Applications of Generating Functional Analysis to Minority Game Models

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- **Past**: brief intro to Minority Games
- **Present**: theory of Inner Product MGs

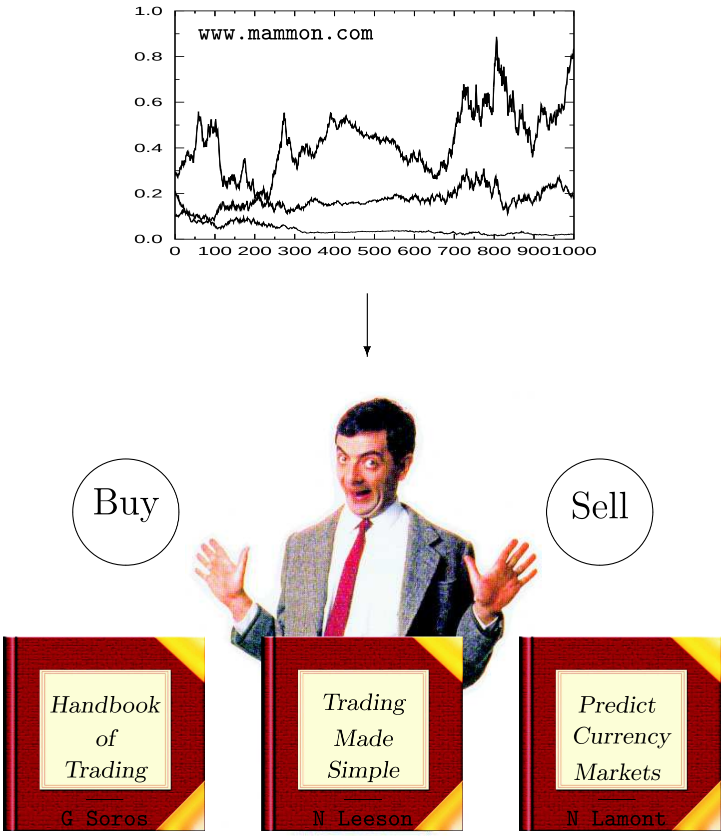
  GFA for generalized MGs, real histories
  history covariance eigenvalue spectrum
  inner product MGs: universality, criticality ...
- **Future**: sharpen tools and questions

  methods: solve GFA eqns for $t < \infty$ and $\alpha < \alpha_c$
  models: MGs interacting with tycoons & regulators
  stylized facts: at criticality only? why critical MGs?
  applicability: do markets behave like MGs?

Brief introduction to MGs

N agents, $i = 1 \ldots N$:

- At each round ℓ:
  all get info $\lambda(\ell)$
  each places a bid $b_i(\ell) \in \mathbb{R}$
- Those in the **minority** group win
  $\sum_j b_j(\ell) > 0 : \quad b_i(\ell) < 0$ wins
  $\sum_j b_j(\ell) < 0 : \quad b_i(\ell) > 0$ wins
- Each agent i has S strategies
  converting $\lambda(\ell)$ into bid $b_i(\ell)$
- dynamics:
  choice of strategy $a = 1 \ldots S$
  as a function of time,
  for all agents,
  based on strategies' performance



overall bid: $A(\ell) = N^{-\frac{1}{2}} \sum_i b_i(\ell)$:

Lookup table MGs (Challet, Zhang, Marsili, ...):

$$\begin{array}{rcl} \text{info}: & m \text{ step history}, & \boldsymbol{\lambda}(\ell) = (\text{sgn}[A(\ell-1)], \ldots, \text{sgn}[A(\ell-m)]) \\ \text{strat}: & \text{lookup table}, & \mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p), \quad p = 2^m \\ \text{bid}: & \text{strategy } a, \text{ info } \boldsymbol{\lambda}, & b_i(\ell) = R_{ia}^{\mu(\boldsymbol{\lambda})} \end{array}$$

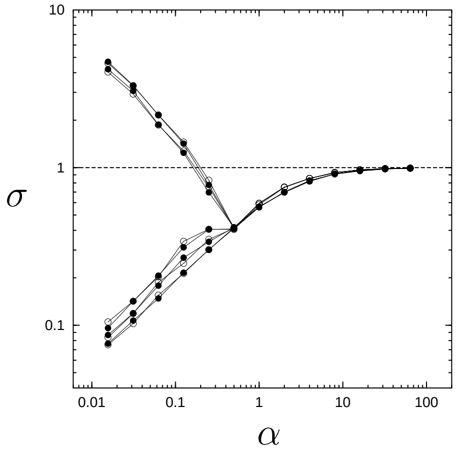
history 'window': $\Delta\ell = \mathcal{O}(\log N)$

