Disordered (Soft) Spin Systems on Random Graphs with Finite Connectivity

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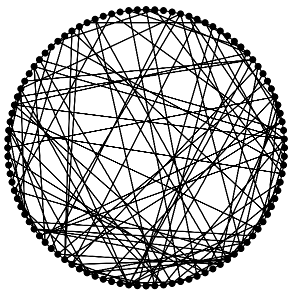


Disordered Many-Particle Systems

Finite Connectivity – Statics

- Attractor Neural Networks
- 'Small World' Models
- Soft Spin RS: Order Parameter Functionals

Finite Connectivity – Dynamics

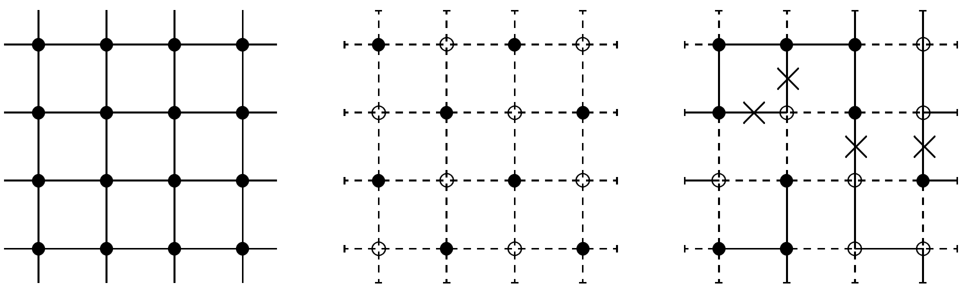
- Generating Functional Analysis

Outlook

DISORDERED MANY-PARTICLE SYSTEMS

- (pseudo-) randomness in microscopic parameters
- no microscopic periodicity
- high degree of frustration, incompatible forces

ferro (___) anti-ferro (- - -) both types



○: spin up ●: spin down

- many relevant time-scales
- many techniques no longer applicable
- even mean-field models are non-trivial

physics:	spin-glasses, glasses
biology:	neural networks, proteins, immune networks
IT:	machine learning, error-correcting codes
economics:	agent-based market models

The Replica Method
(Marc Kac, 1968)

- $x^n = 1 + n \log x + \mathcal{O}(n^2)$

$$\overline{\log Z[\boldsymbol{J}]} = \lim_{n\to 0} \frac{1}{n} \log \overline{Z^n[\boldsymbol{J}]}$$

- n integer:

$$Z^n[\boldsymbol{J}] = \left[\sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}\right]^n = \sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}$$

partition function of n *independent* replicas of system

- disorder average:

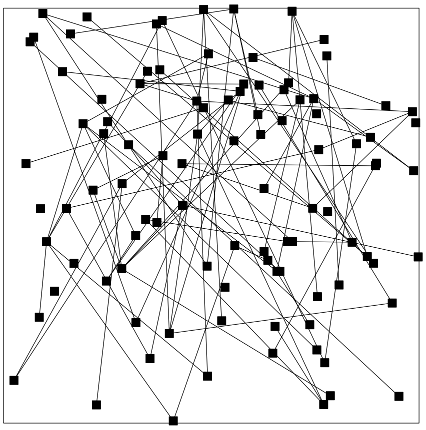
$$\overline{f} = - \lim_{n\to 0} \frac{1}{\beta N n} \log \left[\sum_{\boldsymbol{\sigma}^1} \cdots \sum_{\boldsymbol{\sigma}^n} e^{-\beta H_{\text{eff}}(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n)}\right]$$

$$H_{\text{eff}}(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n) = -\frac{1}{\beta} \log \overline{e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}$$

n *interacting* replicas, *no* disorder

- exchange limits
  $N \to \infty$ versus $n \to 0$

FINITE CONNECTIVITY STATICS



$N = 100,\ c = 2$

N spins on random graph, $c_{ij} \in \{0,1\}$

$$H = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j + \sum_i V(\sigma_i)$$

- $P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1-\frac{c}{N})\delta_{c_{ij},0}, \quad c = \mathcal{O}(N^0)$
- indep random bonds J_{ij}
- disorder: $\{c_{ij}, J_{ij}\}$

Replica theory order parameters

Dependence on connectivity c
(average number of bonds/spin)

connectivity	*variables*	*order param*	*RS ansatz*
$c = N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	discrete	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$1 \ll c \ll N$	continuous	$\{q_{\alpha\beta}\}$	numbers, e.g. q
$c = \mathcal{O}(1)$	discrete	$P(\sigma_1, \ldots, \sigma_n)$	functions, $P(h)$
$c = \mathcal{O}(1)$	continuous	$P(\sigma_1, \ldots, \sigma_n)$	functionals, $W[\{P\}]$

Finite Connectivity Attractor Neural Networks

$$H = -\sum_{i<j}^{N} J_{ij}\sigma_i\sigma_j \qquad J_{ij} = \frac{c_{ij}}{c}\,\phi\left(\sum_{\mu=1}^{p}\xi_i^\mu\xi_j^\mu\right)$$

$$P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + [1-\frac{c}{N}]\delta_{c_{ij},0}$$

