frac{1}{2} \log N + \ldots$
  — perturbation theory in v:
  formula for nr of triangles, $v_c = \mathcal{O}(\log N) \ldots$

- Park & Newman '05

  — $u = \mathcal{O}(1)$ so $\langle k \rangle = \mathcal{O}(N)$
  — mean-field approx:

$$p(\mathbf{c}) \to \mathrm{e}^{u\sum_{ij} c_{ij} + v\sum_{ijk} c_{ij}\langle c_{jk}c_{ki}\rangle}, \quad \textit{eqns for} \;\; m = \langle c_{ij}\rangle, \;\; q = \langle c_{ik}c_{kj}\rangle$$

Generalisation ...

- control closed paths **of all lengths**

$$p(\mathbf{c}) \propto \mathrm{e}^{u\sum_{ij} c_{ij} + \sum_{\ell\geq 3} v_\ell \sum_{i_1\ldots i_\ell} c_{i_1 i_2} c_{i_2 i_3} \ldots c_{i_\ell i_1}}$$

  generating function:
  use $c_{ij} = c_{ij} c_{ji}$

$$\phi(\{v_\ell\}) = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{u\,\mathrm{Tr}(\mathbf{c}^2) + \sum_{\ell\geq 3} v_\ell \mathrm{Tr}(\mathbf{c}^\ell)}$$

$$\langle k\rangle = \partial\phi/\partial u$$

$$\langle m_\ell\rangle = \tfrac{1}{N}\langle \mathrm{Tr}(\mathbf{c}^\ell)\rangle = \partial\phi/\partial v_\ell$$

$$S = \phi - u\langle k\rangle - \textstyle\sum_{\ell\geq 3} v_\ell \langle m_\ell\rangle$$

- since $\mathrm{Tr}(\mathbf{c}^\ell) = N\int\!\mathrm{d}\mu\; \mu^\ell \varrho(\mu|\mathbf{c})$:
  control eigenvalue **spectrum** $\varrho(\mu)$

$$p(\mathbf{c}) \propto \mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})}$$

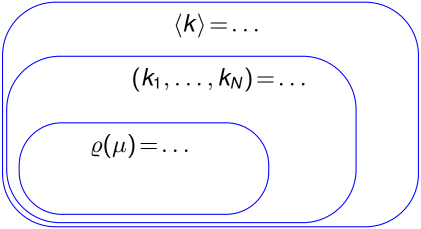
  generating function:

$$\phi[\hat{\varrho}] = \frac{1}{N} \log \sum_{\mathbf{c}} \mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})}$$

$$\varrho(\mu) = \delta\phi/\delta\hat{\varrho}(\mu)$$

$$S = \phi - \int\!\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu)$$

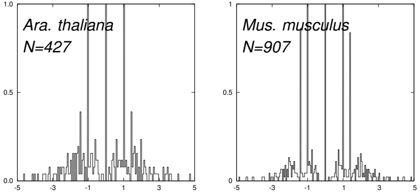
Some interesting questions

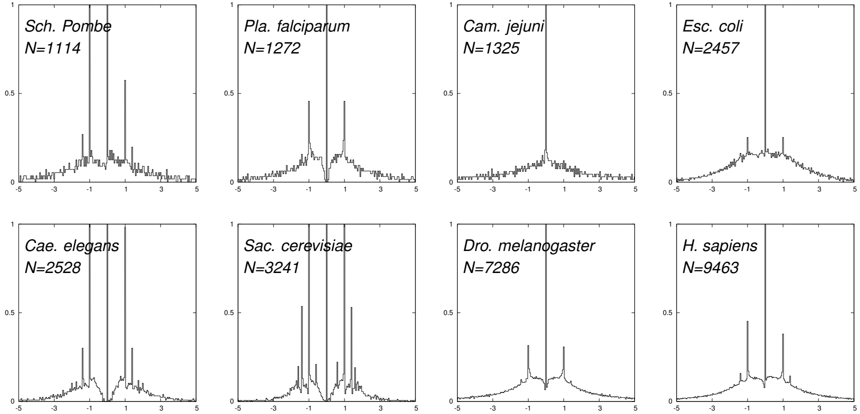


- How informative are spectra of finitely connected graphs?
- How many non-isomorphic graphs are there with given degrees $(k_1, \dots, k_N)$ and a given spectrum $\varrho(\mu)$?
- How similar are processes running on non-isomorphic graphs with the same degrees $(k_1, \dots, k_N)$ and the same spectrum $\varrho(\mu)$?

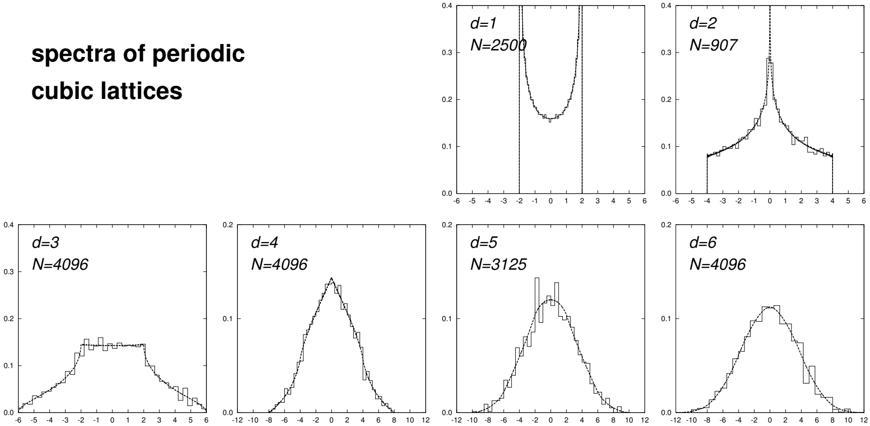
  (spherical spins: free energies identical!)
  (high T expansion: only closed path stats relevant)

spectra of protein interaction networks





spectra of periodic cubic lattices



$$N \to \infty : \qquad \varrho_{d+1}(\mu) = \int_0^1 \mathrm{d}x \; \varrho_d(\mu - 2\cos(\pi x)), \qquad \varrho_1(\mu) = \frac{\theta(2-|\mu|)}{\pi\sqrt{4-\mu^2}}$$

co-spectral graphs

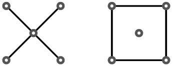
identical nr of edges and closed paths of any length

DS graphs

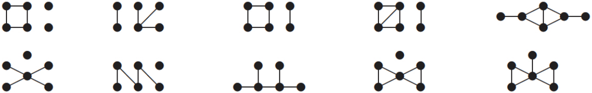
determined fully by their spectrum (modulo isomorphisms)

examples of **non-DS** pairs

$N=5$: one pair



$N=6$: five pairs



$N=10$: regular example pair

$N=13$: example of co-spectral trees

- $N < 5$: all graphs are DS
- $N = 5, 6$: some non-DS, but different degrees
- almost all trees are non-DS
- $N \to \infty$ expectation: nearly all graphs are DS

(Schwenk '73, Van Dam & Haemers '02)

n	# graphs	A
2	2	0
3	4	0
4	11	0
5	34	0.059
6	156	0.064
7	1044	0.105
8	12346	0.139
9	274668	0.186
10	12005168	0.213
11	1018997864	0.211
12	165091172592	0.188

↑ *size* ↑ *non-DS fraction*

Open questions

what happens if we

- restrict ourselves to *sparse* graphs?
- prescribe spectrum *and degree sequence* ?