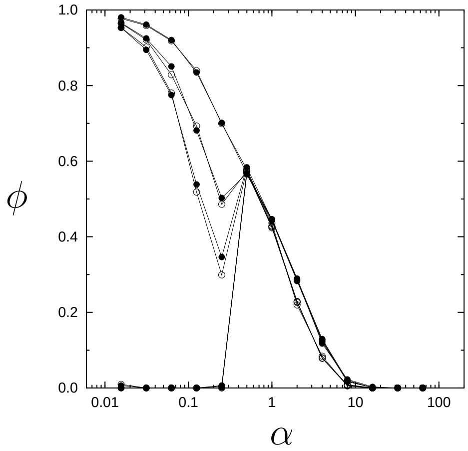
Inner product MGs (Cavagna, Sherrington, Garrahan, ...):

$$\begin{array}{rcl} \text{info}: & p \text{ step history}, & \boldsymbol{\lambda}(\ell) = (f[A(\ell-1)], \ldots, f[A(\ell-p)]) \\ \text{strat}: & \text{linear functional}, & \mathbf{R}_{ia} = (R_{ia}^1, \ldots, R_{ia}^p) \\ \text{bid}: & \text{strategy } a, \text{ info } \boldsymbol{\lambda}, & b_i(\ell) = p^{-\frac{1}{2}} \sum_{\mu \le p} R_{ia}^{\mu} \lambda_{\mu}(\ell) \end{array}$$

history 'window': $\Delta\ell = \mathcal{O}(N)$

$S = 2$,
$\alpha = p/N$,
lookup-table MG:

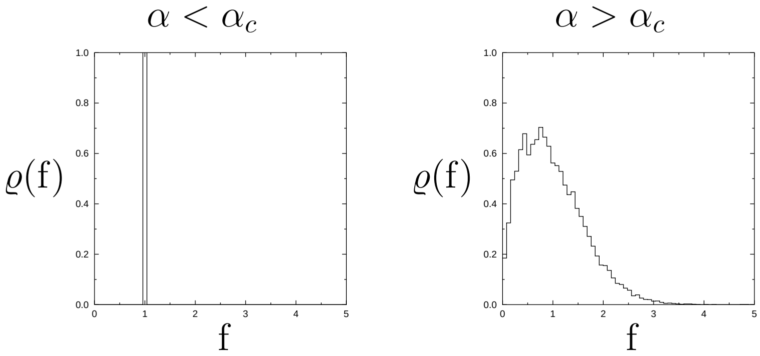




- volatility σ: efficient phase $\sigma < 1$
- ergodicity breaking phase transition
  ϕ fraction of 'frozen' agents
  c persistent correlations in strategy selection
- real vs fake histories, different S: similar phenomenology
- solvable!!

Why study inner product MGs?

- Closer to how real agents predict time series
- Lookup-table MG solved, inner product MG never solved!
- Lookup table MG, real history:
  theory evolves around history frequency distribution $\varrho(\mathrm{f})$
  history strings:
  $$\boldsymbol{\lambda}(\ell) = (\mathrm{sgn}[A(\ell-1)], \ldots, \mathrm{sgn}[A(\ell-m)]) \qquad 2^m = \alpha N$$
  $$\pi_{\boldsymbol{\lambda}} = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell \le L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell)} \qquad \varrho(\mathrm{f}) = \lim_{N\to\infty} 2^{-m} \sum_{\boldsymbol{\lambda}} \delta\left[\mathrm{f} - 2^m \pi_{\boldsymbol{\lambda}}\right]$$
- Inner product MG with
  real market history
  no such thing as $\varrho(\mathrm{f})$,
  history string of length $\mathcal{O}(N)$ ...



Generalized GFA theory for MGs with real history inner product and lookup table as special cases

$S=2$

- real vs fake history:

$$
\begin{aligned}
A(\ell-\mu) &\rightarrow (1-\zeta)A(\ell-\mu)+\zeta Z(\ell,\mu), \qquad \zeta\in[0,1] \\
&\qquad Z(\ell,\mu):\quad \text{zero}-\text{av Gaussian},\ \langle Z(\ell,\mu)Z(\ell',\mu')\rangle=\sigma_{\rm f}^2\delta_{\ell\ell'}\delta_{\mu\mu'}
\end{aligned}
$$

- generating functional ($q_i[.,]$: single agent decision vars)
$\langle\ \ \rangle$: over $\{z_i(\ell),Z(\ell,\lambda)\}$
$\overline{[\ \ ]}$: over $\{R_{ia}^{\mu}\}$

$$
\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} &= \overline{\left\langle e^{-i\sum_{i\ell}\psi_i(\ell)\sigma[q_i(\ell),z_i(\ell)]}\right\rangle} \\
&= \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}G\mathcal{D}\hat{G}\ldots\ e^{N\Psi[C,\hat{C},G,\hat{G},\ldots]}
\end{aligned}
$$

saddle-point eqns for

$$
\begin{aligned}
C(\ell,\ell') &= \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma[q_i(\ell),z_i(\ell)]\sigma[q_i(\ell'),z_i(\ell')]\rangle} \\
G(\ell,\ell') &= \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(\ell')}\overline{\langle\sigma[q_i(\ell),z_i(\ell)]\rangle}
\end{aligned}
$$