$\sigma_i, \xi_i^\mu \in \{-1,1\}$

replica solution:

$$\begin{aligned}\overline{f} &= -\lim_{N\to\infty}(\beta N)^{-1}\overline{\log Z} \\ &= -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta Nn}\log\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} e^{\frac{c}{2N}\sum_{ij}\left[e^{\frac{\beta}{c}\phi(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\sum_\alpha\sigma_i^\alpha\sigma_j^\alpha}-1\right]}\end{aligned}$$

analysis via sub-lattices:

$$I_{\boldsymbol{\xi}} = \{i|\ \boldsymbol{\xi}_i = \boldsymbol{\xi}\}$$

$$\overline{f} = \lim_{n\to 0}\ \mathrm{extr}_{\{P_{\boldsymbol{\xi}}(\boldsymbol{\sigma})\}}\ f[\{P_{\boldsymbol{\xi}}(\boldsymbol{\sigma})\}]$$

$$P_{\boldsymbol{\xi}}(\sigma_1,\ldots,\sigma_n) = \lim_{N\to\infty}\frac{1}{|I_{\boldsymbol{\xi}}|}\sum_{i\in I_{\boldsymbol{\xi}}}\overline{\langle\prod_\alpha\delta_{\sigma_\alpha,\sigma_i^\alpha}\rangle}$$

Replica Symmetric ansatz ?

disorder: $D = \{c_{ij}, J_{ij}\}$
probability: $\mathcal{P}(D)$

$$
\begin{aligned}
P_{\boldsymbol{\xi}}(\sigma_1,\ldots,\sigma_n) &= \lim_{N\to\infty}\frac{1}{|I_{\boldsymbol{\xi}}|}\sum_{i\in I_{\boldsymbol{\xi}}}\overline{\langle\prod_\alpha \delta_{\sigma_\alpha,\sigma_i^\alpha}\rangle} \\
&= \lim_{N\to\infty}\frac{1}{|I_{\boldsymbol{\xi}}|}\sum_{i\in I_{\boldsymbol{\xi}}}\int dD\ \mathcal{P}(D)\ \ \langle\prod_\alpha \delta_{\sigma_\alpha,\sigma_i^\alpha}\rangle\Big|_D
\end{aligned}
$$

Ergodicity:

$$
\langle\prod_\alpha \delta_{\sigma_\alpha,\sigma_i^\alpha}\rangle\Big|_D = \prod_\alpha p(\sigma_\alpha|D,i) = \prod_\alpha\left\{\frac{e^{\beta h(D,i)\sigma_\alpha}}{2\cosh[\beta h(D,i)]}\right\}
$$

so

$$
P_{\boldsymbol{\xi}}(\sigma_1,\ldots,\sigma_n) = \int dh\ W_{\boldsymbol{\xi}}(h)\frac{e^{\beta h\sum_\alpha\sigma_\alpha}}{[2\cosh(\beta h)]^n}
$$

$$
W_{\boldsymbol{\xi}}(h) = \lim_{N\to\infty}\frac{1}{|I_{\boldsymbol{\xi}}|}\sum_{i\in I_{\boldsymbol{\xi}}}\int dD\ \mathcal{P}(D)\delta[h-h(D,i)]
$$

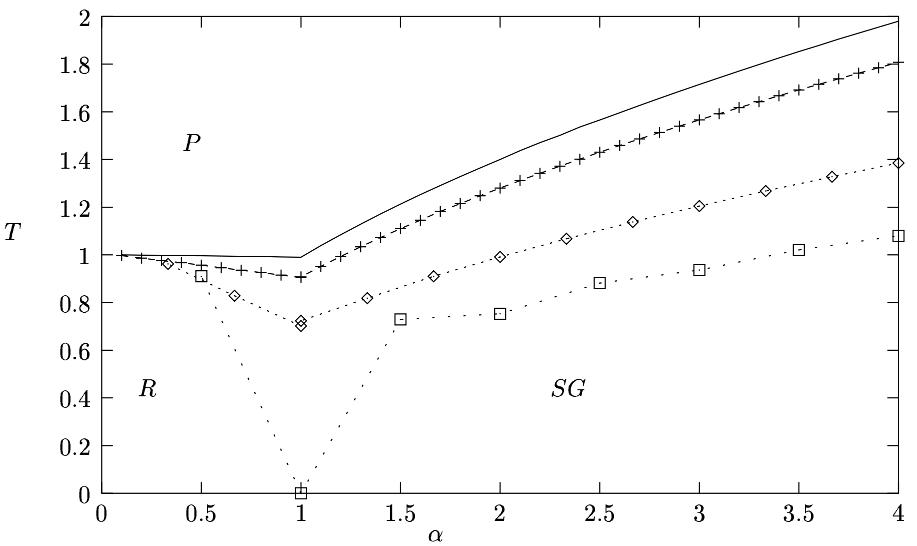
RS order parameters:
2^p effective field distributions $W_{\boldsymbol{\xi}}(h)$

