Coupled Oscillators with Chiral Interactions

ACC Coolen

with

A Tucker, P Sollich, HC Rae

Chiral Kuramoto model

The problem: chiral versus non-chiral

Nonlinear PDE for the free energy

Variational approach

Phase diagram

Some open questions

CHIRAL KURAMOTO MODEL

N phases

$\phi_i \in [0, 2\pi], \quad \boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$

$$H(\boldsymbol{\phi}) = -\frac{J}{N}\sum_{i<j}\cos(\phi_i - \phi_j - \alpha), \qquad J > 0$$

- *oscillators aim for:* $\phi_j = \phi_i - \alpha$ *for all* $i < j$
- *rewrite* H*?*

$$\begin{aligned} H(\boldsymbol{\phi}) &= -\frac{J\cos(\alpha)}{N}\sum_{i<j}\cos(\phi_i - \phi_j) - \frac{J\sin(\alpha)}{N}\sum_{i<j}\sin(\phi_i - \phi_j) \\ &= -\frac{J\cos(\alpha)}{2N}\sum_{i\neq j}\cos(\phi_i - \phi_j) - \frac{J\sin(\alpha)}{N}\sum_{i<j}\sin(\phi_i - \phi_j) \end{aligned}$$

Generalized version

External synchronizing forces,
promoting $\phi_i = \psi$ *for all* i

$$H(\boldsymbol{\phi}) = -\frac{J}{N}\sum_{i<j}\cos(\phi_i - \phi_j - \alpha) - K\sum_i \cos(\phi_i - \psi)$$

partition function Z_N,
free energy per oscillator f_N

$$Z_N = \int d\boldsymbol{\phi}\; e^{-\beta H(\boldsymbol{\phi})} \qquad f_N = -(\beta N)^{-1}\log Z_N$$

CHIRAL VERSUS NON-CHIRAL

notation:
(modified Bessel functions)

$$I_n(z) = (2\pi)^{-1} \int_{-\pi}^{\pi} d\phi \ \cos(n\phi) e^{z\cos(\phi)}$$

Decoupled oscillators: $J = 0$

$$H(\boldsymbol{\phi}) = -K \sum_i \cos(\phi_i - \psi)$$

Here:

$$\begin{aligned} f_N &= -\frac{1}{\beta N} \log \int d\boldsymbol{\phi} \ e^{-\beta H(\boldsymbol{\phi})} \\ &= -\frac{1}{\beta N} \log \int d\boldsymbol{\phi} \ e^{\beta K \sum_i \cos(\phi_i - \psi)} \\ &= -\frac{1}{\beta N} \log \left[\int d\phi \ e^{\beta K \cos(\phi - \psi)} \right]^N \\ &= -\frac{1}{\beta} \log \int d\phi \ e^{\beta K \cos(\phi - \psi)} = -\frac{1}{\beta} \log[2\pi I_0(\beta K)] \end{aligned}$$

- *no collective processes, no singularities in f, no phase transitions ...*

Non-chiral Kuramoto model: $\alpha = 0$

$$H(\boldsymbol{\phi}) = -\frac{J}{N}\sum_{i<j}\cos(\phi_i - \phi_j) - K\sum_i \cos(\phi_i - \psi)$$

solved in five steps:

(i) *symmetrize Hamiltonian:*

$$\begin{aligned} Z_N &= \int d\boldsymbol{\phi}\; e^{\frac{\beta J}{N}\sum_{i<j}\cos(\phi_i-\phi_j)+\beta K\sum_i\cos(\phi_i-\psi)} \\ &= \int d\boldsymbol{\phi}\; e^{\frac{\beta J}{2N}\sum_{ij}\cos(\phi_i-\phi_j)-\frac{1}{2}\beta J+\beta K\sum_i\cos(\phi_i-\psi)} \\ &= e^{-\frac{1}{2}\beta J}\int d\boldsymbol{\phi}\; e^{\frac{\beta J}{2N}[\sum_i\cos(\phi_i)]^2+\frac{1}{2N}\beta J[\sum_i\sin(\phi_i)]^2+\beta K\sum_i\cos(\phi_i-\psi)} \end{aligned}$$