New analytical route

graph ensemble : $$p(\mathbf{c}) = Z^{-1}[\hat{\varrho}]\, \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} \prod_i \delta_{k_i,\sum_j c_{ij}}$$

generating function : $$\Phi[\hat{\varrho}] = \frac{1}{N}\log \sum_{\mathbf{c}} \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} \prod_i \delta_{k_i,\sum_j c_{ij}}$$

$$\varrho(\mu) = \delta\Phi[\hat{\varrho}]/\delta\hat{\varrho}(\mu), \qquad S = \Phi[\hat{\varrho}] - \int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu)$$

- derive $$\Phi[\hat{\varrho}] = \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N}\log \sum_{\mathbf{c}} \Big[\prod_i \delta_{k_i,\sum_j c_{ij}}\Big] \times$$
$$\lim_{n_\mu \to \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\ \lim_{m_\mu \to -n_\mu} \prod_\mu \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{n_\mu}\ \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{\,m_\mu}\Big]$$
$$Z(\mu|\mathbf{c}) = \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\phi}\ \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{I}]\boldsymbol{\phi}}$$
- replica method, steepest descent for $N\to\infty$, analytical continuation to *imaginary* (n_μ, m_μ), limits $\epsilon, \Delta\downarrow 0$
- replica symmetry, bifurcation analysis, phase transitions and entropy, RSB

Origin of the core identity

spectral ensemble constraints,
use Edwards-Jones ('76):

$$\varrho(\mu|\mathbf{c}) = \frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}), \qquad Z(\mu|\mathbf{c}) = \int_{\mathrm{I\!R}^N}\mathrm{d}\boldsymbol{\phi}\;\mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{1}]\boldsymbol{\phi}}$$

insert into $\Phi[\hat{\varrho}]$,

integrate by parts,

discretise integral,

$$\begin{aligned}\mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} &= \mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})} \\ &= \lim_{\varepsilon,\Delta\downarrow 0}\prod_{\mu}\mathrm{e}^{-2\mathrm{Im}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}).\;\frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\end{aligned}$$

$$\mathrm{e}^{-2\;\mathrm{Im}\log z} = z^{\mathrm{i}}.\overline{z}^{\;-\mathrm{i}}$$

$$\Phi[\hat{\varrho}] = \lim_{\varepsilon,\Delta\downarrow 0}\frac{1}{N}\log\sum_{\mathbf{c}}\Big[\prod_{i}\delta_{k_i,\sum_j c_{ij}}\Big]\prod_{\mu}\Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}}\;\overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{-\mathrm{i}}\Big]^{\frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}$$

Flavours of the replica method

the replica dimension n ...

- $n \to 0$: *Kac ('68), Sherrington, Kirkpatrick ('75), Parisi ('79)*

  stat mech of disordered spin systems

$$\overline{\log Z} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n}$$

- $n \in \mathbb{R}, >0$: *Sherrington ('80), ACCC, Penney, Sherrington ('93)*

  'slow' dynamics of parameters in 'fast' spin system
  (partial annealing, $n = T/T'$)

- $n \in \mathbb{R}, <0$: *Dotsenko, Franz, Mezard ('94)*

  slow dynamics evolves to *maximise* free energy of fast system

many applications of finite n replica method,
to heterogeneous many-variable systems

here: $n \notin \mathbb{R}$ ...

Replica analysis of generating function

graph ensemble:

$$p(\mathbf{c}) = Z^{-1}[\hat{\varrho}]\, \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} \prod_i \delta_{k_i,\sum_j c_{ij}}$$

graphicality: $\int \mathrm{d}\mu\ \mu\varrho(\mu|\mathbf{c}) = 0, \quad \int \mathrm{d}\mu\ \mu^2 \varrho(\mu|\mathbf{c}) = \langle k\rangle$

generating function:

$$\Phi[\hat{\varrho}] = \frac{1}{N}\log \sum_{\mathbf{c}} \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} \prod_i \delta_{k_i,\sum_j c_{ij}}$$

$$\varrho(\mu) = \delta\Phi[\hat{\varrho}]/\delta\hat{\varrho}(\mu), \qquad S = \Phi[\hat{\varrho}] - \int \mathrm{d}\mu\ \hat{\varrho}(\mu)\rho(\mu)$$

$\varrho(\mu)$ *prescribed*: $\hat{\varrho}(\mu)$ *to be solved ...*

$\hat{\varrho}(\mu) = \sum_\ell v_\ell \mu^\ell$: *formula for resulting* $\varrho(\mu)$ *...*

transform to average over ER ensemble, use core identity:

$$\Phi[\hat{\varrho}] = \frac{1}{2}\langle k\rangle\Big[\log\Big(\frac{N}{\langle k\rangle}\Big)+1\Big] + \mathcal{O}\Big(\frac{1}{N}\Big)$$

$$+ \lim_{\Delta,\varepsilon\downarrow 0}\ \lim_{n_\mu \to \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\ \lim_{m_\mu\to -n_\mu} \frac{1}{N}\log \int_{-\pi}^{\pi}\Big[\prod_i \frac{\mathrm{d}\omega_i}{2\pi}\mathrm{e}^{\mathrm{i}k_i\omega_i}\Big]$$

$$\times \Big\langle \mathrm{e}^{-\mathrm{i}\sum_{i<j} c_{ij}(\omega_i+\omega_j)} \prod_\mu \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{n_\mu}\, \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{\,m_\mu}\Big]\Big\rangle_{\mathrm{ER}}$$

integer $\{n_\mu, m_\mu\}$:

$$\prod_\mu \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{n_\mu}\, \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{\,m_\mu} \Big] \;=\;$$

$$\prod_\mu \Big\{ \Big[\prod_{\alpha_\mu=1}^{n_\mu} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\phi}_{\mu,\alpha_\mu} \mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\boldsymbol{\phi}^2_{\mu,\alpha_\mu}} \Big] \Big[\prod_{\beta_\mu=1}^{m_\mu} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\psi}_{\mu,\beta_\mu} \mathrm{e}^{-\frac{1}{2}(\varepsilon+\mathrm{i}\mu)\boldsymbol{\psi}^2_{\mu,\beta_\mu}} \Big] \Big\}$$

$$\times \mathrm{e}^{\mathrm{i}\sum_{i<j} c_{ij} \sum_\mu \left[\sum_{\beta_\mu=1}^{m_\mu} \psi^i_{\mu,\beta_\mu}\psi^j_{\mu,\beta_\mu} - \sum_{\alpha_\mu=1}^{n_\mu} \phi^i_{\mu,\alpha_\mu}\phi^j_{\mu,\alpha_\mu} \right]}$$

average over $\mathbf{c}$:

$$\Phi[\hat{\varrho}] \;=\; \frac{1}{2}\langle k\rangle \log\Big(\frac{N}{\langle k\rangle}\Big) + \mathcal{O}\Big(\frac{1}{N}\Big) + \lim_{\Delta,\varepsilon\downarrow 0}\; \lim_{n_\mu \to \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\; \lim_{m_\mu\to -n_\mu}$$

$$\frac{1}{N} \log \Big\{ \prod_i \Big(\int_{-\pi}^{\pi} \frac{\mathrm{d}\omega_i}{2\pi} \mathrm{e}^{\mathrm{i}k_i\omega_i} \int \mathrm{d}\boldsymbol{\phi}^i \mathrm{d}\boldsymbol{\psi}^i \mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}^i\cdot(\varepsilon\mathbf{1}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}^i - \frac{1}{2}\boldsymbol{\psi}^i\cdot(\varepsilon\mathbf{1}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}^i} \Big)$$