RS theory

$$W_{\boldsymbol{\xi}}(h) \;=\; \int\frac{dm}{2\pi}\; e^{-imh}$$
$$\times\exp\left\{c\;\langle\int dh' W_{\boldsymbol{\xi}'}(h')\left[e^{\frac{im}{\beta}\tanh^{-1}[\tanh(\beta h')\tanh(\frac{\beta}{c}\phi(\boldsymbol{\xi}\cdot\boldsymbol{\xi}'))]}-1\right]\rangle_{\boldsymbol{\xi}'}\right\}$$

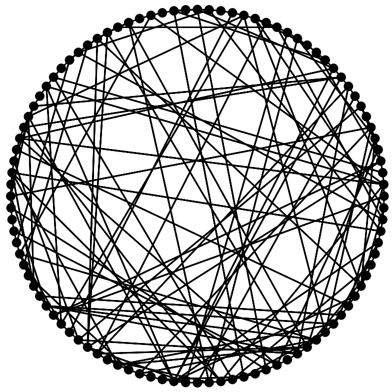
paramagnetic soln: $W_{\boldsymbol{\xi}}(h)=\delta(h) \quad \forall\boldsymbol{\xi}$

$$\mathrm{P}\to\mathrm{R}: \qquad \frac{c}{p}\;2^{-p}\sum_{n=0}^{p}\binom{p}{n}(p-2n)\tanh[\frac{\beta}{c}\phi(p-2n)]=1$$
$$\mathrm{P}\to\mathrm{SG}: \qquad c\;2^{-p}\sum_{n=0}^{p}\binom{p}{n}\tanh^2[\frac{\beta}{c}\phi(p-2n)]=1$$



Fin conn Hopfield model: $\phi(x)=x$, $\alpha=p/c$
$c=2$ ($\Box$), $c=3$ ($\diamond$), $c=10$ ($+$), $c=100$ (line)

'Small World' Spin Models



$N = 100,\ c = 2$

N Ising spins on ring + random graph

$$H = -\sum_i J_i^s \sigma_i \sigma_{i+1} - \frac{1}{c} \sum_{i<j} J_{ij}^\ell c_{ij} \sigma_i \sigma_j$$

- $P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0}$
- indep random interaction energies $\{J_i^s, J_{ij}^\ell\}$
- disorder: $\{c_{ij}, J_{ij}\}$

Replica analysis

$$\overline{f} = \lim_{n\to 0} \frac{1}{n\beta} \text{extr}_{\{P\}} \left\{ \frac{c}{2} \sum_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} P(\boldsymbol{\sigma})P(\boldsymbol{\sigma}')\langle e^{\frac{\beta J_\ell}{c}\sum_\alpha \sigma_\alpha \sigma'_\alpha} - 1\rangle_{J_\ell} - \log \lambda_0(n;P) \right\}$$

$$P(\boldsymbol{\sigma}) = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \prod_\alpha \delta_{\sigma_\alpha, \sigma_i^\alpha} \rangle}, \qquad \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$$

$\lambda_0(n;P)$: largest eigenvalue of $2^n \times 2^n$ replicated transfer matrix

$$\boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{\sigma}'; P) = \left\langle e^{\beta J_s \sum_\alpha \sigma_\alpha \sigma'_\alpha + c \sum_{\boldsymbol{s}} P(\boldsymbol{s}) \langle e^{\frac{\beta J_\ell}{c} \sum_\alpha \sigma_\alpha s_\alpha} - 1 \rangle_{J_\ell}} \right\rangle_{J_s}$$

RS theory

$$\boldsymbol{T}(\boldsymbol{\sigma},\boldsymbol{\sigma}';P) = \langle e^{\beta J_s \sum_\alpha \sigma_\alpha \sigma'_\alpha + c\sum_{\boldsymbol{s}} P(\boldsymbol{s})\langle e^{\frac{\beta J_\ell}{c}\sum_\alpha \sigma_\alpha s_\alpha} - 1\rangle_{J_\ell}}\rangle_{J_s}$$

$$P(\boldsymbol{\sigma}) = \int \mathrm{d}h\, W(h) \frac{e^{\beta h \sum_{\alpha=1}^n \sigma_\alpha}}{[2\cosh(\beta h)]^n}$$

left/right
eigenvectors of $\boldsymbol{T}$:

$$\begin{aligned} v_L(\boldsymbol{\sigma}) &= \int \mathrm{d}y\, \Psi(y) e^{\beta y \sum_\alpha \sigma_\alpha} \\ v_R(\boldsymbol{\sigma}) &= \int \mathrm{d}x\, \Phi(x) e^{\beta x \sum_\alpha \sigma_\alpha} \end{aligned}$$

Final result, $n \to 0$:

$$\begin{aligned} \Phi(x) &= \int \mathrm{d}x'\, \Phi(x') \sum_{k\geq 0} \frac{e^{-c}c^k}{k!} \int \prod_{r\leq k} \mathrm{d}J_{lr}\mathrm{d}h_r\, P(J_{lr})W(h_r) \\ &\qquad \langle \delta[x - \sum_r A(J_{lr}/c, h_r) - A(J_s, x')]\rangle_{J_s} \\ \Psi(y) &= \int \mathrm{d}y'\, \Psi(y') \sum_{k\geq 0} \frac{e^{-c}c^k}{k!} \int \prod_{r\leq k} \mathrm{d}J_{lr}\mathrm{d}h_r\, P(J_{lr})W(h_r) \\ &\qquad \langle \delta[y - A(J_s, y' + \sum_r A(J_{lr}/c, h_r))]\rangle_{J_s} \end{aligned}$$

