workshop programme Thursday morning:
two case studies involving coupled Kuramoto oscillators

9:00 – 10:15

complicated models, new technology

coupled oscillators on complex networks

$$H(\boldsymbol{\phi}) = -J \sum_{i<j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij})$$

10:15 – 10:45

coffee break

10:45 – 12:00

simple models, unexpected problems

coupled oscillators with chiral interactions

$$H(\boldsymbol{\phi}) = -\frac{J}{N} \sum_{i<j} \cos(\phi_i - \phi_j - \alpha)$$

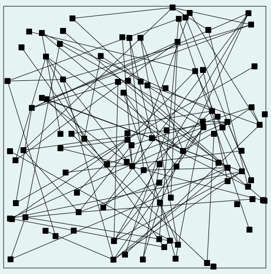
Theory of Coupled Oscillators on Finitely Connected Random Networks

ACC Coolen

with

JPL Hatchett, T Nikoletopoulos, CJ Perez Vicente,

I Perez-Castillo, NS Skantzos, B Wemmenhove



Definitions

Replica calculation of free energy

saddle-point equations
replica-symmetric theory

Phase diagrams

phase transitions
diagrams for different chirality distributions

Theory versus simulations

DEFINITIONS

N Kuramoto-type oscillators,
$\phi_i \in [0, 2\pi]$, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$

$$H(\boldsymbol{\phi}) = -J \sum_{i<j} c_{ij} \cos(\phi_i - \phi_j - \omega_{ij}), \qquad J > 0$$

- *network connectivity: random variables* $c_{ij} \in \{0, 1\}$
  finite connectivity regime: $k_i = \sum_j c_{ij} = \mathcal{O}(N^0)$
  degree distribution: $p_k = N^{-1} \sum_i \delta_{k,k_i}$

  $$P(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0} \quad \text{for all} \;\; i < j$$

  $c = \mathcal{O}(N^0)$

  $p_k = e^{-c}c^k/k!$ *for* $N \to \infty$

- *random interactions between oscillators:* $\omega_{ij} \in [0, 2\pi]$
  independently drawn from $P(\omega)$*, with* $P(-\omega) = P(\omega)$
  e.g.

  $$\omega_{ij} = 0 \quad \rightarrow \quad \text{synchronization of oscillators } (i, j)$$
  $$\omega_{ij} = \pi \quad \rightarrow \quad \text{antisynchronization of oscillators } (i, j)$$

Strategy

- *calculate disorder-averaged free energy per oscillator*

$$\overline{f} = -\lim_{N\to\infty}(\beta N)^{-1}\overline{\log Z}, \qquad Z = \int d\boldsymbol{\phi}\; e^{-\beta H(\boldsymbol{\phi})}$$

$\overline{\cdots}$: *average over* $\{c_{ij}, \omega_{ij}\}$

- *use replica identity*

$$\overline{\log Z} = \lim_{n\to 0} n^{-1}\log \overline{Z^n}$$

- *evaluate* $\overline{Z^n}$ *for integer* n*, in terms of* n *system copies, and evaluate disorder average first:*

$$\overline{Z^n} \;=\; \int d\boldsymbol{\phi}^1 \ldots d\boldsymbol{\phi}^n \overline{e^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\phi}^\alpha)}}$$

- *In result:*
exchange the limits $N\to\infty$ *and* $n\to 0$

$$\begin{aligned}
\overline{f} &= -\lim_{N\to\infty}(\beta N)^{-1}\overline{\log Z} \\
&= -\lim_{N\to\infty}(\beta N)^{-1}\lim_{n\to 0} n^{-1}\log\overline{Z^n} \\
&= -\lim_{N\to\infty}(\beta N)^{-1}\lim_{n\to 0} n^{-1}\log\int d\boldsymbol{\phi}^1 \ldots d\boldsymbol{\phi}^n \overline{e^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\phi}^\alpha)}} \\
&= \lim_{n\to 0} n^{-1}\left\{-\lim_{N\to\infty}(\beta N)^{-1}\log\int d\boldsymbol{\phi}^1 \ldots d\boldsymbol{\phi}^n \overline{e^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\phi}^\alpha)}}\right\}
\end{aligned}$$