Result:

- Effective single agent process:

$$\frac{d}{dt}q(t) \;=\; \theta(t) - \alpha \int_0^t dt'\; R(t,t')\; \sigma[q(t')] + \sqrt{\alpha}\;\eta(t)$$

$$\text{Gaussian } \eta(t), \quad \langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = \Sigma(t,t')$$

- closed eqns:

$$C(t,t') = \langle \sigma[q(t)]\sigma[q(t')] \rangle_\star \qquad G(t,t') = \frac{\delta}{\delta\theta(t')}\langle \sigma[q(t)] \rangle_\star$$

- complications: relation between $\{R, \Sigma\}$ and $\{C, G\}$
involves effective bid process, with Gaussian fields ϕ

$$A(\ell) = A_e(\ell) + \phi_\ell - \frac{1}{2}\tilde{\eta} \sum_{\ell' < \ell} G(\ell,\ell')\overline{W}[\ell,\ell';\{A,Z\}]A(\ell')$$

$$\langle \phi_\ell \rangle_{\{\phi|A,Z\}} = 0, \qquad \langle \phi_\ell \phi_{\ell'} \rangle_{\{\phi|A,Z\}} = \frac{1}{2}[1 + C(\ell,\ell')]\overline{W}[\ell,\ell';\{A,Z\}]$$

- Differences between models: $\overline{W}[\ell,\ell';\{A,Z\}]$
similarity between histories observed at times ℓ and ℓ'

Time-translation invariant stationary states

$$C(t,t') = C(t-t'), \qquad G(t,t') = G(t-t')$$

main static order pars:

$$\chi = \int_0^\infty dt\; G(t), \qquad c = \lim_{t\to\infty} C(t)$$

obeying

$$\begin{aligned}
\phi &= 1 - \mathrm{Erf}[u], \qquad u = \sqrt{\alpha}\chi_{R}\sigma[\infty]/S_0\sqrt{2} \\
c &= \sigma^2[\infty]\Big\{1 - \mathrm{Erf}[u] + \frac{1}{2u^2}\mathrm{Erf}[u] - \frac{1}{u\sqrt{\pi}}e^{-u^2}\Big\} \\
\chi &= \mathrm{Erf}[u]/\alpha\chi_{R}
\end{aligned}$$

closure:

$$\begin{aligned}
\chi_{R} &= \lim_{\delta\to 0}\Big\{\overline{W}[0,0;\{A,Z\}] + \sum_{\ell=1}^{\infty}\frac{\partial}{\partial A(0)}\big\langle\big\langle \overline{W}[\ell,0;\{A,Z\}]A(\ell)\big\rangle\big\rangle_{\{A,Z\}}\Big\} \\
S_0^2 &= \lim_{\delta\to 0}\lim_{L\to\infty}\frac{\tilde{\eta}}{L^2\delta}\sum_{\ell,\ell'=1}^{L}\big\langle\big\langle \overline{W}[\ell,\ell';\{A,Z\}]A(\ell)A(\ell')\big\rangle\big\rangle_{\{A,Z\}}
\end{aligned}$$

Inner product MG with fake market history

- Put $\zeta \to 1$, calculate kernels (diagrammatically)
- Inner product and lookup table MGs
  differ only in characteristic amplitude,
  that can be transformed away:

$$G_{\rm IP}(t,t') = \kappa^{-1} G_{\rm LU}(t,t'), \quad \sigma^2_{\rm IP} = \kappa \sigma^2_{\rm LU}$$

  all other (time-dep) order parameters identical to
  those of look-up table MG

The real problem: generalized MGs with true market history

- define random matrix $\mathbf{B}(\ell)$:

$$B_{\lambda\lambda'}(\ell) = \mathcal{F}_\lambda[\ell, A, Z]\mathcal{F}_{\lambda'}[\ell, A, Z]$$

$$\text{LU}: \quad \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\ \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \qquad \boldsymbol{\lambda} \in \{-1, 1\}^m$$

$$\text{IP}: \quad \mathcal{F}_\lambda[\ell, A, Z] = f[(1-\zeta)A(\ell-\lambda) + \zeta Z(\ell, \lambda)] \quad \lambda \in \{1, \ldots, p\}$$