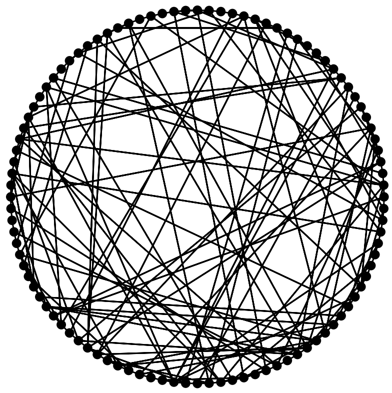
$$W(h) = \int \mathrm{d}x\mathrm{d}y\, \Phi(x)\Psi(y)\delta[h - x - y]$$

with

$$A(u,z) = \frac{1}{2\beta}\log\left[\frac{\cosh(\beta(u+z))}{\cosh(\beta(u-z))}\right]$$

$$B(u,z) = \frac{1}{2\beta}\log\left[4\cosh(\beta(u+z))\cosh(\beta(u-z))\right]$$

Example: 'Small World' Magnets



$$H = -J_0 \sum_i \sigma_i \sigma_{i+1} - \frac{J}{c} \sum_{i<j} c_{ij} \sigma_i \sigma_j$$

(connectivity disorder only)

$$\Phi(x) = \int \mathrm{d}x' \, \Phi(x') \sum_{k\geq 0} \frac{e^{-c} c^k}{k!} \int \prod_{r\leq k} [\mathrm{d}h_r \, W(h_r)] \times \delta[x - \sum_r A(\frac{J}{c}, h_r) - A(J_0, x')]$$

$$\Psi(y) = \int \mathrm{d}y' \, \Psi(y') \sum_{k\geq 0} \frac{e^{-c} c^k}{k!} \int \prod_{r\leq k} [\mathrm{d}h_r \, W(h_r)] \times \delta[y - A(J_0, y' + \sum_r A(\frac{J}{c}, h_r))]$$

$$W(h) = \int \mathrm{d}x\mathrm{d}y \, \Phi(x)\Psi(y)\delta[h - x - y]$$

paramagnetic soln:

$$W(h) = \Phi(h) = \Psi(h) = \delta(h)$$

2nd order transitions away from P:
expand in moments of $W(h)$ and identify instabilities

$$\mathrm{P} \to \mathrm{F}: \qquad 1 = c \tanh(\frac{\beta J}{c}) \left[\frac{1 + \tanh(\beta J_0)}{1 - \tanh(\beta J_0)}\right]$$

$\mathrm{P} \to \mathrm{SG}$: does not happen

$$\text{P} \rightarrow \text{F}: \qquad \beta J = \frac{1}{2} c \; \log \left[\frac{c + e^{-2\beta J_0}}{c - e^{-2\beta J_0}} \right]$$

limits:

$$c \rightarrow \infty: \quad \beta J = e^{-2\beta J_0}$$

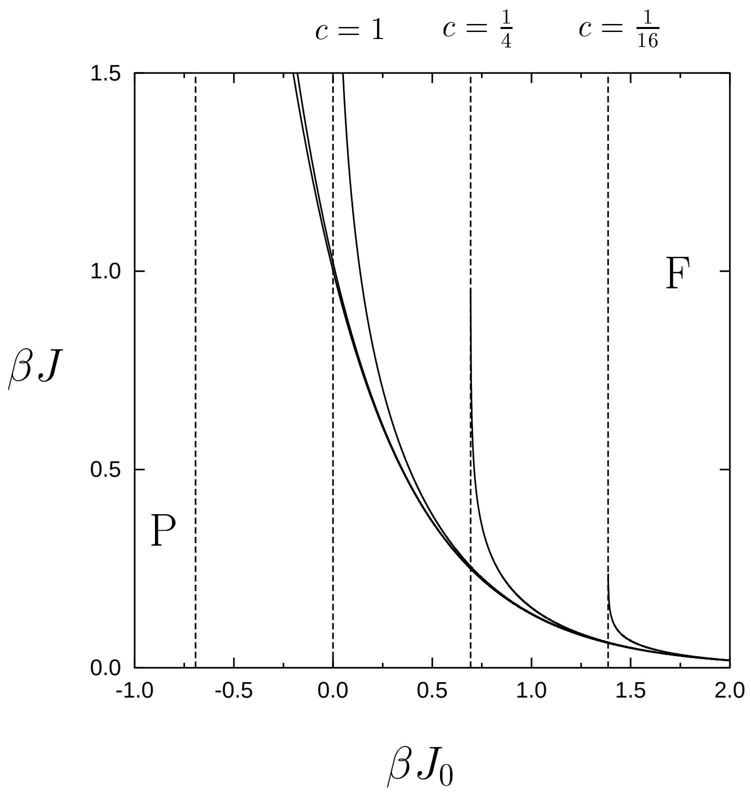
$$J_0 = 0: \quad \text{order possible for } c > 1$$

$$J = 0 \text{ or } c = 0: \quad \text{never order}$$

combination:

$$c > 0,\; J_0 > 0: \quad \text{order for } T < T_c \;\; \text{with} \;\; T_c(J_0, J, c) > 0$$