REPLICA CALCULATION OF THE DISORDER-AVERAGED FREE ENERGY

Disorder-averaged free energy per oscillator:

$$\overline{f} \;=\; \lim_{n\to 0}\frac{1}{n}\left\{-\lim_{N\to\infty}\frac{1}{\beta N}\log\int d\boldsymbol{\phi}^1\ldots d\boldsymbol{\phi}^n\,\overline{e^{-\beta\sum_{\alpha=1}^{n}H(\boldsymbol{\phi}^{\alpha})}}\right\}$$

Disorder average:

$$\overline{e^{-\beta\sum_{\alpha=1}^{n}H(\boldsymbol{\phi}^{\alpha})}} \;=\; \prod_{i<j}\overline{e^{\beta J\sum_{\alpha=1}^{n}c_{ij}\cos(\phi_i^\alpha-\phi_j^\alpha-\omega_{ij})}}$$

$$= \exp\left\{\frac{c}{2N}\sum_{ij}\left[\int d\omega\;P(\omega)e^{\beta J\sum_{\alpha=1}^{n}\cos(\phi_i^\alpha-\phi_j^\alpha-\omega)}-1\right]+\mathcal{O}(N^0)\right\}$$

replica-order parameter:
$\boldsymbol{\phi}=(\phi_1,\ldots,\phi_n)$, $\boldsymbol{\phi}_i=(\phi_i^1,\ldots,\phi_i^n)$

$$P(\boldsymbol{\phi})=\frac{1}{N}\sum_i\delta[\boldsymbol{\phi}-\boldsymbol{\phi}_i]$$

$$\overline{e^{-\beta\sum_{\alpha=1}^{n}H(\boldsymbol{\phi}^{\alpha})}} \;=$$

$$\exp\left\{\frac{cN}{2}\int d\boldsymbol{\phi}d\boldsymbol{\phi}'P(\boldsymbol{\phi})P(\boldsymbol{\phi}')\left[\int d\omega\;P(\omega)e^{\beta J\sum_\alpha\cos(\phi_\alpha-\phi'_\alpha-\omega)}-1\right]+\mathcal{O}(N^0)\right\}$$

minor technicalities

- *discretize domain* $[0, 2\pi]^n$ *of* $\boldsymbol{\phi}$
- *insert appropriate functional* δ*-distributions*
  to isolate order parameter function $P(\boldsymbol{\phi})$
  integral representations: conjugate functions $\hat{P}(\boldsymbol{\phi})$
- *take continuum limit for domain* $[0, 2\pi]^n$ *of* $\boldsymbol{\phi}$*,*
  gives path integral measure:

$$\prod_{\boldsymbol{\phi}}[dP(\boldsymbol{\phi})d\hat{P}(\boldsymbol{\phi})/2\pi] = \{dPd\hat{P}\}$$

$$\begin{aligned}\overline{f} \;\; = \;\; &-\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{\beta N n}\log\int\{dPd\hat{P}\}\; e^{iN\int d\boldsymbol{\phi}\;\, P(\boldsymbol{\phi})\hat{P}(\boldsymbol{\phi})+N\log\int d\boldsymbol{\phi}\;\, e^{-i\hat{P}(\boldsymbol{\phi})}} \\ &\times\exp\left\{\frac{cN}{2}\int d\boldsymbol{\phi}d\boldsymbol{\phi}' P(\boldsymbol{\phi})P(\boldsymbol{\phi}')[\int d\omega\;\, P(\omega)e^{\beta J\sum_\alpha\cos(\phi_\alpha-\phi'_\alpha-\omega)}-1]\right\}\end{aligned}$$

$N \to \infty$*:*
saddle-point integration

$$\begin{aligned}\overline{f} \;\; = \;\; &-\lim_{n\to 0}\frac{1}{\beta n}\text{extr}_{\{P,\hat{P}\}}\left\{i\int d\boldsymbol{\phi}\;\, P(\boldsymbol{\phi})\hat{P}(\boldsymbol{\phi})+\log\int d\boldsymbol{\phi}\;\, e^{-i\hat{P}(\boldsymbol{\phi})}\right. \\ &\left.+\frac{1}{2}c\int d\boldsymbol{\phi}d\boldsymbol{\phi}' P(\boldsymbol{\phi})P(\boldsymbol{\phi}')[\int d\omega\;\, P(\omega)e^{\beta J\sum_\alpha\cos(\phi_\alpha-\phi'_\alpha-\omega)}-1]\right\}\end{aligned}$$