$$\times \mathrm{e}^{\frac{\langle k\rangle}{2N}\sum_{ij} \mathrm{e}^{-\mathrm{i}(\omega_i+\omega_j)+\mathrm{i}(\boldsymbol{\psi}^i\cdot\boldsymbol{\psi}^j - \boldsymbol{\phi}^i\cdot\boldsymbol{\phi}^j)}} \Big\}$$

notation:

$$\boldsymbol{\phi}^i = \{\phi^i_{\mu,\alpha_\mu}\}, \quad \boldsymbol{\phi}^i\cdot\boldsymbol{\phi}^j = \sum_\mu \sum_{\alpha_\mu=1}^{n_\mu} \phi^i_{\mu,\alpha_\mu}\phi^j_{\mu,\alpha_\mu}, \quad \boldsymbol{\psi}^i = \{\psi^i_{\mu,\beta_\mu}\}, \quad \boldsymbol{\psi}^i\cdot\boldsymbol{\psi}^j = \sum_\mu \sum_{\beta_\mu=1}^{m_\mu} \psi^i_{\mu,\beta_\mu}\psi^j_{\mu,\beta_\mu}$$

M: *matrix with entries* $M_{\mu,\alpha;\mu',\alpha'} = \mu\delta_{\mu\mu'}\delta_{\alpha\alpha'}$

Steepest decent form

order parameter:

$$\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)=\frac{1}{N}\sum_i \delta(\boldsymbol{\phi}-\boldsymbol{\phi}^i)\delta(\boldsymbol{\psi}-\boldsymbol{\psi}^i)\delta(\omega-\omega_i)\Big]$$

leads to path integral representation:

$$\Phi[\hat{\varrho}]=\frac{1}{2}\langle k\rangle\log\Big(\frac{N}{\langle k\rangle}\Big)+\mathcal{O}\big(\frac{1}{N}\big)$$
$$+\lim_{\Delta,\varepsilon\downarrow 0}\ \lim_{n_\mu\to\frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\ \lim_{m_\mu\to -n_\mu}\frac{1}{N}\log\int\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\}\ \mathrm{e}^{N\left[\Psi[\mathcal{P},\hat{\mathcal{P}}]+\epsilon_N\right]}$$

with

$$\begin{aligned}\Psi[\mathcal{P},\hat{\mathcal{P}}] &= \mathrm{i}\int \mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\omega\ \hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\\ &+\frac{1}{2}\langle k\rangle\int \mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\omega\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\mathrm{d}\omega'\ \mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\mathcal{P}(\boldsymbol{\phi}',\boldsymbol{\psi}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')+\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\\ &+\sum_k p(k)\log\int_{-\pi}^{\pi}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}k\omega}\int \mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\ \mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{I}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{I}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)}\end{aligned}$$

$\boldsymbol{\phi}=\{\phi_{\mu,\alpha_\mu}\}$, $\boldsymbol{\psi}=\{\psi_{\mu,\beta_\mu}\}$

$\lim_{N\to\infty}\epsilon_N=0$

work out saddle point eqns for $\{\mathcal{P},\hat{\mathcal{P}}\}$:

$$\Phi[\hat{\varrho}] = \frac{1}{2}\langle k\rangle\Big[\log\Big(\frac{N}{\langle k\rangle}\Big)+1\Big] + \sum_k p(k)\log\tilde{p}(k) + \epsilon_N$$
$$+\lim_{\Delta,\varepsilon\downarrow 0}\ \lim_{n_\mu\to\frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\ \lim_{m_\mu\to -n_\mu}\sum_k p(k)\log\int\mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\ \mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{1}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{1}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}}$$
$$\times\Big[\int\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\ \mathcal{W}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\Big]^k$$

$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi})$ solved from

$$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi}) = \sum_k\frac{k}{\langle k\rangle}p(k)$$
$$\times\frac{\mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{1}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{1}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}}\Big[\int\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\ \mathcal{W}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\Big]^{k-1}}{\int\mathrm{d}\boldsymbol{\phi}''\mathrm{d}\boldsymbol{\psi}''\mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}''\cdot(\varepsilon\mathbf{1}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}''-\frac{1}{2}\boldsymbol{\psi}''\cdot(\varepsilon\mathbf{1}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}''}\Big[\int\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\ \mathcal{W}(\boldsymbol{\phi}',\boldsymbol{\psi}')\mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}''\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}''\cdot\boldsymbol{\phi}')}\Big]^k}$$

next:

– ansatz for $\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi})$
– take limits $m_\mu\to -n_\mu$ and $n_\mu\to\frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$
– take limits $\Delta,\varepsilon\downarrow 0$

Replica symmetry ansatz

$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi})$ symmetric under all permutations of $\{\phi_{\mu,1},\ldots,\phi_{\mu,n_\mu}\}$ and $\{\psi_{\mu,1},\ldots,\psi_{\mu,m_\mu}\}$

De Finetti:

$$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mathcal{C}\int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}]\Big[\prod_{\mu}\prod_{\alpha_\mu=1}^{n_\mu}\pi(\phi_{\mu,\alpha_\mu}|\mu)\Big]\Big[\prod_{\mu}\prod_{\beta_\mu=1}^{m_\mu}\overline{\pi(\psi_{\mu,\beta_\mu}|\mu)}\Big]$$

$\mathcal{W}[\{\pi\}]$:
measure on the space of conditioned distributions $\pi(x|\mu)$

$\int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}] = 1$,

$\mathcal{W}[\{\pi\}] > 0$ only if $\int\mathrm{d}x\ \pi(x|\mu) = 1$

– insert ansatz into saddle-point eqn
– derive closed eqns for $\mathcal{C}$ and $\mathcal{W}[\{\pi\}]$
– insert into generation function $\Phi[\hat{\varrho}]$

closed eqns:

$$\mathcal{W}[\{\pi\}] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}$$

$$\times \frac{\Big[\prod_{\ell<k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_{k-1}\}]\delta_{\mathrm{F}}\Big[\pi(.|\mu)-\pi(.|\mu,\pi_1,\ldots,\pi_{k-1})\Big]}{\Big[\prod_{\ell\leq k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_k\}]}$$

with

$$\pi(\phi|\mu,\pi_1,\ldots,\pi_k) = \frac{\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)}{\int\mathrm{d}x\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)x^2}\prod_{\ell\leq k}\hat{\pi}_\ell(x|\mu)}$$

$$\mathcal{A}[\{\pi_1,\ldots,\pi_k\}] = \prod_{\mu}\Big[\Big(\int\mathrm{d}x\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)x^2}\prod_{\ell\leq k}\hat{\pi}_\ell(x|\mu)\Big)^{n_\mu} \times\Big(\overline{\int\mathrm{d}x\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)x^2}\prod_{\ell\leq k}\hat{\pi}_\ell(x|\mu)}\Big)^{m_\mu}\Big]$$

$\hat{\pi}(\phi|\mu) = \int\mathrm{d}x\;\mathrm{e}^{-\mathrm{i}x\phi}\;\pi(x|\mu)$,

normalisation constant

$$\mathcal{C}^2 = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\Big[\prod_{\ell<k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_{k-1}\}]}{\Big[\prod_{\ell\leq k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_k\}]}$$