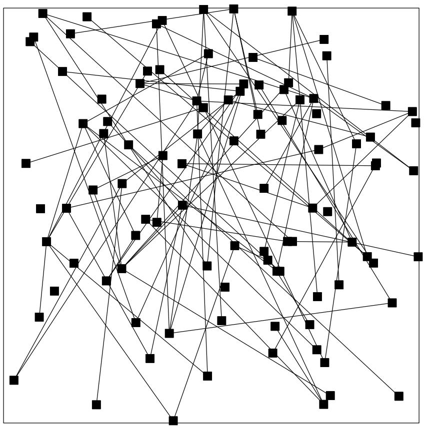
$$\text{even for} \;\; c \ll 1 \;\; !!$$



solid:
P→F transition
for $c = \frac{1}{16}, \frac{1}{4}, 1, 4, 16$

dashed:
$\beta J_0 > \log(1/\sqrt{c})$
for $c = \frac{1}{16}, \frac{1}{4}, 1, 4, 16$

Soft Spin Models



N soft spins $\sigma_i \in \mathbb{R}$

$$H = -\sum_{i<j} c_{ij} J_{ij} \sigma_i \sigma_j + \sum_i V(\sigma_i)$$

- $P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0}, \quad c = \mathcal{O}(N^0)$
- indep random bonds J_{ij}

Replica theory

order parameter:

$$P(\sigma_1, \ldots, \sigma_n) = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \prod_\alpha \delta[\sigma_\alpha - \sigma_i^\alpha] \rangle}$$

saddle-point:

$$P(\boldsymbol{\sigma}) = \frac{e^{c\int d\boldsymbol{\sigma}'\ P(\boldsymbol{\sigma}')[\int dJ\ P(J) e^{\beta J \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'} - 1] - \beta\sum_\alpha V(\sigma_\alpha)}}{\int d\boldsymbol{\sigma}'\ e^{c\int d\boldsymbol{\sigma}''\ P(\boldsymbol{\sigma}'')[\int dJ\ P(J) e^{\beta J \boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}''} - 1] - \beta\sum_\alpha V(\sigma'_\alpha)}}$$

Replica Symmetric ansatz ?

disorder: $D = \{c_{ij}, J_{ij}\}$
probability: $\mathcal{P}(D)$

$$
\begin{aligned}
P(\sigma_1, \ldots, \sigma_n) &= \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\prod_\alpha \delta[\sigma_\alpha - \sigma_i^\alpha]} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_i \int dD\, \mathcal{P}(D) \;\; \langle \prod_\alpha \delta[\sigma_\alpha - \sigma_i^\alpha] \rangle \Big|_D
\end{aligned}
$$

Ergodicity:

$$
\langle \prod_\alpha \delta[\sigma_\alpha - \sigma_i^\alpha] \rangle \Big|_D = \prod_\alpha P(\sigma_\alpha | D, i)
$$

so

$$
P_{\mathrm{RS}}(\sigma_1, \ldots, \sigma_n) = \int \{dP\}\; W[\{P\}] \;\; \prod_\alpha P(\sigma_\alpha)
$$

$$
W[\{P\}] = \lim_{N\to\infty} \frac{1}{N} \sum_i \int dD\, \mathcal{P}(D) \;\; \prod_\sigma \delta \left[P(\sigma) - \prod_\alpha P(\sigma | D, i) \right]
$$

RS order parameter for soft spins:
functional $W[\{P\}]$ on space of single-spin distributions $P(\sigma)$

saddle-point eqn, $n \to 0$:

$$
\begin{aligned}
W[\{P\}] &= \sum_{\ell \geq 0} \frac{c^\ell}{\ell!} e^{-c} \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] dJ_k P(J_k)] \\
&\quad \times \prod_\sigma \delta \left[P(\sigma) - \frac{e^{-\beta V(\sigma)} \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma \sigma'}}{\int d\sigma'' e^{-\beta V(\sigma'')} \prod_{k=1}^{\ell} \int d\sigma' P_k(\sigma') e^{\beta J_k \sigma'' \sigma'}} \right]
\end{aligned}
$$

Application: Chiral Soft Spin Models

N interacting vectorial soft spins
$\boldsymbol{\sigma}_i \in S_d$ (unit sphere in $\mathbb{R}^d$)

$$H = -J \sum_{i<j} c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{U}_{ij} \boldsymbol{\sigma}_j + \sum_i V(\boldsymbol{\sigma}_i)$$