(ii) *Gaussian linearization:*

$$\begin{aligned} Z_N &= e^{-\frac{1}{2}\beta J}\int d\boldsymbol{\phi}\; e^{\beta K\sum_i\cos(\phi_i-\psi)}\int\frac{dx}{\sqrt{2\pi}}\; e^{-\frac{1}{2}x^2+x\sqrt{\beta J/N}\sum_i\cos(\phi_i)} \\ &\qquad\times\int\frac{dy}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2+y\sqrt{\beta J/N}\sum_i\sin(\phi_i)} \end{aligned}$$

$$
\begin{aligned}
Z_N &= e^{-\frac{1}{2}\beta J}\int\frac{dxdy}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \\
&\qquad\times\left\{\int d\phi\ e^{x\sqrt{\beta J/N}\cos(\phi)+y\sqrt{\beta J/N}\sin(\phi)+\beta K\cos(\phi-\psi)}\right\}^N \\
&= e^{-\frac{1}{2}\beta J}\int\frac{dxdy}{2\pi}e^{-\frac{1}{2}(x^2+y^2)} \\
&\qquad\times\left\{\int d\phi\ e^{\cos(\phi)\sqrt{[x\sqrt{\beta J/N}+\beta K\cos(\psi)]^2+[y\sqrt{\beta J/N}+\beta K\sin(\psi)]^2}}\right\}^N \\
&= \beta JN(2\pi)^N e^{-\frac{1}{2}\beta J}\int\frac{dxdy}{2\pi}e^{-\frac{1}{2}\beta JN(x^2+y^2)} \\
&\qquad\times\left\{I_0\left(\beta\sqrt{[Jx+K\cos(\psi)]^2+[Jy+K\sin(\psi)]^2}\right)\right\}^N
\end{aligned}
$$

(iv) switch to polar coordinates:
$(x,y)=r(\cos\theta,\sin\theta)$

$$
\begin{aligned}
Z_N &= \beta JN(2\pi)^N e^{-\frac{1}{2}\beta J}\int_0^\infty dr\ re^{-\frac{1}{2}\beta JNr^2}\int\frac{d\theta}{2\pi} \\
&\qquad\times\left\{I_0\left(\beta\sqrt{[Jr\cos(\theta)+K\cos(\psi)]^2+[Jr\sin(\theta)+K\sin(\psi)]^2}\right)\right\}^N \\
&= \beta JN(2\pi)^N e^{-\frac{1}{2}\beta J}\int_0^\infty dr\ re^{-\frac{1}{2}\beta JNr^2}\int\frac{d\theta}{2\pi} \\
&\qquad\times\left\{I_0\left(\beta\sqrt{J^2r^2+K^2+2JKr\cos(\theta-\psi)}\right)\right\}^N
\end{aligned}
$$

$$Z_N = \int dr d\theta \; e^{N\left\{\log\left[2\pi I_0\left(\beta\sqrt{J^2r^2+K^2+2JKr\cos(\theta)}\right)\right]-\frac{1}{2}\beta Jr^2\right\}+\mathcal{O}(\log N)}$$

(v) limit $N \to \infty$ by steepest descent:
(Laplace)

$$\begin{aligned}
f &= \lim_{N\to\infty} f_N = -\lim_{N\to\infty}\frac{1}{\beta N}\log Z_N \\
&= -\frac{1}{\beta}\max_{r,\theta}\left\{\log\left[2\pi I_0\left(\beta\sqrt{J^2r^2+K^2+2JKr\cos(\theta)}\right)\right]-\frac{1}{2}\beta Jr^2\right\} \\
&= \min_{q,\theta}\left\{\frac{1}{2}Jq^2-\frac{1}{\beta}\log\left[2\pi I_0\left(\beta\sqrt{J^2q^2+K^2+2JKq\cos(\theta)}\right)\right]\right\} \\
&= \min_{q}\left\{\frac{1}{2}Jq^2-\frac{1}{\beta}\log\left[2\pi I_0\left(\beta\sqrt{J^2q^2+K^2+2JKq}\right)\right]\right\} \\
&= \min_{q}\left\{\frac{1}{2}Jq^2-\frac{1}{\beta}\log\left[2\pi I_0[\beta(K+Jq)]\right]\right\}
\end{aligned}$$

summary of solution for non-chiral Kuramoto model

$$f = \min_{q \geq 0} f(q)$$

$$f(q) = \frac{1}{2}Jq^2 - \frac{1}{\beta}\log\left[2\pi I_0[\beta(K+Jq)]\right]$$