saddle-points equations: functional variation with respect to $\hat{P}(\boldsymbol{\phi})$ and $P(\boldsymbol{\phi})$

$$P(\boldsymbol{\phi}) = \frac{e^{-i\hat{P}(\boldsymbol{\phi})}}{\int d\boldsymbol{\phi}' e^{-i\hat{P}(\boldsymbol{\phi}')}}$$

$$\hat{P}(\boldsymbol{\phi}) = ic \int d\boldsymbol{\phi}' \; P(\boldsymbol{\phi}')[\int d\omega \; P(\omega) e^{\beta J \sum_\alpha \cos(\phi_\alpha - \phi'_\alpha - \omega)} - 1]$$

eliminate $\hat{P}$:

$$P(\boldsymbol{\phi}) = \frac{e^{c\int d\boldsymbol{\phi}' P(\boldsymbol{\phi}')[\int d\omega \; P(\omega) e^{\beta J \sum_\alpha \cos(\phi_\alpha - \phi'_\alpha - \omega)} - 1]}}{\int d\boldsymbol{\phi}' e^{c\int d\boldsymbol{\phi}'' \; P(\boldsymbol{\phi}'')[\int d\omega \; P(\omega) e^{\beta J \sum_\alpha \cos(\phi'_\alpha - \phi''_\alpha - \omega)} - 1]}}$$

$$\begin{aligned} \overline{f} \; &= \\ &\lim_{n\to 0} \frac{1}{\beta n} \left\{ \frac{1}{2} c \int d\boldsymbol{\phi} d\boldsymbol{\phi}' P(\boldsymbol{\phi}) P(\boldsymbol{\phi}')[\int d\omega P(\omega) e^{\beta J \sum_\alpha \cos(\phi_\alpha - \phi'_\alpha - \omega)} - 1] \right. \\ &\qquad \left. - \log \int d\boldsymbol{\phi} \; e^{c\int d\boldsymbol{\phi}' P(\boldsymbol{\phi}')[\int d\omega P(\omega) e^{\beta J \sum_\alpha \cos(\phi_\alpha - \phi'_\alpha - \omega)} - 1]} \right\} \end{aligned}$$

Replica symmetric theory

$$P(\phi_1,\ldots,\phi_n)=\frac{1}{N}\sum_i\prod_{\alpha=1}^{n}\delta[\phi_\alpha-\phi_i^\alpha]$$

next:
limit $n\to 0$ in saddle-point eqns
RS ansatz for continuous variables?

- *let $P[\phi|\boldsymbol{\mu}]$ denote a complete parametrized family of functions on $[0,2\pi]$*
  $\boldsymbol{\mu}=(\mu_0,\mu_1,\mu_2,\ldots)$
  $$P_{\rm RS}(\phi_1,\ldots,\phi_n)=\int d\boldsymbol{\mu}\; w(\boldsymbol{\mu})\prod_\alpha P[\phi_\alpha|\boldsymbol{\mu}],\qquad \int d\boldsymbol{\mu}\; w(\boldsymbol{\mu})=1$$
- *representation-independent formulation:*
  $$P_{\rm RS}(\phi_1,\ldots,\phi_n)=\int\{dP\}\; W[\{P\}]\;\prod_\alpha P(\phi_\alpha)$$

RS order parameter:

$$\text{functional measure}\;\; W[\{P\}]$$

physical interpretation:

$$\int\{dP\}\; W[\{P\}]\prod_\alpha\left[\int d\phi\; P(\phi)f_\alpha(\phi)\right]=\lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\prod_\alpha\langle f_\alpha(\phi_i)\rangle}$$

insert RS ansatz:

$$
\begin{aligned}
W[\{P\}] \;&=\; \sum_{k\geq 0} p_k \int \prod_{\ell\leq k} [\{dP_\ell\} W[\{P_\ell\}] d\omega_\ell P(\omega_\ell)] \\
&\quad \times \prod_{\phi\in[0,2\pi]} \delta\left[P(\phi) - \frac{\prod_{\ell=1}^{k}\int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi-\phi'-\omega_\ell)}}{\int d\phi'' \prod_{\ell=1}^{k}\int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi''-\phi'-\omega_\ell)}}\right]
\end{aligned}
$$