- $P(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + (1 - \frac{c}{N}) \delta_{c_{ij},0}$
- indep random unitary matrices $\boldsymbol{U}_{ij}$,
  $P(\boldsymbol{U}) = P(\boldsymbol{U}^\dagger)$

$$W[\{P\}] \ = \ \sum_{\ell \geq 0} \frac{c^\ell}{\ell!} e^{-c} \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] d\boldsymbol{U}_k P(\boldsymbol{U}_k)]$$
$$\times \prod_{\boldsymbol{\sigma} \in S_d} \delta \left[P(\boldsymbol{\sigma}) - \frac{e^{-\beta V(\boldsymbol{\sigma})} \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{U}_k \boldsymbol{\sigma}'}}{\int d\boldsymbol{\sigma}'' e^{-\beta V(\boldsymbol{\sigma}'')} \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma}'' \cdot \boldsymbol{U}_k \boldsymbol{\sigma}'}} \right]$$

d=2: Finitely Connected XY Spins with Random Chiral Interactions

$\boldsymbol{\sigma}_i \to (\cos\phi_i, \sin\phi_i)$,
$P(\boldsymbol{U}) \to P(\omega)$:

$$H = -J \sum_{i<j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij})$$

$$\begin{aligned} W[\{P\}] \;=\; & \sum_{\ell \geq 0} \frac{c^\ell}{\ell!} e^{-c} \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] d\omega_k P(\omega_k)] \\ & \times \prod_{\phi \in [0,2\pi]} \delta\left[P(\phi) - \frac{\prod_{k=1}^{\ell} \int d\phi' P_k(\phi') e^{\beta J \cos(\phi - \phi' - \omega_k)}}{\int d\phi'' \prod_{k=1}^{\ell} \int d\phi' P_k(\phi') e^{\beta J \cos(\phi'' - \phi' - \omega_k)}}\right] \end{aligned}$$

paramagnetic soln:

$$W[\{P\}] = \prod_{\phi \in [0,2\pi]} \delta\left[P(\phi) - \frac{1}{2\pi}\right]$$

2nd order transitions away from P:

- transform $P(\phi) \to \frac{1}{2\pi} + \Delta(\phi)$, with $W[\{P\}] \to \tilde{W}[\{\Delta\}]$
- expand in functional moments $\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \ldots \Delta(\phi_r)$
- identify instabilities

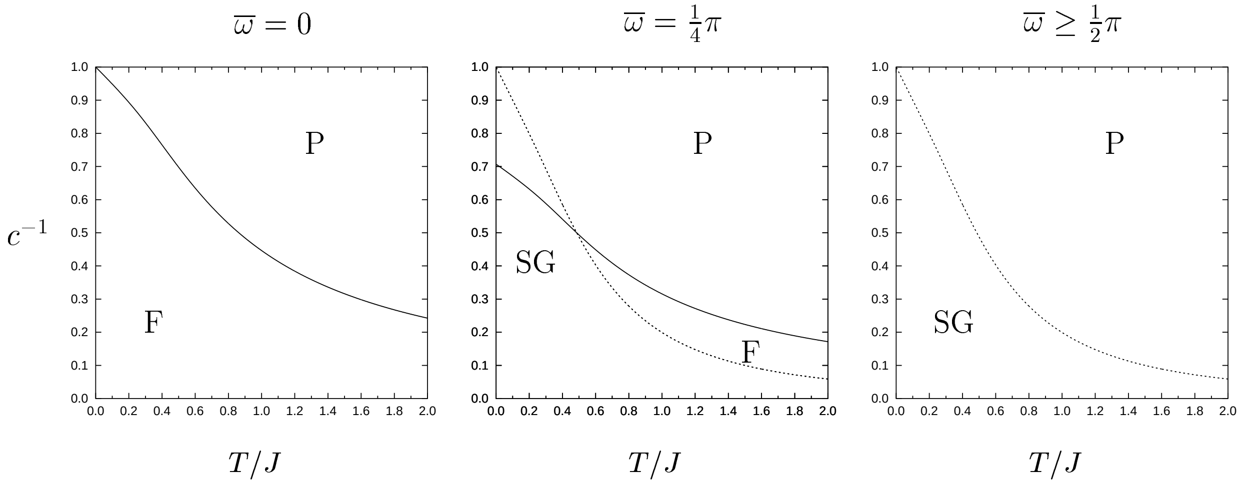
$$\text{P} \to \text{F}: \qquad c^{-1} = \frac{I_1(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega P(\omega) \cos(\omega)$$