- *$f'(q) = 0$:*

$$q = \frac{I_1[\beta(K+Jq)]}{I_0[\beta(K+Jq)]}$$

- *physical meaning:*

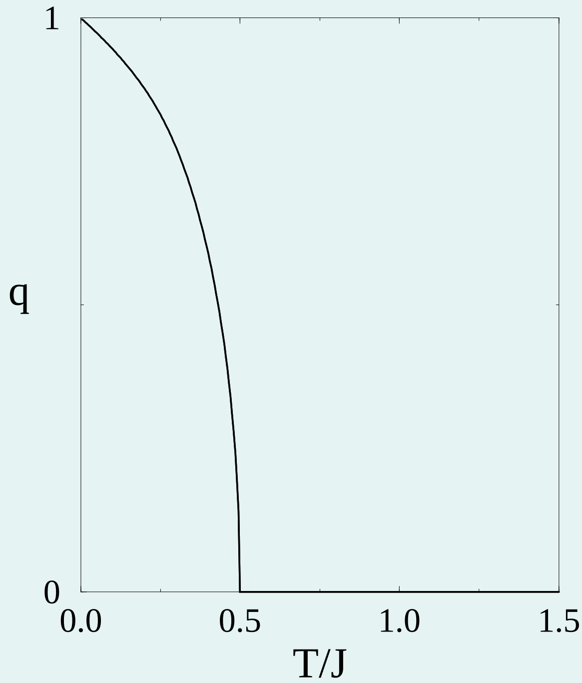
$$q^2 = \lim_{N\to\infty}\left[\langle[N^{-1}\sum_i \cos(\phi_i)]^2\rangle + \langle[N^{-1}\sum_i \sin(\phi_i)]^2\rangle\right]$$

$K > 0$: no phase transition

$K = 0$: transition at $\beta J = 2$

$T/J > \frac{1}{2}$: no synchronization

$T/J < \frac{1}{2}$: synchronization



EQN FOR FREE ENERGY PER OSCILLATOR

Effect of adding one oscillator

N oscillators:

$$Z_N[K,J,\psi] = \int d\boldsymbol{\phi}\ e^{\frac{\beta J}{N}\sum_{i<j}^{N}\cos(\phi_i-\phi_j-\alpha)+\beta K\sum_{i=1}^{N}\cos(\phi_i-\psi)}$$

$N+1$ oscillators:

$$Z_{N+1}[K,J,\psi] = \int d\phi_{N+1}\int d\boldsymbol{\phi}\ e^{\frac{\beta J}{N+1}\sum_{i<j}^{N}\cos(\phi_i-\phi_j-\alpha)+\beta K\sum_{i=1}^{N+1}\cos(\phi_i-\psi)} \times\ e^{\frac{\beta J}{N+1}\sum_{i=1}^{N}\cos(\phi_i-\phi_{N+1}-\alpha)}$$

$$= \int d\phi\ e^{\beta K\cos(\phi-\psi)}\ \times\ Z_N\left[\sqrt{K^2+[\frac{J}{N+1}]^2+\frac{2KJ}{N+1}\cos(\psi-\phi-\alpha)},\frac{JN}{N+1},\arctan[\ldots]\right]$$

$$= \int d\phi\ e^{\beta K\cos(\phi)}\ \times\ Z_N\left[\sqrt{K^2+\frac{J^2}{(N+1)^2}+\frac{2KJ}{N+1}\cos(\phi+\alpha)},\frac{JN}{N+1},\arctan[\ldots]\right]$$

note:
Z_N *indep of* ψ*, so:*

$$Z_{N+1}[K,J] = \int d\phi\; e^{\beta K\cos(\phi)} \times Z_N\left[\sqrt{K^2+\frac{J^2}{(N+1)^2}+\frac{2KJ}{N+1}\cos(\phi+\alpha)},\frac{NJ}{N+1}\right]$$

in terms of

$$f_N[K,J] = -(\beta N)^{-1}\log Z_N[K,J]$$

one finds

$$f_{N+1}[K,J] \;=\; -\frac{1}{\beta(N+1)}\log\left\{\int d\phi\; e^{\beta K\cos(\phi)} \times e^{-\beta N f_N[\sqrt{K^2+J^2/(N+1)^2+2KJ\cos(\phi+\alpha)/(N+1)},JN/(N+1)]}\right\}$$

Large N

$$f_N[K\sqrt{1+2J\cos(\phi+\alpha)/KN+\mathcal{O}(N^{-2})},\frac{JN}{N+1}] = f_N[K,J]+\frac{J}{N}\left\{\cos(\phi+\alpha)\frac{\partial}{\partial K}f_N[K,J]-\frac{\partial}{\partial J}f_N[K,J]\right\}+\mathcal{O}(\frac{1}{N^2})$$