Exploiting the nature of the propagation

order parameter eqn for $\mathcal{W}[\{\pi\}]$ describes stationary state of stochastic propagation of complex conditioned distributions:

$$\pi(\phi|\mu) \;\; \rightarrow \;\; \pi(\phi|\mu,\pi_1,\ldots,\pi_{k-1}) = \frac{\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)}{\int\!\mathrm{d}x\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)x^2}\prod_{\ell\leq k}\hat{\pi}_\ell(x|\mu)}$$

propagation shape-preserving for $\pi(\phi|\mu)$ of the form

$$\pi(\phi|x,u) = \frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}x\phi^2+\mathrm{i}u\phi}}{\left(\frac{2\pi}{\mathrm{i}x}\right)^{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}\mathrm{i}u^2/x}}$$

$x(\mu)$, $u(\mu)$: complex functions on $\mathbb{R}$,
$\mathrm{Im}\;x(\mu) < 0$ for all $\mu\in\mathbb{R}$

$$\pi(\phi|\mu,\pi_1,\ldots,\pi_{k-1}) = \pi(\phi|x'(\mu),u'(\mu))$$

$$x'(\mu) = -\mathrm{i}\varepsilon-\mu-\sum_{\ell<k}\frac{1}{x_\ell(\mu)}, \qquad u'(\mu) = -\sum_{\ell<k}\frac{u_\ell(\mu)}{x_\ell(\mu)}$$

If $\mathrm{Im}x_\ell(\mu)<0$: also $\mathrm{Im}x'(\mu)<0$,
for $\varepsilon\to 0$: $\;x(\mu)\in\mathbb{R}$

work out math details,
define $u(\mu) = y(\mu) + \mathrm{i}z(\mu)$,
so $x(\mu), y(\mu), z(\mu)$ all real-valued

$\mathcal{A}[\{x, y, z\}]$: induced by the loops

$$\mathcal{W}[\{x,y,z\}] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\mathcal{A}[\{x,y,z\}]\mathcal{F}_{k-1}[\{x,y,z\}]}{\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{A}[\{x',y',z'\}]\mathcal{F}_k[\{x',y',z'\}]}$$

$$\mathcal{C}^2 = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\int\{\mathrm{d}x\mathrm{d}y\mathrm{d}z\}\mathcal{A}[\{x,y,z\}]\mathcal{F}_{k-1}[\{x,y,z\}]}{\int\{\mathrm{d}x\mathrm{d}y\mathrm{d}z\}\mathcal{A}[\{x,y,z\}]\mathcal{F}_k[\{x,y,z\}]}$$

with

$$\mathcal{F}_k[\{x,y,z\}] = \Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\mathrm{d}y_\ell\mathrm{d}z_\ell\}\ \mathcal{W}[\{x_\ell,y_\ell,z_\ell\}]\Big]\delta_{\mathrm{F}}\begin{bmatrix} x - F[x_1,\ldots,x_k] \\ y - G[x_1,y_1,\ldots,x_k,y_k] \\ z - G[x_1,z_1,\ldots,x_k,z_k]\end{bmatrix}$$

$$F[\mu|x_1,\ldots,x_{k-1}] = -\mu - \sum_{\ell<k} 1/x_\ell(\mu)$$

$$G[\mu|x_1,y_1,\ldots,x_{k-1},y_{k-1}] = -\sum_{\ell<k} y_\ell(\mu)/x_\ell(\mu)$$

$$\mathcal{A}[\{x,y,z\}] = \mathrm{e}^{-\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\frac{1}{2}\mathrm{sgn}\left[x(\mu)\right] - \frac{1}{\pi}\frac{y^2(\mu)-z^2(\mu)}{x(\mu)}\right\}}$$

Equation for the spectrum

Evaluate and differentiate generating function $\Phi[\hat{\varrho}]$:

$$\varrho(\mu) = -\frac{\mathrm{d}}{\mathrm{d}\mu}\Big\{\sum_k p(k)\frac{\int\{\mathrm{d}x\mathrm{d}y\mathrm{d}z\}\,\mathcal{A}[\{x,y,z\}]\mathcal{F}_k[\{x,y,z\}]\Big[\frac{1}{2}\mathrm{sgn}\big[x(\mu)\big]-\frac{y^2(\mu)-z^2(\mu)}{\pi x(\mu)}\Big]}{\int\{\mathrm{d}x\mathrm{d}y\mathrm{d}z\}\,\mathcal{A}[\{x,y,z\}]\mathcal{F}_k[\{x,y,z\}]}$$

$$+\frac{1}{2}\langle k\rangle\mathcal{C}^2\int\{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{W}[\{x,y,z\}]\mathcal{W}[\{x',y',z'\}]\mathcal{B}[\{x,y,z\},\{x',y',z'\}]$$

$$\times\Big[\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]$$

$$+\frac{1}{\pi}\frac{[y'^2(\mu)-z'^2(\mu)]/x'(\mu)+[y^2(\mu)-z^2(\mu)]/x(\mu)-2[y(\mu)y'(\mu)-z(\mu)z'(\mu)]}{x(\mu)x'(\mu)-1}\Big]\Big\}$$

with

$$\mathcal{B}[\{x,y,z\},\{x',y',z'\}] = \mathrm{e}^{\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]\right\}}$$

$$\times\mathrm{e}^{\frac{1}{\pi}\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left\{\frac{[y'^2(\mu)-z'^2(\mu)]/x'(\mu)+[y^2(\mu)-z^2(\mu)]/x(\mu)-2[y(\mu)y'(\mu)-z(\mu)z'(\mu)]}{x(\mu)x'(\mu)-1}\right\}}$$

Further symmetries and bifurcations

reflections in imaginary and real axis of
centres of propagated functions $\pi(\phi|x,y,z)$

$$\mathcal{W}[\{x,y,z\}] \to \mathcal{W}[\{x,-y,z\}], \qquad \mathcal{W}[\{x,y,z\}] \to \mathcal{W}[\{x,y,-z\}]$$

Strongly invariant saddle-point:
$\mathcal{W}[\{x,y,z\}] = \mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}]$

$$\mathcal{W}[\{x\}] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x\}]}$$

$$\mathcal{C}^2 = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\int\{\mathrm{d}x\}\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x\}\mathcal{A}[\{x\}]\mathcal{F}_k[\{x\}]}$$

with

$$\mathcal{F}_k[\{x\}] = \Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\}\ \mathcal{W}[\{x_\ell\}]\Big]\delta_{\mathrm{F}}\big[x - F[x_1,\ldots,x_k]\big]$$

$$\mathcal{A}[\{x\}] = \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{sgn}[x(\mu)]}$$

$$\mathcal{B}[\{x\},\{x'\}] = \mathrm{e}^{\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]\right]}$$

symmetry-breaking transitions

$$\mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}] \quad \to \quad \mathcal{W}[\{x,y,z\}] \neq \mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}]$$

continuous bifurcations located via
functional moment (Guzai) expansion:

- type I: $\int\{\mathrm{d}y\mathrm{d}z\}\mathcal{W}[\{y,z|x\}]y \neq 0$ or $\int\{\mathrm{d}y\mathrm{d}z\}\mathcal{W}[\{y,z|x\}]z \neq 0$

  $\exists$ nontrivial soln of

$$f(\mu|\{x\}) \quad = \quad -\int\{\mathrm{d}x''\}\mathcal{W}[\{x''\}]\frac{f(\mu|\{x''\})}{x''(\mu)}\frac{\sum_{k>1}p(k)k(k-1)\,\frac{\mathcal{F}_{k-2}[\{x+1/x''\}]}{\int\{\mathrm{d}x'\}\,\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}{\sum_{k>0}p(k)k\,\frac{\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}$$