$$\text{P} \to \text{SG}: \qquad c^{-1} = I_1^2(\beta J) / I_0^2(\beta J)$$

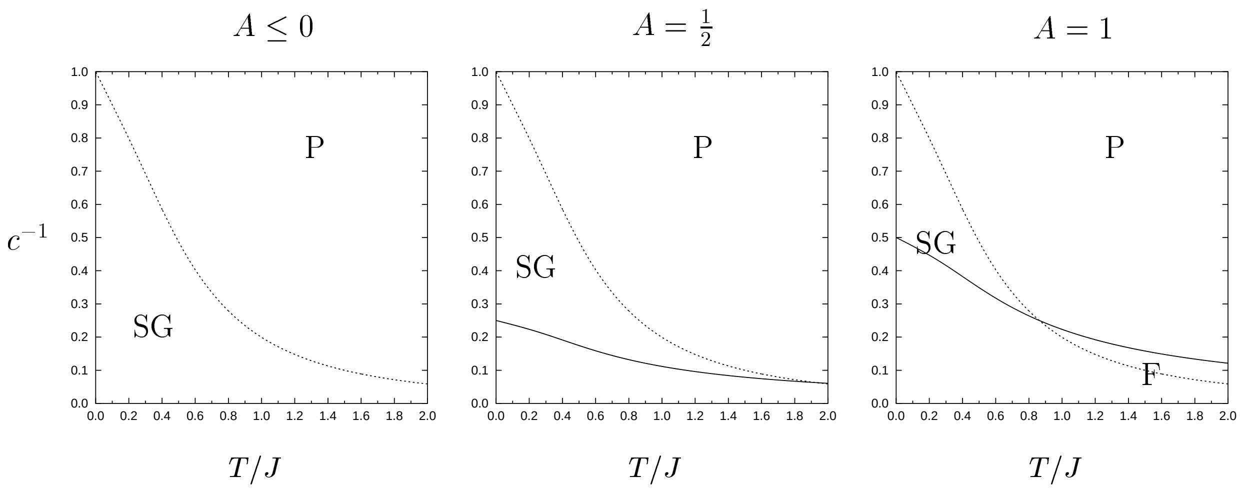
modified Bessel func $\qquad I_n(z) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \cos(n\phi) e^{z\cos(\phi)}$

solid lines: P→F
dotted lines: P→SG

$$H = -J \sum_{i<j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij})$$

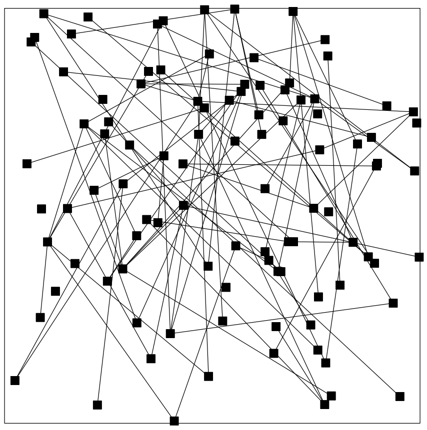


$$P(\omega) = \tfrac{1}{2}\delta(\omega - \overline{\omega}) + \tfrac{1}{2}\delta(\omega + \overline{\omega})$$



$$P(\omega) = \tfrac{1}{2\pi}[1 + A\cos(\ell\phi)]$$

Generating Functional Analysis For Finite Connectivity Models



finitely connected Ising spin model with parallel stochastic dynamics:

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W_t[\boldsymbol{\sigma};\boldsymbol{\sigma}']p_t(\boldsymbol{\sigma}')$$

$$W_t[\boldsymbol{\sigma};\boldsymbol{\sigma}'] = \prod_i \frac{e^{\beta\sigma_i h_i(\boldsymbol{\sigma}';t)}}{2\cosh[\beta h_i(\boldsymbol{\sigma}';t)]}$$

local fields:

$$h_i(\boldsymbol{\sigma};t) = \sum_{j\neq i} c_{ij}J_{ij}\sigma_j + \theta(t)$$

controlled symmetry:

$$i<j: \qquad \mathrm{Prob}(c_{ij}) = W(c_{ij})$$
$$i>j: \qquad \mathrm{Prob}(c_{ij}) = \epsilon_1\delta_{c_{ij},c_{ji}} + (1-\epsilon_1)W(c_{ij})$$

$$i<j: \qquad \mathrm{Prob}(J_{ij}) = P(J_{ij})$$
$$i>j: \qquad \mathrm{Prob}(J_{ij}) = \epsilon_2\delta[J_{ij}-J_{ji}] + (1-\epsilon_2)P(J_{ij})$$

$$W(x) = \frac{c}{N}\delta_{x,1} + (1-\frac{c}{N})\delta_{x,0} \qquad c = \mathcal{O}(N^0)$$

detailed balance & equil stat mech: $\quad \epsilon_1 = \epsilon_2 = 1$

generating functional

$$Z[\boldsymbol{\psi}] = \overline{\langle e^{-i\sum_{i\leq N}\sum_{t<t_m}\psi_i(t)\sigma_i(t)}\rangle}$$
$$= \sum_{\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t_m)} e^{-i\sum_i\sum_t\psi_i(t)\sigma_i(t)} p_0(\boldsymbol{\sigma}(0))\overline{\prod_{t<t_m} W_t[\boldsymbol{\sigma}(t+1);\boldsymbol{\sigma}(t)]}$$

insert

$$1 = \prod_{it}\int dh_i(t)\ \delta[h_i(t) - \sum_{j\neq i} c_{ij}J_{ij}\sigma_j(t) - \theta(t)]$$

disorder average,
standard manipulations:

effective single spin problem

$P(\boldsymbol{\sigma}|\boldsymbol{\theta})$: fraction of sites i which exhibit
a single spin path $\boldsymbol{\sigma} = (\sigma(0), \sigma(1), \sigma(2), \ldots)$
given a field path $\boldsymbol{\theta} = (\theta(0), \theta(1),\ \theta(2), \ldots)$

