Mathematical tools

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ACC Coolen

King's College London and Saddle Point Science

The delta distribution

Gaussian integrals

Steepest descent integration

The delta distribution

- intuitive definition of $\delta(x)$:

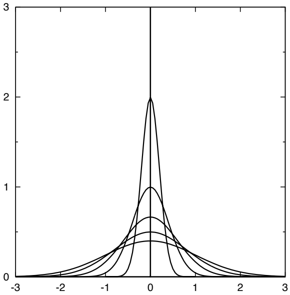
  prob distribution for a 'random' variable x
  that is always zero

  $$\langle f \rangle = \int_{-\infty}^{\infty} \mathrm{d}x\; f(x)\delta(x) = f(0) \qquad \textit{for any } f$$

  for instance

  $$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2\sigma^2}$$

  not a function: $\delta(x \neq 0) = 0,\;\; \delta(0) = \infty$



- status of $\delta(x)$:

  $\delta(x)$ only has a meaning when appearing *inside an integration*,
  one takes the limit $\sigma \downarrow 0$ *after* performing the integration

  $$\int_{-\infty}^{\infty} \mathrm{d}x\; f(x)\delta(x) = \lim_{\sigma \downarrow 0} \int_{-\infty}^{\infty} \mathrm{d}x\; f(x) \frac{\mathrm{e}^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} = \lim_{\sigma \downarrow 0} \int_{-\infty}^{\infty} \mathrm{d}x\; f(x\sigma) \frac{\mathrm{e}^{-x^2/2}}{\sqrt{2\pi}} = f(0)$$

- differentiation of $\delta(x)$:

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\delta'(x) = \int_{-\infty}^{\infty} \mathrm{d}x \left\{ \frac{\mathrm{d}}{\mathrm{d}x}\Big(f(x)\delta(x)\Big) - f'(x)\delta(x) \right\}$$

$$= \lim_{\sigma\downarrow 0} \left[f(x)\frac{\mathrm{e}^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \right]_{x=-\infty}^{x=\infty} - f'(0) = -f'(0)$$

generalization:

$$\int_{-\infty}^{\infty} \mathrm{d}x\ f(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x) = (-1)^n \lim_{x\to 0} \frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \qquad (n = 0, 1, 2, \ldots)$$

- integration of $\delta(x)$: $\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}\theta(x)$ $\qquad \theta(x<0) = 0$, $\theta(x>0) = 1$

Proof: both sides have same effect in integrals

$$\int \mathrm{d}x \left\{ \delta(x) - \frac{\mathrm{d}}{\mathrm{d}x}\theta(x) \right\} f(x) = f(0) - \lim_{\epsilon\downarrow 0} \int_{-\epsilon}^{\epsilon} \mathrm{d}x \left\{ \frac{\mathrm{d}}{\mathrm{d}x}\Big(\theta(x)f(x)\Big) -$$

$$f(0) - \lim_{\epsilon\downarrow 0} [f(\epsilon) - 0] + \lim_{\epsilon\downarrow 0} \int_{0}^{\epsilon} \mathrm{d}x\ f'(x) = 0$$

- generalization to vector arguments: $\mathbf{x} \in \mathbb{R}^N$: $\qquad \delta(\mathbf{x}) = \prod_{i=1}^{N} \delta(x_i)$

- Integral representation of $\delta(x)$

  use defns of Fourier transforms and their inverse:

$$\hat{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-2\pi \mathrm{i} kx} f(x) \qquad f(x) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{2\pi \mathrm{i} kx} \hat{f}(k) \quad \Rightarrow \quad f(x) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{2\pi \mathrm{i} kx} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-2\pi \mathrm{i} ky} f(y)$$

apply to $\delta(x)$: $\quad \delta(x) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{2\pi \mathrm{i} kx} = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i} kx}$

- invertible functions of x as arguments: $\quad \delta\left[g(x) - g(a)\right] = \frac{\delta(x-a)}{|g'(a)|}$

  Proof: both sides have same effect in integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f(x) \left\{ \delta\left[g(x) - g(a)\right] - \frac{\delta(x-a)}{|g'(a)|} \right\} = \int_{-\infty}^{\infty} \mathrm{d}x\, g'(x) \frac{f(x)}{g'(x)} \delta\left[g(x) - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= \int_{g(-\infty)}^{g(\infty)} \mathrm{d}k\, \frac{f(g^{\mathrm{inv}}(k))}{g'(g^{\mathrm{inv}}(k))} \delta\left[k - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= sgn[g'(a)] \int_{-\infty}^{\infty} \mathrm{d}k\, \frac{f(g^{\mathrm{inv}}(k))}{g'(g^{\mathrm{inv}}(k))} \delta\left[k - g(a)\right] - \frac{f(a)}{|g'(a)|}$$

$$= sgn[g'(a)] \frac{f(a)}{g'(a)} - \frac{f(a)}{|g'(a)|} = 0$$

Gaussian integrals

- one-dimensional:

$$\int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2/\sigma^2} = 1, \qquad \int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\,x\,\mathrm{e}^{-\frac{1}{2}x^2/\sigma^2} = 0, \qquad \int \frac{\mathrm{d}x}{\sigma\sqrt{2\pi}}\,x^2\mathrm{e}^{-\frac{1}{2}x^2/\sigma^2} = \sigma^2$$

$$\int \frac{\mathrm{d}x}{\sqrt{2\pi}}\,\mathrm{e}^{kx-\frac{1}{2}x^2} = e^{\frac{1}{2}k^2} \quad (k \in \mathbb{C})$$