Insert:

$$f_{N+1}[K,J] \;=\; \frac{N}{(N+1)} f_N[K,J] - \frac{1}{\beta(N+1)} \log \int d\phi\; e^{\beta K \cos(\phi) - \beta J \left[\cos(\phi+\alpha) \frac{\partial}{\partial K} f_N[K,J] - \frac{\partial}{\partial J} f_N[K,J]\right] + ..}$$

$F_N = N f_N$:

$$F_{N+1}[K,J] = F_N[K,J] - \beta^{-1} \log \int d\phi\; e^{\beta K \cos(\phi) - \beta J \left[\cos(\phi+\alpha) \frac{\partial}{\partial K} f_N[K,J] - \frac{\partial}{\partial J} f_N[K,J]\right]} + \mathcal{O}(\frac{1}{N})$$

iterate from N up to $2N$:

$$F_{2N}[K,J] - F_N[K,J] \;=\; -\frac{N}{\beta} \log \int d\phi\; e^{\beta K \cos(\phi) - \beta J \left[\cos(\phi+\alpha) \frac{\partial}{\partial K} f_N[K,J] - \frac{\partial}{\partial J} f_N[K,J]\right]} + \mathcal{O}(N^0)$$

$$2 f_{2N}[K,J] - f_N[K,J] \;=\; -\frac{1}{\beta} \log \int d\phi\; e^{\beta K \cos(\phi) - \beta J \left[\cos(\phi+\alpha) \frac{\partial}{\partial K} f_N[K,J] - \frac{\partial}{\partial J} f_N[K,J]\right]} + \mathcal{O}(N^{-1})$$

$f = \lim_{N\to\infty} f_N[K, J]$:

$$
\begin{aligned}
f &= -\frac{1}{\beta}\log\int d\phi\ e^{\beta K\cos(\phi)-\beta J\left[\cos(\phi+\alpha)\frac{\partial f}{\partial K}-\frac{\partial f}{\partial J}\right]} \\
&= -J\frac{\partial f}{\partial J} - \frac{1}{\beta}\log\int d\phi\ e^{\beta K\cos(\phi)-\beta J\cos(\phi+\alpha)\frac{\partial f}{\partial K}} \\
&= -J\frac{\partial f}{\partial J} - \frac{1}{\beta}\log\int d\phi\ e^{\beta\cos(\phi)\sqrt{K^2+J^2(\partial f/\partial K)^2-2KJ\cos(\alpha)(\partial f/\partial K)}}
\end{aligned}
$$

Final result

$$
f = -J\frac{\partial f}{\partial J} - \frac{1}{\beta}\log\left\{2\pi I_0\left[\beta\sqrt{K^2+J^2(\frac{\partial f}{\partial K})^2-2KJ\cos(\alpha)\frac{\partial f}{\partial K}}\right]\right\}
$$

exact for any α*!*

special case:

non-chiral Kuramoto model

$\alpha = 0$

$$f = -J\frac{\partial f}{\partial J} - \frac{1}{\beta}\log\left\{2\pi I_0[\beta(K - J\frac{\partial f}{\partial K})]\right\}$$

solved by

$$f[K,J] = \min_{q\geq 0}\left\{\frac{1}{2}Jq^2 - \frac{1}{\beta}\log\left[2\pi I_0[\beta(K+Jq)]\right]\right\}$$

VARIATIONAL APPROACH

Strategy

for any trial Hamiltonian $\tilde{H}$,
with equilibrium averages $\langle \ldots \rangle_0$
and free energy $\tilde{f}$

$$f \;\leq\; \tilde{f} + N^{-1}\langle H(\boldsymbol{\phi}) - \tilde{H}(\boldsymbol{\phi})\rangle_0$$

variational approximation:

$$f^\star = \lim_{N\to\infty} \min_{\tilde{H}} \{\tilde{f} + N^{-1}\langle H(\boldsymbol{\phi}) - \tilde{H}(\boldsymbol{\phi})\rangle_0\}$$

Simple trial Hamiltonian

$$\tilde{H}(\boldsymbol{\phi}) = \sum_i V(\phi_i) : \qquad f^\star = \lim_{N\to\infty} \min_{\{V_i\}} \{\tilde{f} + N^{-1}\langle H(\boldsymbol{\phi}) - \tilde{H}(\boldsymbol{\phi})\rangle_0\}$$