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}) = p_0(\sigma(0))e^{-c} \times$$
$$\Bigg\{\prod_t \left[\frac{e^{\beta\sigma(t+1)\theta(t)}}{2\cosh[\beta\theta(t)]}\right] + \sum_{k>0}\frac{c^k}{k!}\int dJ_1 P(J_1)\ldots dJ_k P(J_k) \sum_{\boldsymbol{\sigma}'_1\cdots\boldsymbol{\sigma}'_k}$$
$$\times \prod_{\ell=1}^{k}\Big[(1-\epsilon_1)P(\boldsymbol{\sigma}'_\ell|\mathbf{0}) + \epsilon_1\left[\epsilon_2 P(\boldsymbol{\sigma}'_\ell|J_\ell\boldsymbol{\sigma}) + (1-\epsilon_2)\langle P(\boldsymbol{\sigma}'_\ell|J'\boldsymbol{\sigma})\rangle_{J'}\right]\Big]$$
$$\times \prod_t \frac{e^{\beta\sigma(t+1)[\theta(t)+\sum_{\ell\leq k}J_\ell\sigma'_\ell(t)]}}{2\cosh[\beta[\theta(t)+\sum_{\ell\leq k}J_\ell\sigma'_\ell(t)]]}\Bigg\}$$

$\epsilon_1 \in [0,1]$: connect symm

$\epsilon_2 \in [0,1]$: bond value symm

Simplest case: uniform bonds

$P(J') = \delta(J - J')$

$$P(\boldsymbol{\sigma}|\boldsymbol{\theta}) \;=\; p_0(\sigma(0)) \sum_{k\geq 0} \frac{e^{-c}c^k}{k!} \sum_{\boldsymbol{\sigma}'_1,\ldots,\boldsymbol{\sigma}'_k} \prod_{\ell\leq k} \Big[(1-\epsilon_1)P(\boldsymbol{\sigma}'_\ell|\mathbf{0}) + \epsilon_1 P(\boldsymbol{\sigma}'_\ell|J\boldsymbol{\sigma})\Big]$$
$$\times \prod_{t<t_m} \frac{e^{\beta\sigma(t+1)[\theta(t)+J\sum_{\ell\leq k}\sigma'_\ell(t)]}}{2\cosh[\beta[\theta(t)+J\sum_{\ell\leq k}\sigma'_\ell(t)]]}$$

- asymmetric connectivity, $\epsilon_1 = 0$:

  let $W(\boldsymbol{\sigma}) \equiv P(\boldsymbol{\sigma}|\mathbf{0})$

$$W(\boldsymbol{\sigma}) \;=\; p_0(\sigma(0)) \sum_{k\geq 0} \frac{e^{-c}c^k}{k!} \sum_{\boldsymbol{\sigma}'_1,\ldots,\boldsymbol{\sigma}'_k} \prod_{\ell\leq k} W(\boldsymbol{\sigma}'_\ell)$$
$$\times \prod_{t<t_m} \frac{e^{\beta J\sigma(t+1)\sum_{\ell\leq k}\sigma'_\ell(t)}}{2\cosh[\beta J\sum_{\ell\leq k}\sigma'_\ell(t)]}$$

  (→ Derrida, Gardner & Zippelius)

- symmetric connectivity, $\epsilon_1 = 1$:

  let $W(\boldsymbol{\sigma}|\boldsymbol{\tau}) \equiv P(\boldsymbol{\sigma}|J\boldsymbol{\tau})$

$$W(\boldsymbol{\sigma}|\boldsymbol{\tau}) \;=\; p_0(\sigma(0)) \sum_{k\geq 0} \frac{e^{-c}c^k}{k!} \sum_{\boldsymbol{\sigma}'_1,\ldots,\boldsymbol{\sigma}'_k} \prod_{\ell\leq k} W(\boldsymbol{\sigma}'_\ell|\boldsymbol{\sigma})$$
$$\times \prod_{t<t_m} \frac{e^{\beta J\sigma(t+1)[\tau(t)+\sum_{\ell\leq k}\sigma'_\ell(t)]}}{2\cosh[\beta J[\tau(t)+\sum_{\ell\leq k}\sigma'_\ell(t)]]}$$

OUTLOOK

statics

- low temperature solutions and F→SG transitions, beyond variational ansätze ?
- replica symmetry breaking, generalization of Parisi scheme, or something else ?
- 'small world' spin glasses, with RKKY-type bonds
  nearest neighbour: $J_0 > 0$
  random finite conn: J_{ij} zero average Gaussian

dynamics

- solutions of GFA order parameter equations
- exact dynamical replica theory ?
  (single time observables rather than path statistics !)

other

- finite connectivity Gardner calculations,
  (storage capacity of attractor networks)
- finite connectivity quantum systems,
  e.g. $H = -J\sum_{i<j} c_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$ (with Pauli matrices)