- $\{\chi_R, S_0\}$ can be written in terms of

$$\Delta_{r+1}(\ell_0, \ldots, \ell_r) = \frac{1}{p}\ \mathrm{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\Big\rangle\Big\rangle_{\{A,Z\}}$$

$$\tilde{\Delta}_{r+r'+2}(\ell_0, \ldots, \ell_r; \ell'_0, \ldots, \ell'_{r'}) = \frac{1}{p}\ \mathrm{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1)\mathbf{B}(\ell'_0)\Big\rangle\Big\rangle_{\{A,Z\}}$$

$$\tilde{\tilde{\Delta}}_{r+r'+1}(\ell_0, \ell_1 \ldots, \ell_r; \ell_0, \ell'_1, \ldots, \ell'_{r'}) = \frac{1}{p}\ \mathrm{Tr}\Big\langle\Big\langle \mathbf{B}(\ell_0)\mathbf{B}(\ell_1)\ldots\mathbf{B}(\ell_r)\mathbf{B}(\ell'_{r'})\ldots\mathbf{B}(\ell'_1)\Big\rangle\Big\rangle_{\{A,Z\}}$$

Short history correlation times

$$\prod_{i=1}^{r}\mathbf{B}(\ell_i)\to\mathbf{B}^r(A,Z):\qquad \mathbf{B}(A,Z)=\lim_{L\to\infty}\frac{1}{L}\sum_{\ell\le L}\mathbf{B}(\ell),\quad \text{eigenv spectrum } \varrho(\mu)$$

- Generalized theory:

$$u = \frac{\omega\sigma[\infty]\sqrt{\alpha}}{\sqrt{2(1+c)}},\qquad \frac{1-\phi}{\alpha}=\int_0^\infty \frac{d\mu\;\varrho(\mu)}{1+(\mu\chi)^{-1}},\qquad \phi=1-\mathrm{Erf}[u]$$

$$c = \sigma^2[\infty]\left\{1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}e^{-u^2}\right\}$$

$$\omega = \frac{\int d\mu\;\varrho(\mu)\mu\chi/(1+\mu\chi)}{\sqrt{\int d\mu\;\varrho(\mu)[\mu\chi/(1+\mu\chi)]^2}}\;\in\;[0,1]$$

- if $\omega=1$: fake history theory, if and only if $\varrho(\mu)$ is delta-distribution
- Phase transition point α_c indep of whether histories real or fake
- Look-up table MG: spectrum $\varrho(\mu)$ reduces to history frequency distr

Inner product MGs with true market history

$$\text{bids}: \qquad b_i(\ell) = p^{-\frac{1}{2}} \sum_{\mu \le p} R^{\mu}_{ia}\; f[(1-\zeta)A(\ell-\mu)+\zeta Z(\ell,\mu)]$$

need spectrum of

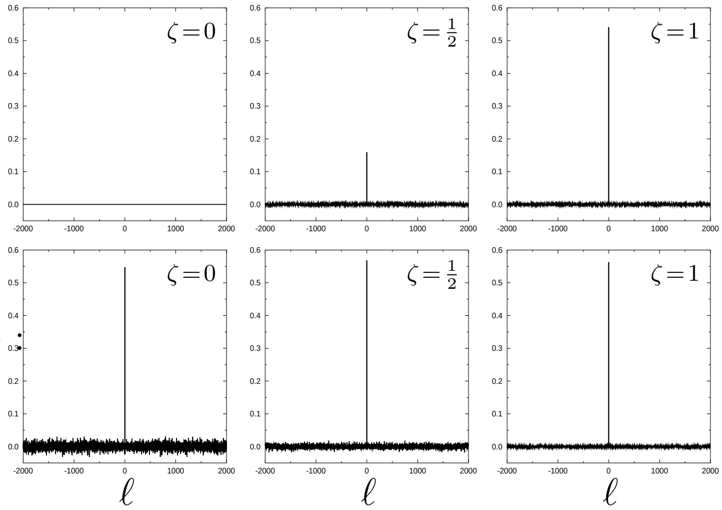
$$B_{\lambda\lambda'}(A,Z) = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell \le L} f[(1-\zeta)A(\ell-\lambda)+\zeta Z(\ell,\lambda)]f[(1-\zeta)A(\ell-\lambda')+\zeta Z(\ell,\lambda')]$$

$$\Xi(\ell) = \langle A(t+\ell)A(t)\rangle$$

$(N = 1025,\ \alpha = 2)$

$f[A] = A :$

$f[A] = \text{sgn}(A) :$



Calculation of spectrum $\varrho(\mu)$

short-hands:

$$
\begin{aligned}
Q_0 &= \int Dz\ f^2[z\sqrt{(1-\zeta)^2\sigma^2+\zeta^2\sigma_{\rm f}{}^2}] \\
Q_1 &= \int Dz\ \Big[\int Dy\ f[(1-\zeta)\sigma z+\zeta\sigma_{\rm f} y]\Big]^2
\end{aligned}
$$

facts:

(i) always $\int d\mu\ \varrho(\mu)\mu = Q_0$
(ii) if $\zeta = 1$: **B** self-averaging over $\{A, Z\}$ and Toeplitz