- *minimum:*

$$V_i(\phi) = -\lambda_i \cos(\phi - \omega_i)$$

- *large N:*

$$\lambda_i \to \lambda(x), \quad \omega_i \to \omega(x), \quad x = i/N$$

- *solve eqns:*

$$\lambda(x) = \lambda, \quad \omega(x) = \omega_0 - 2\Omega x$$

resulting variational approximation

$$f^\star[K,J] \;=\; \min_{\{\lambda,\omega_0,\Omega\}}\left\{-\frac{1}{\beta}\log[2\pi I_0(\beta\lambda)]+\lambda\frac{I_1(\beta\lambda)}{I_0(\beta\lambda)}+\frac{K}{2\Omega}[\sin(\omega_0-2\Omega)-\sin(\omega_0)]\right.$$
$$\left.+\frac{J}{2\Omega^2}\frac{I_1^2(\beta\lambda)}{I_0^2(\beta\lambda)}[\sin(\Omega)\sin(\alpha-\Omega)-\Omega\sin(\alpha)]\right\}$$

observation:

$f^\star[K,J]$ *solves the exact eqn*

$$f=-J\frac{\partial f}{\partial J}-\frac{1}{\beta}\log\left\{2\pi I_0\left[\beta\sqrt{K^2+J^2(\frac{\partial f}{\partial K})^2-2KJ\cos(\alpha)\frac{\partial f}{\partial K}}\right]\right\}$$

hence $f[K,J]=f^\star[K,J]$

solution found!

PHASE DIAGRAM

order parameter eqns

$$\tan(\Omega) = \tan(\omega_0), \qquad \sin(\alpha - \Omega) = \frac{K \sin(\alpha)}{\lambda}, \qquad \Omega = \frac{J \sin(\alpha)}{\lambda} \frac{I_1(\beta\lambda)}{I_0(\beta\lambda)}$$

minimum for $K = 0$*:*

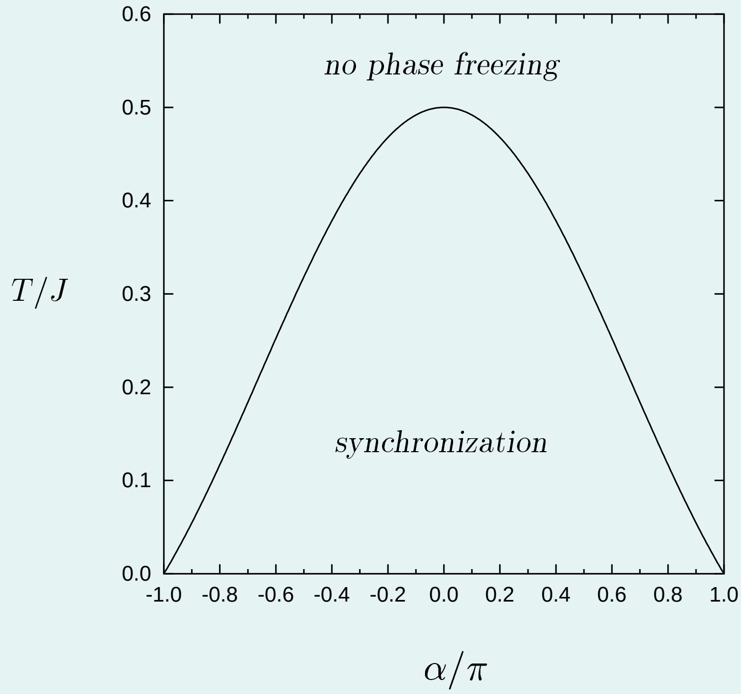
$$\frac{I_1(\beta\lambda)}{I_0(\beta\lambda)} = \frac{\alpha\lambda}{J \sin(\alpha)}$$

bifurcation of $\lambda \neq 0$*:*
(synchronization transition)

$$T_c/J = \frac{\sin(\alpha)}{2\alpha}$$

phase diagram of
chiral Kuramoto model:

$$H = -\frac{J}{N} \sum_{i<j} \cos(\phi_i - \phi_j - \alpha)$$



Some open questions

- *Are models of coupled oscillators with chiral interactions relevant?*
- *Is there a simpler (more direct) method for solving them?*
- *Does the present method also work for more complicated cases?*
  (derive non-linear PDE for the free energy, find variational solution, show that it solves the PDE)
- *Are saddle-point solutions with higher winding numbers physically relevant?*
  (local minima perhaps, similar to e.g. 'butterfly' solutions in Kohonen maps)
- *Dynamics?*