- type II: $\int\{\mathrm{d}y\mathrm{d}z\}\mathcal{W}[\{y,z|x\}]y = \int\{\mathrm{d}y\mathrm{d}z\}\mathcal{W}[\{y,z|x\}]z = 0$

  $\exists$ nontrivial soln of

$$f(\mu,\nu|\{x\}) \quad = \quad \int\{\mathrm{d}x''\}\mathcal{W}[\{x''\}]\frac{f(\mu,\nu|\{x''\})}{x''(\mu)x''(\nu)}\frac{\sum_{k>0}p(k)k(k-1)\,\frac{\mathcal{F}_{k-2}[\{x+1/x''\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}{\sum_{k>0}p(k)k\,\frac{\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}$$

Limit of locally tree-like graphs

$\hat{\varrho}(\mu) \to 0$ for all μ: $p(\mathbf{c}) \propto \prod_i \delta_{k_i, \sum_j c_{ij}}$

$\mathcal{A}[\{x,y,z\}] = \mathcal{B}[\{x,y,z\},\{x',y',z'\}] = \mathcal{C} = 1$

- entropy per node?

$$S = \frac{1}{2}\langle k\rangle\Big[\log\Big(\frac{N}{\langle k\rangle}\Big)+1\Big] + \sum_k p(k)\log\tilde{p}(k) + \epsilon_N \qquad \checkmark$$

- spectra $\varrho(\mu)$?

  simplest form $\mathcal{W}[\{x,y,z\}] = \mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}]$:

$$\mathcal{W}[\{x\}] = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\Big[\prod_{\ell<k}\int\{\mathrm{d}x_\ell\}\mathcal{W}[\{x_\ell\}]\Big]\delta_{\mathrm{F}}\big[x - F[x_1,\ldots,x_{k-1}]\big]$$

$$\varrho(\mu) = -\frac{\mathrm{d}}{\mathrm{d}\mu}\Big\{\frac{1}{2}\sum_k p(k)\int\{\mathrm{d}x\}\mathcal{F}_k[\{x\}]\mathrm{sgn}\big[x(\mu)\big]\Big]$$

$$+\frac{1}{2}\langle k\rangle\int\{\mathrm{d}x\mathrm{d}x'\}\mathcal{W}[\{x\}]\mathcal{W}[\{x'\}]\;\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]\Big\}$$

Regular locally tree-like graphs

$$\mathcal{W}[\{x\}] = \Big[\prod_{\ell<k}\int\{\mathrm{d}x_\ell\}\mathcal{W}[\{x_\ell\}]\Big]\delta_{\mathrm{F}}\big[x - F[x_1,\ldots,x_{k-1}]\big]$$

$$\mathcal{W}[\{x\}] = \prod_\mu \mathcal{W}(x(\mu)|\mu), \qquad \mathcal{W}(x|\mu) = \Big[\prod_{\ell<k}\int \mathrm{d}x_\ell \mathcal{W}(x_\ell|\mu)\Big]\delta\big[x+\mu+\sum_{\ell<k}\frac{1}{x_\ell}\big]$$

- $k=1$:

$$\mathcal{W}(x|\mu) = \delta(x+\mu), \qquad \varrho(\mu) = \frac{1}{2}\delta(\mu-1) + \frac{1}{2}\delta(\mu+1) \quad \checkmark$$

- $k \geq 2$:

$$|\mu| < 2\sqrt{k-1}: \qquad \mathcal{W}(x|\mu) = \frac{1}{\pi}\frac{\sqrt{k-1-\frac{1}{4}\mu^2}}{(x+\frac{1}{2}\mu)^2 + k-1-\frac{1}{4}\mu^2}$$

$$|\mu| > 2\sqrt{k-1}: \qquad \mathcal{W}(x|\mu) = \delta\Big[x+\frac{1}{2}\mu+\frac{1}{2}\mu\sqrt{1-4(k-1)/\mu^2}\Big]$$

gives McKay's '81 formula:

$$\varrho(\mu) = \theta\big[2\sqrt{k-1}-|\mu|\big]\ \frac{k\sqrt{4(k-1)-\mu^2}}{2\pi(k^2-\mu^2)} \qquad \checkmark$$

Order parameter equations of Rodgers-Bray and Dorogovtsev et al

Present order par eqn:

$$\mathcal{W}(x|\mu) \;=\; \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\Big[\prod_{\ell<k}\int \mathrm{d}x_\ell\; \mathcal{W}(x_\ell|\mu)\Big]\delta\Big[x+\mu+\sum_{\ell<k}\frac{1}{x_\ell}\Big] \qquad (1)$$

short-hands

$$\Phi(u)=\sum_{k>0} p(k)(k/\langle k\rangle)u^{k-1} \qquad G(z|\mu)=\int \mathrm{d}t\; \mathcal{W}(t|\mu)\mathrm{e}^{\mathrm{i}z/t}$$

gives:

$$G(z|\mu)=\int \mathrm{d}y\; \mathrm{e}^{\mathrm{i}y\mu}\;\Phi\big(G(y|\mu)\big)\lambda(y,z) \qquad \lambda(y,z)=\int\frac{\mathrm{d}x}{2\pi}\;\mathrm{e}^{\mathrm{i}(z/x+yx)}$$

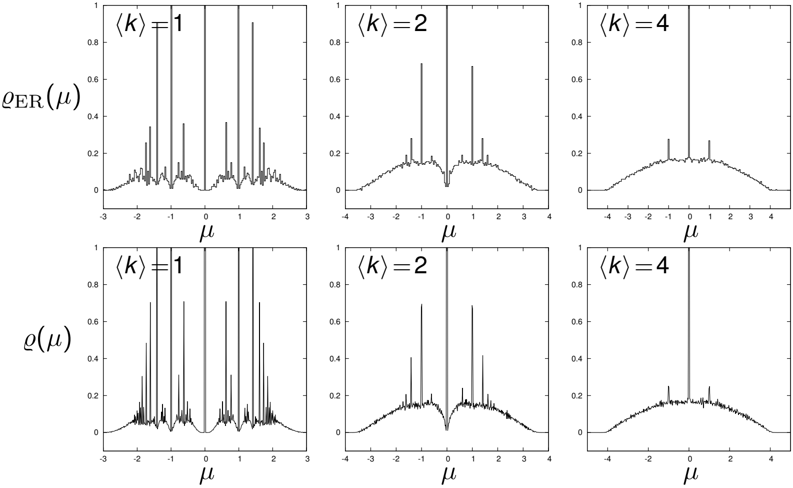
integral:

$$\lambda(y,z) \;=\; \delta(y)-\theta(y)\frac{\sqrt{z}}{\sqrt{y}}J_1(2\sqrt{yz})$$

result:

$$G(z|\mu) \;=\; 1-\sqrt{z}\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\;\mathrm{e}^{\mathrm{i}y\mu}\;\Phi\big(G(y|\mu)\big)J_1(2\sqrt{yz}) \qquad \checkmark$$

Poissonnian locally tree-like graphs



Interpretation and solution of eqns

loopy graph ensembles

$$\mathcal{W}[\{x,y,z\}] = \frac{\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\mathcal{A}[\{x,y,z\}]\mathcal{F}_{k-1}[\{x,y,z\}]}{\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{A}[\{x',y',z'\}]\mathcal{F}_k[\{x',y',z'\}]}}{\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{A}[\{x',y',z'\}]\mathcal{F}_{k-1}[\{x',y',z'\}]}{\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{A}[\{x',y',z'\}]\mathcal{F}_k[\{x',y',z'\}]}}$$

$$\mathcal{F}_k[\{x,y,z\}] = \Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\mathrm{d}y_\ell\mathrm{d}z_\ell\}\ \mathcal{W}[\{x_\ell,y_\ell,z_\ell\}]\Big]\delta_{\mathrm{F}}\begin{bmatrix} x - F[x_1,\ldots,x_k] \\ y - G[x_1,y_1,\ldots,x_k,y_k] \\ z - G[x_1,z_1,\ldots,x_k,z_k]\end{bmatrix}$$

tree-like limit:

$\hat{\varrho}(\mu)\to 0$:

$\mathcal{A}[\{x,y,z\}]\to 1$

$$\mathcal{W}[\{x,y,z\}] = \sum_{k>0} p(k)\frac{k}{\langle k\rangle}\mathcal{F}_{k-1}[\{x,y,z\}]$$

structure of message-passing algorithms, e.g. belief propagation, cavity method

meaning of $\mathcal{A}[\{x,y,z\}]\neq 1$?

define stochastic
message passing process:

(i) Draw degree k at random with probability $P(k) = p(k)k/\langle k\rangle$

(iii) Draw new state $\{x', y', z'\}$ at random according to $\mathcal{F}_{k-1}[\{x', y', z'\}]$

(iii) Accept $\{x', y', z'\}$ with probability $\mathcal{P}[\{x', y', z'\}|k]$, otherwise stay at $\{x, y, z\}$

(iv) Return to (i)

with

$$\mathcal{F}_{k-1}[\{x,y,z\}] = \Big[\prod_{\ell<k}\int\{\mathrm{d}x_\ell \mathrm{d}y_\ell \mathrm{d}z_\ell\}\ \mathcal{W}[\{x_\ell, y_\ell, z_\ell\}]\Big]\delta_{\mathrm{F}}\begin{bmatrix} x - F[x_1,\ldots,x_k] \\ y - G[x_1,y_1,\ldots,x_k,y_k] \\ z - G[x_1,z_1,\ldots,x_k,z_k]\end{bmatrix}$$

posterior measure $\mathcal{W}'[\{x,y,z\}]$ after one iteration:

$$\mathcal{W}'[\{x,y,z\}] = \sum_k p(k)\frac{k}{\langle k\rangle}\mathcal{P}[\{x,y,z\}|k]\mathcal{F}_{k-1}[\{x,y,z\}] + \mathcal{W}[\{x,y,z\}]\Big[1-\sum_k p(k)\frac{k}{\langle k\rangle}\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{P}[\{x',y',z'\}|k]\mathcal{F}_{k-1}[\{x',y',z'\}]\Big]$$

invariant measure:

$$\mathcal{W}[\{x,y,z\}] = \frac{\sum_k p(k)\frac{k}{\langle k\rangle}\mathcal{P}[\{x,y,z\}|k]\mathcal{F}_{k-1}[\{x,y,z\}]}{\sum_k p(k)\frac{k}{\langle k\rangle}\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{P}[\{x',y',z'\}|k]\mathcal{F}_{k-1}[\{x',y',z'\}]}$$

comparison with
present RS theory:

order parameter equation
=
stationarity condition for process of the above form
with move acceptance probabilities

$$\mathcal{P}[\{x,y,z\}|k] \propto \frac{\mathcal{A}[\{x,y,z\}]}{\int\{\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'\}\mathcal{A}[\{x',y',z'\}]\mathcal{F}_k[\{x',y',z'\}]}$$

- tells us how to solve eqn via population dynamics algorithm
- standard (tree-like) belief propagation: $\mathcal{A}[\{x,y,z\}] = 1$, accept all moves
- correct loopy belief propagation: nontrivial message acceptance probs

Regular loopy graphs

simplest soln:

$$\mathcal{W}[\{x,y,z\}] = \mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}], \qquad \mathcal{W}[\{x\}] = \frac{\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_{k-1}[\{x'\}]}$$

spectrum:

$$\varrho(\mu) = -\frac{\mathcal{C}^2}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\int\{\mathrm{d}x\mathrm{d}x'\}\mathcal{W}[\{x\}]\mathcal{W}[\{x'\}]\mathcal{B}[\{x\},\{x'\}]$$
$$\times\Big\{\theta[x(\mu)x'(\mu)]\mathrm{sgn}\big[x(\mu)+x'(\mu)\big]\Big[1+(k-2)\theta[1-x(\mu)x'(\mu)]\Big]\Big\}$$

with

$$\mathcal{C}^2 = \frac{\int\{\mathrm{d}x\}\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x\}\mathcal{A}[\{x\}]\mathcal{F}_{k}[\{x\}]}, \qquad \mathcal{F}_k[\{x\}] = \Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\}\ \mathcal{W}[\{x_\ell\}]\Big]\delta_{\mathrm{F}}\big[x-F[x_1,\ldots,x_k]\big]$$

$$\mathcal{A}[\{x\}] = \mathrm{e}^{-\frac{1}{2}\int\mathrm{d}\mu\ \hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{sgn}[x(\mu)]}$$

$$\mathcal{B}[\{x\},\{x'\}] = \mathrm{e}^{\int\mathrm{d}\mu\ \hat{\varrho}(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]\right]}$$

can also be written in terms of marginal distributions $\mathcal{W}(x|\mu)$

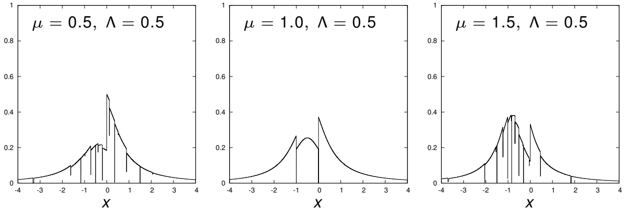
for each $\mu \in \mathbb{R}$:

$$\mathcal{W}(x) = \frac{e^{\Lambda\,\mathrm{sgn}(x)}\mathcal{F}(x)}{\int \mathrm{d}x'\, e^{\Lambda\,\mathrm{sgn}(x')}\mathcal{F}(x')} \qquad \mathcal{F}(x) = \int \prod_{\ell<k}\Big[\mathrm{d}x_\ell\; \mathcal{W}(x_\ell)\Big]\,\delta\Big[x+\mu+\sum_{\ell<k}\frac{1}{x_\ell}\Big]$$

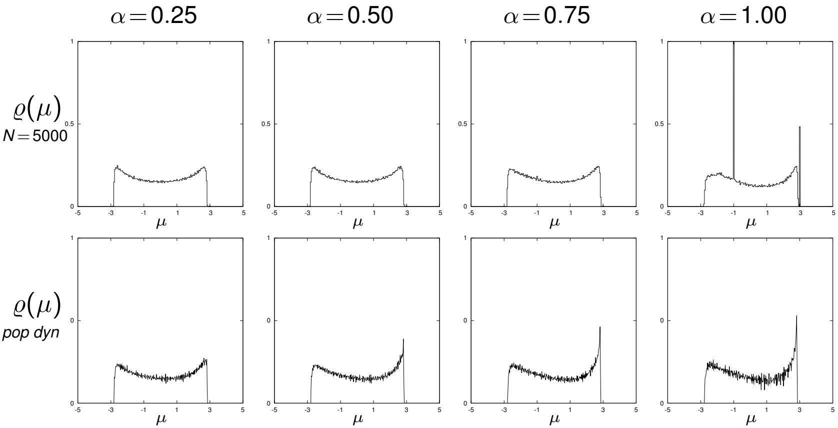
- $|\mu| > 2\sqrt{k-1}$: $$\mathcal{W}(x) = \delta\Big[x+\frac{1}{2}\mu+\frac{1}{2}\mu\sqrt{1-4(k-1)/\mu^2}\Big]$$