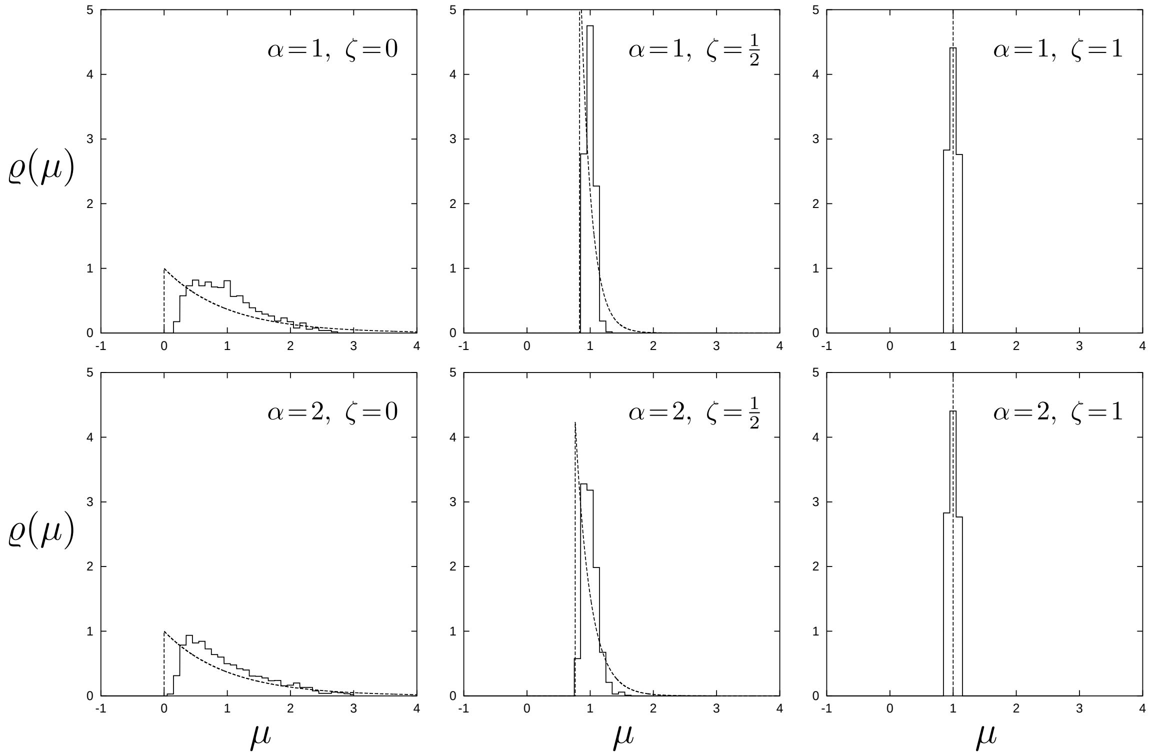
assumptions:

(i) **B** self-averaging over $\{Z\}$, but not over $\{A\}$
(ii) **B** is Toeplitz
(iii) Bid correlations $\Xi(\ell \neq 0)$ small

$$
\varrho(\mu) = Q_1^{-1}\theta[\mu - Q_0 + Q_1]e^{-(\mu-Q_0+Q_1)/Q_1}
$$

$\int d\mu\ \mu\varrho(\mu) = Q_0$, OK
$\lim_{\zeta\to 1}\varrho(\mu) = \delta[\mu - Q_0]$, OK

origin of nontrivial $\varrho(\mu)$: *consistency* of the histories



Histograms: spectra measured in simulations ($N = 1025$). Dotted lines: prediction, with σ as measured in simulation (formula for $\varrho(\mu)$ involves σ)

formula for $\varrho(\mu)$ involves volatility σ ...

The volatility

Short history correlation times, ergodic stationary states:

$$\sigma^2 =$$

$$\lim_{\delta_N \to 0} \left\{ \frac{1}{2} \sum_{r,r' \geq 0} (-\delta_N)^{r+r'} \int_0^\infty d\mu \; \varrho(\mu) \; \mu^{r+r'+1} \sum_{\ell_0 \ldots \ell_r} G(\ell_0, \ell_1) \ldots G(\ell_{r-1}, \ell_r) \right.$$

$$\left. \times \sum_{\ell'_0 \ldots \ell'_{r'}} G(\ell'_0, \ell'_1) \ldots G(\ell'_{r'-1}, \ell'_{r'}) \; \left[1 + C(\ell_r, \ell'_{r'})\right] \delta_{\ell,\ell_0} \delta_{\ell,\ell'_0} \right\}$$

contains (as always) short-time observables,
so approximate $C(\ell) \to c + \delta_{\ell 0}(1-c)$

$$\sigma^2 \;=\; \frac{1}{2}(1+c) \int_0^\infty d\mu \; \varrho(\mu) \frac{\mu}{(1+\mu\chi)^2} + \frac{1}{2}(1-c) Q_0$$

Final static theory for inner product MGs

with $E_1(z)=\int_z^\infty dt\ t^{-1}e^{-t}$:

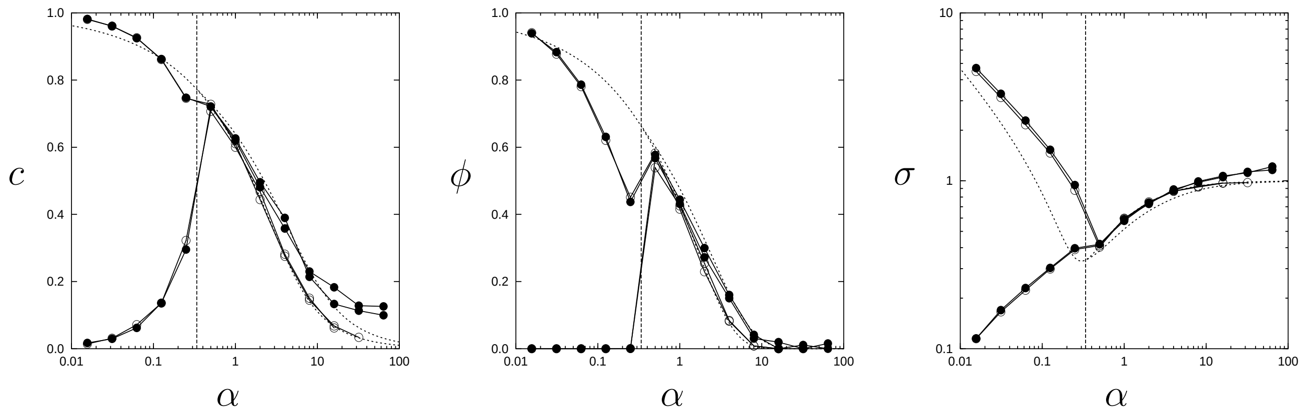
$$u=\omega\sigma[\infty]\sqrt{\alpha}/\sqrt{2(1+c)}$$
$$\phi=1-\mathrm{Erf}[u]$$
$$\mathrm{Erf}[u]=\alpha\Big\{1-\frac{1}{\chi Q_1}\ e^{[1+\chi(Q_0-Q_1)]/\chi Q_1}\ E_1\Big(\frac{1+\chi(Q_0-Q_1)}{\chi Q_1}\Big)\Big\}$$
$$c=\sigma^2[\infty]\Big\{1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}e^{-u^2}\Big\}$$
$$\omega=\frac{\chi Q_1-e^{[1+\chi(Q_0-Q_1)]/\chi Q_1}\ E_1(\frac{1+\chi(Q_0-Q_1)}{\chi Q_1})}{\sqrt{(\chi Q_1)^2-e^{[1+\chi(Q_0-Q_1)]/\chi Q_1}\ E_1(\frac{1+\chi(Q_0-Q_1)}{\chi Q_1})\big(2\chi Q_1+1\big)+\frac{\chi Q_1}{1+\chi(Q_0-Q_1)}}}$$
$$\sigma^2=\frac{1+c}{2Q_1\chi^2}\left\{(1+Q_1\chi)\frac{\alpha-1+\phi}{\alpha}-\frac{1}{1+\chi(Q_0-Q_1)}\right\}+\frac{1}{2}(1-c)Q_0$$
$$Q_0=\int Dz\ f^2[z\sqrt{(1-\zeta)^2\sigma^2+\zeta^2\sigma_{\mathrm{f}}{}^2}],\qquad Q_1=\int Dy\ \Big[\int Dz\ f[(1-\zeta)\sigma y+\zeta\sigma_{\mathrm{f}}z]\Big]^2$$

Universality for $\zeta \to 0$, strictly real histories

$\lim_{\zeta\to 0} Q_1 = \lim_{\zeta\to 0} Q_0 = \int Dz\ f^2[\sigma z]$
Let $\tilde{\chi} = Q_0\chi$:

$$u = \omega\sigma[\infty]\sqrt{\alpha}/\sqrt{2(1+c)}$$
$$\phi = 1 - \mathrm{Erf}[u]$$
$$c = \sigma^2[\infty]\Big\{1 - \mathrm{Erf}[u] + \frac{1}{2u^2}\mathrm{Erf}[u] - \frac{1}{u\sqrt{\pi}}e^{-u^2}\Big\}$$
$$\omega = \frac{\tilde{\chi} - e^{1/\tilde{\chi}}\ E_1(1/\tilde{\chi})}{\sqrt{\tilde{\chi}^2 - e^{1/\tilde{\chi}}\ E_1(1/\tilde{\chi})\big(2\tilde{\chi}+1\big) + \tilde{\chi}}}$$
$$\mathrm{Erf}[u] = \alpha\Big\{1 - \tilde{\chi}^{-1}e^{1/\tilde{\chi}}\ E_1(1/\tilde{\chi})\Big\}$$

values of $\{c, \phi, \tilde{\chi}\}$ depend only on α, not on $f[A]$!