$p_k = e^{-c}c^k/k!$

free energy per oscillator:

$$
\begin{aligned}
\overline{f}_{\rm RS} \;&=\; \frac{c}{2\beta}\int \{dP_1 dP_2\}\; W[\{P_1\}]W[\{P_2\}] \int d\omega P(\omega) \log \int d\phi d\phi' P_1(\phi) P_2(\phi') e^{\beta J\cos(\phi-\phi'-\omega)} \\
&\quad -\frac{1}{\beta}\sum_{k\geq 0} p_k \int \prod_{\ell=1}^{k} [\{dP_\ell\} W[\{P_\ell\}] d\omega_\ell P(\omega_\ell)] \log \int d\phi \prod_{\ell=1}^{k} \left[\int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi-\phi'-\omega_\ell)}\right]
\end{aligned}
$$

PHASE DIAGRAMS

bifurcation analysis
of order parameter eqn:

$$W[\{P\}] \;=\; \sum_{k\geq 0} p_k \int \prod_{\ell\leq k} [\{dP_\ell\} W[\{P_\ell\}] d\omega_\ell P(\omega_\ell)]$$
$$\times \prod_{\phi\in[0,2\pi]} \delta\left[P(\phi) - \frac{\prod_{\ell=1}^{k}\int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi-\phi'-\omega_\ell)}}{\int d\phi'' \prod_{\ell=1}^{k}\int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi''-\phi'-\omega_\ell)}}\right]$$

$\beta = 0$:

$$\text{paramagnetic state}: \quad W[\{P\}] = \prod_{\phi\in[0,2\pi]} \delta\left[P(\phi) - \frac{1}{2\pi}\right]$$

Phase transitions

Continuous bifurcations away from paramagnetic state
located by Guzai (i.e. functional moment) expansion

- *transform:*

$$P(\phi) \to \frac{1}{2\pi} + \Delta(\phi), \qquad W[\{P\}] \to \tilde{W}[\{\Delta\}]$$

constraint:

$$\tilde{W}[\{\Delta\}] = 0 \quad \text{if} \quad \int_0^{2\pi} d\phi\ \Delta(\phi) \neq 0$$

- *expand saddle-point eqns*
  in functional moments

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \ldots \Delta(\phi_r)$$

- *assume: close to continuous bifurcation*
  $\exists \epsilon \ll 1$ *such that*

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \ldots \Delta(\phi_r) = \mathcal{O}(\epsilon^r)$$

Lowest order bifurcation ϵ^1

$$\Psi(\phi) = \frac{c}{2\pi I_0(\beta J)} \int_0^{2\pi} d\phi' \int d\omega \ P(\omega) e^{\beta J \cos(\phi-\phi'-\omega)} \Psi(\phi') \qquad \int_0^{2\pi} d\phi \ \Psi(\phi) = 0$$

$c = \sum_k p_k k$

$I_k(z)$: *modified Bessel functions*

- *solution: Fourier modes* $\Psi(\phi) = e^{ik\phi}$
  transition:
  $$c = \min_{k>0} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \ P(\omega) \cos(k\omega) \right\}^{-1}$$
- *bifurcating state:*
  $$\lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \delta[\phi - \phi_i] \rangle} = \frac{1}{2\pi} \left[1 + \epsilon \cos(k\phi - \lambda) + \ldots \right]$$
  $$\lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \begin{pmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{pmatrix} \rangle} = \frac{1}{2}\epsilon \ \delta_{k1} \begin{pmatrix} \cos(\lambda) \\ \sin(\lambda) \end{pmatrix} + \ldots$$

- $k = 1$: *global synchronization*

  $k > 1$: *no global synchronization*

$$\text{P} \rightarrow \text{F}: \qquad c = \left\{ \frac{I_1(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(\omega) \right\}^{-1}$$

$$\text{KT}: \qquad c = \min_{k>1} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(k\omega) \right\}^{-1}$$