- $|\mu| < 2\sqrt{k-1},\ \Lambda = 0$: $$\mathcal{W}(x) = \frac{1}{\pi}\frac{\sqrt{k-1-\frac{1}{4}\mu^2}}{(x+\frac{1}{2}\mu)^2+k-1-\frac{1}{4}\mu^2}$$

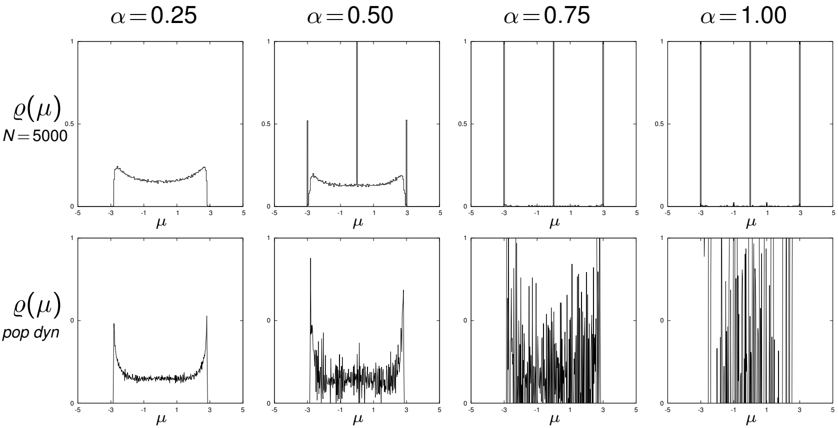
- $k = 2$: $$\mathcal{W}(x) = \frac{1}{Z}\frac{e^{\Lambda\,\mathrm{sgn}(x)}}{(x+\mu)^2}\mathcal{W}\Big(\frac{-1}{x+\mu}\Big)$$



control triangles : $p(\mathbf{c}) \propto e^{\alpha \mathrm{Tr}(\mathbf{c}^3)} \prod_i \delta_{3,\sum_j c_{ij}}$



$$\textit{control squares}: \quad p(\mathbf{c}) \propto \mathrm{e}^{\alpha \mathrm{Tr}(\mathbf{c}^4)} \prod_i \delta_{3,\sum_j c_{ij}}$$



Ising models on loopy graphs

Hamiltonian and free energy density for interacting spins on graph **c**:

$$H(\sigma_1,\ldots,\sigma_N|\mathbf{c}) = -J\sum_{i<j} c_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i$$

$$f(\mathbf{c}) = -\frac{1}{\beta N}\log\sum_{\sigma_1\ldots\sigma_N}\exp[-\beta H(\sigma_1,\ldots,\sigma_N|\mathbf{c})]$$

average free energy density

(use $\overline{\log Z} = \lim_{n\to 0}\frac{1}{n}\log\overline{Z^n}$)

$$\bar{f} = -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N n}\log\sum_{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_N} \mathrm{e}^{\beta h\sum_{\alpha=1}^{n}\sum_i \sigma_i^\alpha - \beta N E_{\rm eff}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N)}$$

effective interaction energy for replicated spins $\boldsymbol{\sigma}_i = (\sigma_i^1,\ldots,\sigma_i^n)$

$$E_{\rm eff}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N) = -\frac{1}{\beta N}\log\sum_{\mathbf{c}} p(\mathbf{c})\mathrm{e}^{\beta J\sum_{i<j} c_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha}$$

$$p(\mathbf{c}) \propto \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})}\prod_i \delta_{k_i,\sum_j c_{ij}}$$

new generating function

$$\Phi_K[\hat{\varrho},\{\boldsymbol{\sigma}\}] = \frac{1}{N}\log\sum_{\mathbf{c}} \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c}) + K\sum_{i<j} c_{ij}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j} \prod_{i=1}^{N} \delta_{k_i,\sum_j c_{ij}}$$

- $\lim_{K\to 0}\Phi_K[\hat{\varrho},\{\boldsymbol{\sigma}\}] = \Phi[\hat{\varrho}]$
- $N\to\infty$: dependence of Φ_K on spins only via $\mathcal{D}(\boldsymbol{\sigma},k) = \frac{1}{N}\sum_i \delta_{k,k_i}\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i}, \qquad \boldsymbol{\sigma}\in\{-1,1\}^n$
- graph problem coupled to spin problem, connected via $\mathcal{D}$:

$$-\beta E_{\rm eff}[\mathcal{D}] = \Phi_K[\hat{\varrho},\mathcal{D}] - \Phi[\hat{\varrho}]$$

$$-\beta\bar{f} = \lim_{n\to 0}\mathrm{extr}_{\{\mathcal{D},\hat{\mathcal{D}}\}}\frac{1}{n}\Big\{\mathrm{i}\sum_{\boldsymbol{\sigma},k}\hat{\mathcal{D}}(\boldsymbol{\sigma},k)\mathcal{D}(\boldsymbol{\sigma},k) - \beta E_{\rm eff}[\mathcal{D}] + \sum_k p(k)\log\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\beta h\sum_\alpha \sigma_\alpha - \mathrm{i}\hat{\mathcal{D}}(\boldsymbol{\sigma},k)}\Big\}$$

- two types of replicas and analytical continuations:

  $\alpha_\mu = 1\ldots n_\mu$, $\beta_\mu = 1\ldots m_\mu$: as before ($n_\mu, m_\mu$ imaginary)
  $\alpha = 1\ldots n$: spin related, $n\to 0$

spin part of the problem

- replica symmetric ansatz
$$\mathcal{D}_{\rm RS}(\boldsymbol{\sigma},k)=\sum_k p(k)\int\!{\rm d}x\;W_k(x)\frac{{\rm e}^{\beta x\sum_{\alpha=1}^n\sigma_\alpha}}{[2\cosh(\beta x)]^n}$$

  $W_k(x)$: distr of effective fields
  at sites with degree k

- simple manipulations,
  replica limit $n\to 0$ where possible:

$$-\beta\bar{f}_{\rm RS}={\rm extr}_{\{W_k\}}\Big\{\sum_k p(k)\int\!{\rm d}x\;W_k(x)\Big[\beta h\tanh(\beta x)+\log[2\cosh(\beta x)]\Big]$$
$$-\lim_{n\to 0}\frac{\beta}{n}E_{\rm eff}[\mathcal{D}_{\rm RS}]-\sum_k p(k)\int\!{\rm d}x\;\log\cosh(\beta x)\int\!\frac{{\rm d}\hat{x}}{2\pi}{\rm e}^{{\rm i}\hat{x}x}\;\hat{W}_k(\hat{x})\log\hat{W}_k(\hat{x})\Big\}$$
$$\hat{W}_k(\hat{x})=\int\!{\rm d}x\;W_k(x){\rm e}^{-{\rm i}\hat{x}x}$$

- physical observables
$$m=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i\rangle}=\sum_k p(k)\int\!{\rm d}x\;W_k(x)\tanh(\beta x)$$
$$q=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i\rangle^2}=\sum_k p(k)\int\!{\rm d}x\;W_k(x)\tanh^2(\beta x)$$

resulting theory

$$-\beta E[\{W_k\}] = \text{complicated formula involving order parameter } \mathcal{W}_K[\{x,y,z\},v]$$

$$\mathcal{W}_K[\{x,y,z\},v] = \text{soln of complicated order parameter eqn}$$

simplest solns:

$\mathcal{W}_K[\{x,y,z\},v] = \mathcal{W}_K[\{x\},v]\delta[\{y\}]\delta[\{z\}]$

$$\mathcal{W}_K[\{x\},v] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\int \mathrm{d}u\; W_k(u)\int\frac{\mathrm{d}\hat{v}}{2\pi}\;\mathrm{e}^{\mathrm{i}\hat{v}(v-u)}$$
$$\times\frac{\mathcal{A}[\{x\}]\Big[\prod_{\ell<k}\int\{\mathrm{d}x_\ell\}\mathrm{d}v_\ell\;\mathcal{W}_K[\{x_\ell\},v_\ell]\mathrm{e}^{-\mathrm{i}\hat{v}H(v_\ell)}\Big]\delta_{\mathrm{F}}\big[x-F[x_1,\ldots,x_{k-1}]\big]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\}\mathrm{d}v_\ell\;\mathcal{W}_K[\{x_\ell\},v_\ell]\mathrm{e}^{-\mathrm{i}\hat{v}H(v_\ell)}\Big]\delta_{\mathrm{F}}\big[x'-F[x_1,\ldots,x_k]\big]}$$

$$H(v) = \frac{1}{\beta}\mathrm{atanh}[\tanh(\beta v)\tanh(K)]$$

note: $\int\mathrm{d}v\;\mathcal{W}_K[\{x,y,z\},v] = \mathcal{W}[\{x,y,z\}]$

$\mathcal{W}_K[v|\{x,y,z\}]$: effect of local topology on magnetic ordering

Test: disconnected graph and spin variables

regular loopy graphs,
with $k = 2d$, but

$$\mathcal{W}_K[\{x,y,z\},v] = \mathcal{W}[\{x,y,z\}]\mathcal{W}_K(v)$$

saddle point eqn:

$$\mathcal{W}_K(v) = \int \mathrm{d}u\, W(u) \int \frac{\mathrm{d}\hat{v}}{2\pi}\, \mathrm{e}^{\mathrm{i}\hat{v}(v-u)} \Big(\int \mathrm{d}v'\, \mathcal{W}_K(v')\mathrm{e}^{-\mathrm{i}\hat{v}H(v')} \Big)^{-1}$$

inverse soln:

$$\hat{W}(\hat{v}) = \hat{\mathcal{W}}_K(\hat{v}) \int \mathrm{d}v\, \mathcal{W}_K(v)\mathrm{e}^{-\mathrm{i}\hat{v}H(v)}$$

interaction energy:

$$\begin{aligned} -\beta E[\{W\}] = {} & d \log\cosh(K) + 2d \int \frac{\mathrm{d}v\mathrm{d}\hat{v}}{2\pi} \mathrm{e}^{\mathrm{i}\hat{v}v} \hat{W}(\hat{v}) \log\cosh(\beta v) \log \Big[\hat{W}(\hat{v})/\hat{\mathcal{W}}_K(\hat{v}) \Big] \\ & + d \int \mathrm{d}v'\, \mathcal{W}_K(v') \log \Big(\frac{1+\sinh^2(\beta v')/\cosh^2(K)}{\cosh^2(\beta v')} \Big) \\ & - d \int \mathrm{d}v\mathrm{d}v'\, \mathcal{W}_K(v)\mathcal{W}_K(v') \log[1+\tanh(K)\tanh(\beta v)\tanh(\beta v')] \end{aligned}$$

in homogeneous systems: soln of spin eqn of the form

$$W(x) = \delta[x - \beta^{-1}\mathrm{atanh}(m)]$$

giving

$$\bar{f}_{\mathrm{RS}} = \mathrm{extr}_m \Big\{ E[m] - hm - \frac{1}{\beta}\log 2 + \frac{1}{2\beta}\log(1-m^2) + \frac{m\,\mathrm{atanh}(m)}{\beta} \Big\}$$

resulting eqns for m and $\mathcal{W}_K(v)$:

$$m = \tanh\left[\beta\left(h - \mathrm{d}E[m]/\mathrm{d}m\right)\right]$$

$$\mathrm{e}^{-\mathrm{i}\hat{v}\ \mathrm{atanh}(m)/\beta} = \hat{\mathcal{W}}_K(\hat{v}) \int \mathrm{d}v\ \mathcal{W}_K(v) \mathrm{e}^{-\mathrm{i}\hat{v}H(v)}$$

soln:

$$\mathcal{W}_K(v) = \delta(v - v^\star), \qquad m = \frac{\tanh(\beta v^\star)[1 + \tanh(K)]}{1 - \tanh^2(\beta v^\star)\tanh(K)}$$

$E[m]$ decouples from spectral features of the graph:

$$-\beta E[m] = 2dm\left[\mathrm{atanh}(m) - \beta v^\star\right] + d\log\left(\frac{\cosh^2(\beta v^\star) + \sinh^2(K)}{\cosh(K)\cosh^2(\beta v^\star) + \sinh(K)\sinh^2(\beta v^\star)}\right)$$

zero field phase transition: recovers Bethe lattice result

$$T_c = 2J/\log\left[d/(d-1)\right] \qquad \checkmark$$

Discussion and summary

- new analytical approach to (processes on) loopy networks, based on max entropy graph ensembles characterised by *degrees and spectrum*

- replica formula for tricky constraint that allows sum over graphs to be done (via Edwards-Jones)

$$\mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} = \lim_{\varepsilon,\Delta\downarrow 0}\prod_{\mu}\Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}n(\mu)}\ \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{\ -\mathrm{i}n(\mu)}\Big]$$

$$Z(\mu|\mathbf{c}) = \int \mathrm{d}\boldsymbol{\phi}\ \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbb{1}]\boldsymbol{\phi}}, \qquad n(\mu) = \frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$$

- if spectrum imposed via hard constraint: same eqns, but ensemble entropy reduced by a (diverging) constant

- closed explicit order parameter eqns in replica language

- RS order parameter equations for *loopy* graphs interpreted as stationary state of message passing with nontrivial acceptance probabilities

Work to do ...

- Use of graphicality conditions $\int d\mu\ \mu\varrho(\mu)=0$ and $\int d\mu\ \mu^2\varrho(\mu)=\langle k\rangle$ to identify physical saddle-point
- Analytical solution of order parameter eqn for regular loopy graphs?

$$\mathcal{W}(x) = \frac{[1-\tau\ \mathrm{sgn}(x)]\mathcal{F}_{k-1}(x)}{1-\tau\int dx'\ \mathrm{sgn}(x')]\mathcal{F}_{k-1}(x')}, \qquad \tau\in(-1,1)$$

$$\mathcal{F}_{k-1}(x) = \Big[\int\prod_{\ell<k} dx_\ell \mathcal{W}(x_\ell)\Big]\delta\Big[x+\mu+\sum_{\ell<k}\frac{1}{x_\ell}\Big]$$

- Analytical solution of $\hat{\varrho}(\mu)$ for regular cubic lattice spectra? Predicted zero field transition temperatures, critical exponents?
- Proof that f is self-averaging, i.e. $\lim_{N\to\infty}\big[\overline{f^2(\mathbf{c})}-\overline{f(\mathbf{c})}^2\big]=0$?
- transitions to $\mathcal{W}[\{x,y,z\}]\neq\mathcal{W}[\{x\}]\delta[\{y\}]\delta[\{z\}]$?
- replica symmetry breaking transitions? (two types)