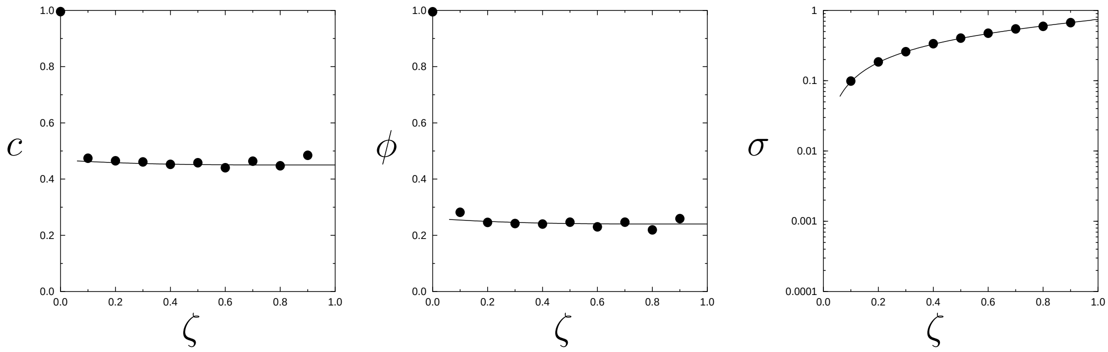
$f[A] = \mathrm{sgn}(A)$ and $S = 1$.
$\zeta = 0$ (strictly real histories, connected full circles), biased/unbiased init cond.
$\zeta = 1$ (strictly fake histories, connected open circles), biased/unbiased init cond.
Dotted: theoretical predictions for $\zeta = 0$ and $\zeta = 1$.

Degenerate case at $\zeta = 0$: $f[A] = A$

$\lim_{\zeta\to 0} Q_0 = \lim_{\zeta\to 0} Q_1 = \sigma^2$:

$$\sigma^2\Big\{1 - \frac{1}{\tilde{\chi}^2}\Big[(1+\tilde{\chi})\frac{\alpha-1+\phi}{\alpha} - 1\Big]\Big\} = 0$$

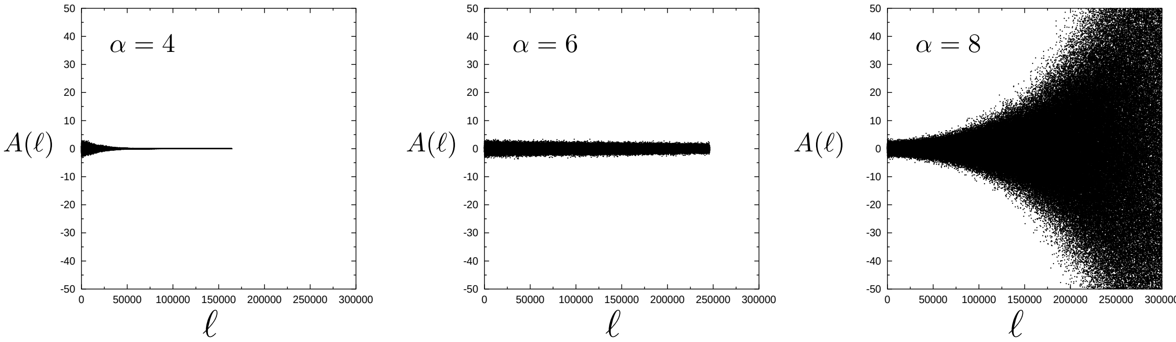
only stationary solution: $\sigma = 0$



$f[A] = A$, $\alpha = 2$, and $S = 1$. Simulations (markers) vs theory (dotted lines).

but if $Q_0 = \sigma^2 = 0$, then $\chi = \tilde{\chi}/Q_0 = \infty$

$f[A] = A$ system *always* critical & ergodicity *always* broken



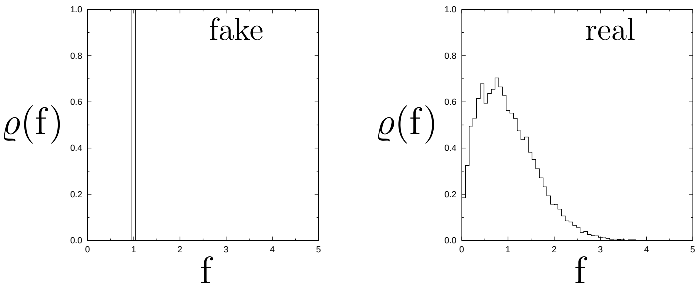
solution $\sigma = 0$ is reached only for $\alpha < \tilde{\alpha}_c$, where $6 < \tilde{\alpha}_c < 8$

system unstable for $\alpha > \tilde{\alpha}_c$!