Lowest order bifurcation ϵ^2

$$\Psi(\phi_1,\phi_2) \;=\; c\int \frac{d\phi_1' d\phi_2'}{[2\pi I_0(\beta J)]^2}\left[\int d\omega\; P(\omega) e^{\beta J\cos(\phi_1-\phi_1'-\omega)+\beta J\cos(\phi_2-\phi_2'-\omega)}\right]\Psi(\phi_1',\phi_2')$$

$$\int d\phi_1\;\Psi(\phi_1,\phi_2)=\int d\phi_2\;\Psi(\phi_1,\phi_2)=0$$

- *solution: Fourier modes* $\Psi(\phi_1,\phi_2)=e^{i(k_1\phi_1+k_2\phi_2)}$
  transition:

$$c=\min_{k_1\neq 0,\; k_2\neq 0}\left\{\frac{I_{k_1}(\beta J)I_{k_2}(\beta J)}{I_0^2(\beta J)}\int d\omega\; P(\omega)\cos[(k_1+k_2)\omega]\right\}^{-1}$$

  min: $k_1=-k_2=1$

$$\mathrm{P}\to\mathrm{SG}: \qquad c=I_0^2(\beta J)/I_1^2(\beta J)$$

- *bifurcating state:*

$$\lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\delta[\phi-\phi_i]\rangle}=\frac{1}{2\pi}$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\delta[\phi-\phi_i]\rangle\langle\delta[\phi'-\phi_i]\rangle}=\frac{1}{4\pi^2}\left[1+\epsilon\cos(\phi-\phi'+\psi)+\ldots\right]$$

- *no global synchronization, yet*

$$\lim_{N\to\infty} \frac{1}{N} \sum_i \left[\overline{\langle \cos(\phi_i) \rangle^2} + \overline{\langle \sin(\phi_i) \rangle^2} \right] > 0$$

- *P→SG bifurcation precedes P→KT*
  physical transitions away from P: $P \to F$, $P \to SG$

Phase diagrams

phases:

P: *paramagnetic state, no freezing of oscillator phases*

F: *synchronized state, coherent oscillations*

SG: *spin-glass state, frozen relative phases but incoherent*

phase transitions:

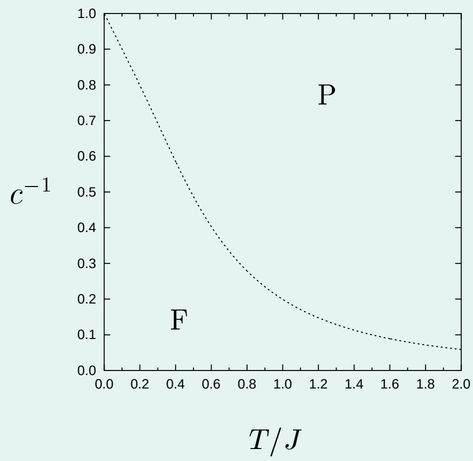
P → F : $c^{-1} = [I_1(\beta J)/I_0(\beta J)] \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(\omega)$

P → SG : $c^{-1} = [I_1(\beta J)/I_0(\beta J)]^2$

F → SG : cannot yet calculate ...
Parisi − Toulouse hypothesis ?

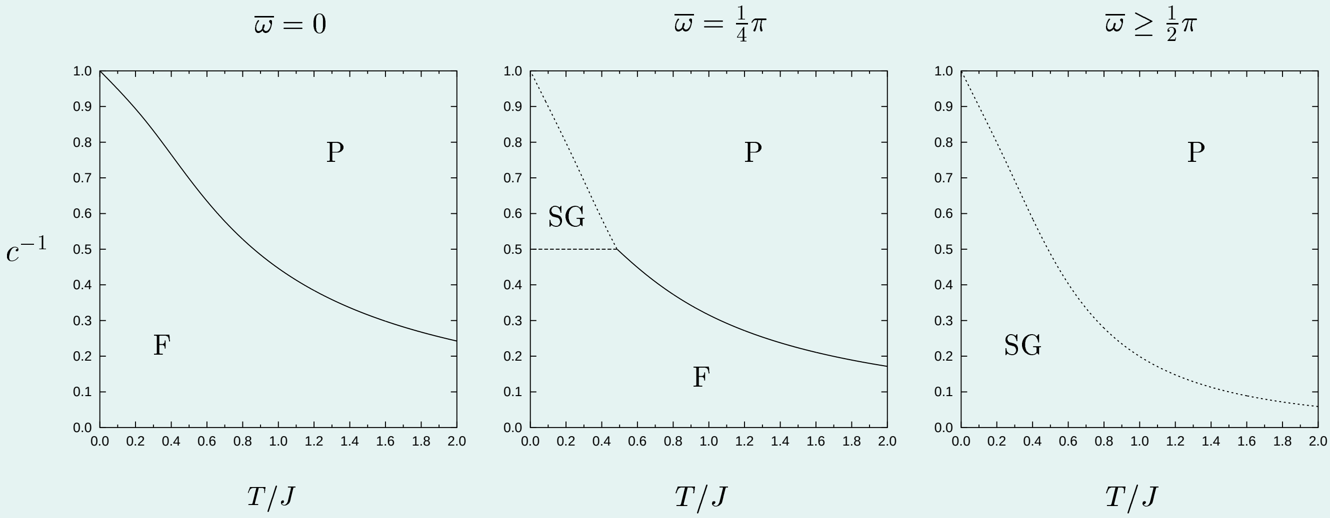
Example:

$P(\omega) = 1/2\pi$



Example:

$$P(\omega) = \frac{1}{2}\delta(\omega - \overline{\omega}) + \frac{1}{2}\delta(\omega + \overline{\omega})$$



THEORY VERSUS SIMULATIONS

Numerical solution of order parameter equations
(via truncated parametrizations) versus simulations

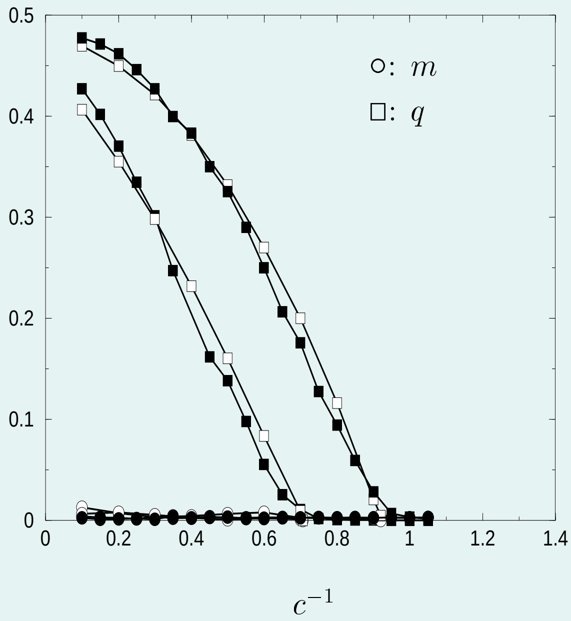
$$m^2 = \left[\frac{1}{N}\sum_i \overline{\langle\cos(\phi_i)\rangle}\right]^2 + \left[\frac{1}{N}\sum_i \overline{\langle\sin(\phi_i)\rangle}\right]^2$$

$$q = \frac{1}{2N}\sum_i \left[\overline{\langle\cos(\phi_i)\rangle^2} + \overline{\langle\sin(\phi_i)\rangle^2}\right]$$

$$\begin{aligned} \text{P} &: \quad q = m = 0 \\ \text{F} &: \quad q > 0,\ m > 0 \\ \text{SG} &: \quad q > 0,\ m = 0 \end{aligned}$$

Example:

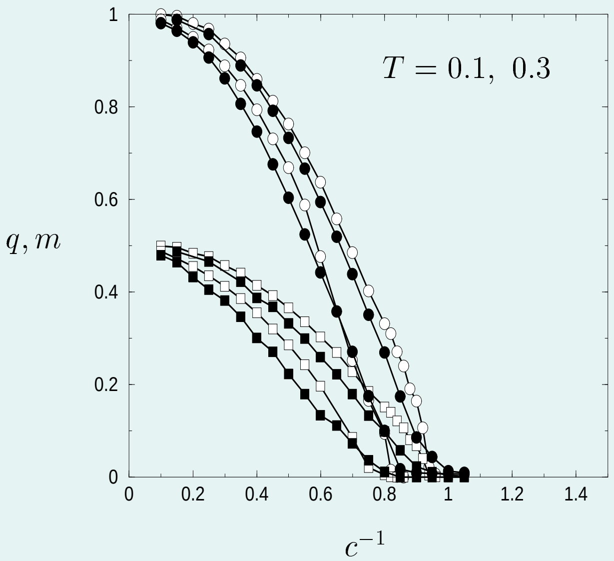
$P(\omega) = 1/2\pi$

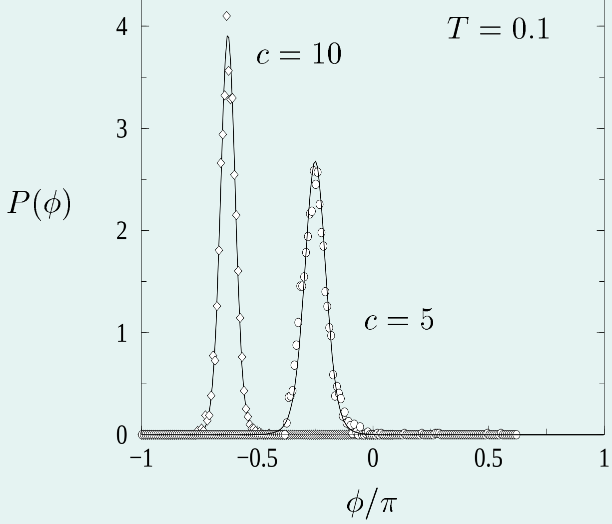


upper: $T = 0.1$

lower: $T = 0.3$

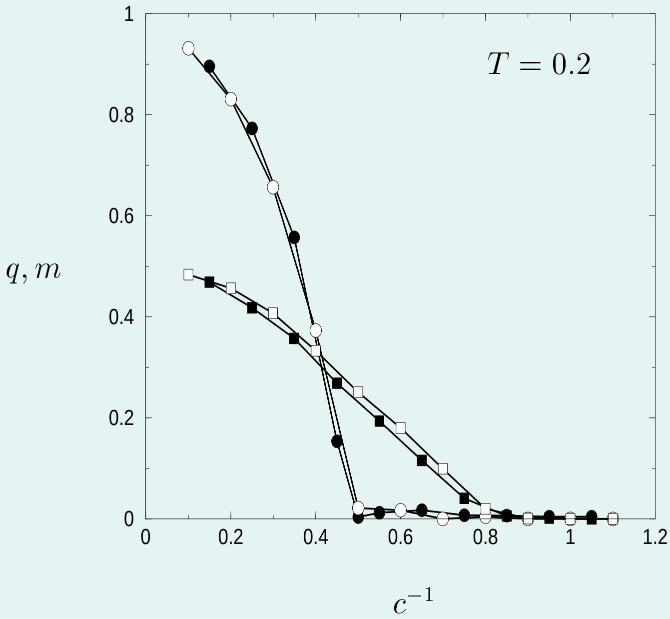
Example: $P(\omega) = \delta(\omega)$



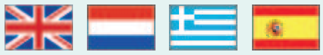


Example:

$P(\omega) = \frac{1}{2}\delta(\omega - \frac{\pi}{4}) + \frac{1}{2}\delta(\omega + \frac{\pi}{4})$



confirms Parisi-Toulouse
for F→SG transition

The GUZAI Initiative (2003–present) 

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statics

- *Finite connectivity attractor neural networks. J. Phys. A 36 (2003) 9617*
- *The Little-Hopfield model on a sparse random graph. J. Phys. A 37 (2004) 9087*
- *Analytic solution of attractor neural networks on scale-free graphs. J. Phys. A 37 (2004) 8789*
- *Replicated transfer matrix analysis of Ising spin models on 'small world' lattices. J. Phys. A 37 (2004) 6455*
- *Finitely connected vector spin systems with random matrix interactions. preprint* `cond-mat/0504690`

dynamics

- *Parallel dynamics of disordered Ising spin systems on finitely connected random graphs. J. Phys. A 37 (2004) 6201*
- *Dynamical replica analysis of disordered Ising spin systems on finitely connected random graphs. preprint* `cond-mat/0504313`

systems with evolving bonds

- *Slowly evolving connectivity in recurrent neural networks I: the extreme dilution regime. J. Phys. A 37 (2004) 7653*
- *Slowly evolving random graphs II: adaptive geometry in finite connectivity Hopfield models. J. Phys. A 37 (2004) 7843*