- N-dimensional:

$$\int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\,\mathrm{e}^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = 1, \qquad \int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\,x_i\mathrm{e}^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = 0,$$

$$\int \frac{\mathrm{d}\mathbf{x}}{\sqrt{(2\pi)^N \det\mathbf{C}}}\,x_i x_j\mathrm{e}^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}} = C_{ij}$$

- multivariate Gaussian distribution:

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N \det\mathbf{C}}}\,\mathrm{e}^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{C}^{-1}\mathbf{x}}$$

$$\int \mathrm{d}\mathbf{x}\; p(\mathbf{x}) x_i x_j = C_{ij}, \qquad \int \mathrm{d}\mathbf{x}\; p(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \mathrm{e}^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{C}\mathbf{k}}$$

Steepest descent integration

Objective of steepest descent
(or 'saddle-point') integration:
large N behavior of integrals of the type $I_N = \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x})\; \mathrm{e}^{-Nf(\mathbf{x})}$

- $f(\mathbf{x})$ real-valued, smooth, bounded from below, and with unique minimum at $\mathbf{x}^\star$

  expand f around minimum:

  $$f(\mathbf{x}) = f(\mathbf{x}^\star) + \frac{1}{2}\sum_{ij=1}^{p} A_{ij}(x_i - x_i^\star)(x_j - x_j^\star) + \mathcal{O}(|\mathbf{x}-\mathbf{x}^\star|^3) \qquad A_{ij} = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}^\star}$$

  Insert into integral,
  transform $\mathbf{x} = \mathbf{x}^\star + \mathbf{y}/\sqrt{N}$:

  $$\begin{aligned} I_N &= \mathrm{e}^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x}) \mathrm{e}^{-\frac{1}{2}N\sum_{ij}(x_i - x_i^\star)A_{ij}(x_j - x_j^\star) + \mathcal{O}(N|\mathbf{x}-\mathbf{x}^\star|^3)} \\ &= N^{-\frac{p}{2}} \mathrm{e}^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g\left(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}}\right) \mathrm{e}^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})} \end{aligned}$$

$$\int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; g(\mathbf{x})\; e^{-Nf(\mathbf{x})} = N^{-\frac{p}{2}} e^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g\Big(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}}\Big)\; e^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}$$

- first result, for $p \ll N/\log N$: (condition can be sharpened)

$$\begin{aligned}
-\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\; \mathrm{e}^{-Nf(\mathbf{x})} \\
&= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} - \frac{1}{N} \log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; \mathrm{e}^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})} \Big] \\
&= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} - \frac{1}{2N} \log\Big(\frac{(2\pi)^p}{\det \mathbf{A}}\Big) - \frac{1}{N}\log\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)\Big] \\
&= f(\mathbf{x}^\star) + \lim_{N\to\infty} \Big[\frac{p\log N}{2N} + \mathcal{O}(\frac{p}{N}) + \mathcal{O}(\frac{p^{3/2}}{N^{3/2}})\Big] = f(\mathbf{x}^\star)
\end{aligned}$$

- second result, for $p \ll \sqrt{N}$: (condition can be sharpened)

$$\lim_{N\to\infty} \frac{\int \mathrm{d}\mathbf{x}\; g(\mathbf{x}) e^{-Nf(\mathbf{x})}}{\int \mathrm{d}\mathbf{x}\; e^{-Nf(\mathbf{x})}} = \lim_{N\to\infty} \Bigg[\frac{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; g(\mathbf{x}^\star + \mathbf{y}/\sqrt{N})\; \mathrm{e}^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}}{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{y}\; \mathrm{e}^{-\frac{1}{2}\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(|\mathbf{y}|^3/\sqrt{N})}} \Bigg]$$

$$= \frac{g(\mathbf{x}^\star)\Big(1 + \mathcal{O}(\frac{p}{\sqrt{N}})\Big)\sqrt{\frac{(2\pi)^p}{\det \mathbf{A}}}\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)}{\sqrt{\frac{(2\pi)^p}{\det \mathbf{A}}}\Big(1 + \mathcal{O}(\frac{p^{3/2}}{\sqrt{N}})\Big)} = g(\mathbf{x}^\star)$$

- $f(\mathbf{x})$ complex-valued:

  – deform integration path in complex plane, using Cauchy's theorem, such that along deformed path the imaginary part of $f(\mathbf{x})$ is constant, and preferably zero

  – proceed using Laplace's argument, and find the leading order in N by extremization of the real part of $f(\mathbf{x})$

  similar fomulae,
  but with (possibly complex) extrema
  that need no longer be maxima:

$$
\begin{aligned}
-\lim_{N\to\infty}\frac{1}{N}\log\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\ \mathrm{e}^{-Nf(\mathbf{x})} &= \mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p}f(\mathbf{x}) \\
\lim_{N\to\infty}\frac{\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\ g(\mathbf{x})\mathrm{e}^{-Nf(\mathbf{x})}}{\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\ \mathrm{e}^{-Nf(\mathbf{x})}} &= g\Big(\arg\ \mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p}f(\mathbf{x})\Big)
\end{aligned}
$$