Future/ongoing MG theory research based on GFA methods: sharper tools and sharper questions

1. Methods - solving GFA equations

- No rigorous *time dependent* solns $C(t,t')$ and $G(t,t')$ of GFA eqns
  only ad-hoc tricks and ansätze ...
  (except spherical MGs, and first few time steps in batch MGs)
  hence also no exact formulas for fluctuation observables (volatility, bid covariance)
- No rigorous solns yet for $\alpha < \alpha_c$
  not even stationary states, only ad-hoc tricks and ansätze ...
  (except spherical MGs, and first few time steps in batch MGs)
- Analysis of real history MGs where
  history strings are far from random
  (all analysis so far:
  history sampling weakly non-random)



2. Models - 'open' MGs, response to tycoons and regulators

How do MGs react to $\mathcal{O}(1)$ time-dependent external contributions to the overall bid?
(e.g. George Soros, large pension funds, market regulators, ...)

$$A(\ell) \;\rightarrow\; \frac{1}{\sqrt{N}}\sum_i b_i(\ell) + A_e(\ell)$$

Fake history batch MG

usual effective single-agent process

$$q(t+1) \;=\; q(t) + \theta(t) - \alpha \sum_{t'\leq t} (\mathbb{1}+G)^{-1}_{tt'} \sigma[q(t'), z(t')] + \sqrt{\alpha}\,\eta(t)$$

$\eta(t)$: zero-mean Gaussian noise, $\langle \eta(t)\eta(t')\rangle = \Sigma_{tt'}$,

$$\Sigma = (\mathbb{1}+G)^{-1} D[A_e] (\mathbb{1}+G^{\dagger})^{-1}, \qquad D[A_e]_{tt'} = 1 + C_{tt'} + 2A_e(t)A_e(t')$$

- bid statistics: $\overline{\langle A_\mu(t)} = A_e(t) + \overline{\langle A_\mu(t)}_{\rm int}$,

$$\overline{\langle A_\mu(\ell)}_{\rm int} = -\sum_{t'} [G(1\!\!1 + G)^{-1}]_{tt'} A_e(t')$$

$$\frac{1}{p}\sum_\mu \overline{\langle A_\mu(t) A_\mu(t')\rangle} - \frac{1}{p^2}\sum_{\mu\nu} \overline{\langle A_\mu(t)\rangle\langle A_\nu(t')\rangle} = \frac{1}{2}[(1\!\!1 + G)^{-1} D (1\!\!1 + G^\dagger)^{-1}]_{tt'}$$

no change in bid volatility!

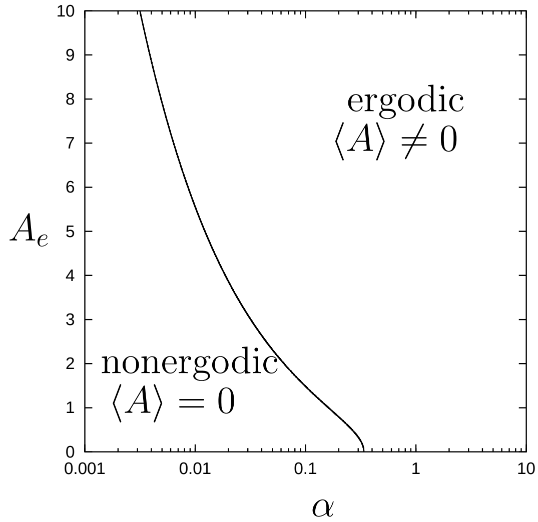
- simplest case: $A_e(\ell) = A_e$

stationary state equivalent to ordinary MG with *multiplicative* decision noise, of strength

$$T_{\rm eff} = \frac{1}{\sqrt{2}\ {\rm erf}^{\rm inv}[1/\sqrt{1+2A_e^2}]}$$

bids:

$$\overline{\langle A_\mu(t)\rangle}_{\rm int} = -\chi A_e/(1+\chi)$$



MG response to time-dependent $A_e(t)$?

Fake history spherical MG

$$
\begin{aligned}
A(\ell) &\rightarrow \frac{1}{\sqrt{N}}\sum_i b_i(\ell) + A_e(\ell) \\
\mathrm{sgn}[q_i(t)] &\rightarrow \lambda(t)q_i(t), \qquad \lambda(t) \text{ such that } \frac{1}{N}\sum_i \lambda^2(t)q_i^2(t) = 1
\end{aligned}
$$

GFA order parameter eqns:

$$
\begin{aligned}
\lambda(t+1)C(t+1,t') &= \lambda(t)C(t,t') - \alpha[(1\!\!1+G)^{-1}C](t,t') + \alpha[\Sigma G^\dagger](t,t') \\
\lambda(t+1)G(t+1,t') &= \lambda(t)G(t,t') - \alpha[(1\!\!1+G)^{-1}G](t,t') + \delta_{tt'} \\
& C(t,t) = 1, \qquad \Sigma = (1\!\!1+G)^{-1}D[A_e](1\!\!1+G^\dagger)^{-1} \\
& D[A_e](t,t') = 1 + C(t,t') + 2A_e(t)A_e(t')
\end{aligned}
$$

to be solved via Fourier transforms ...

3. Nature and source of fluctuations in MGs - 'stylized facts'

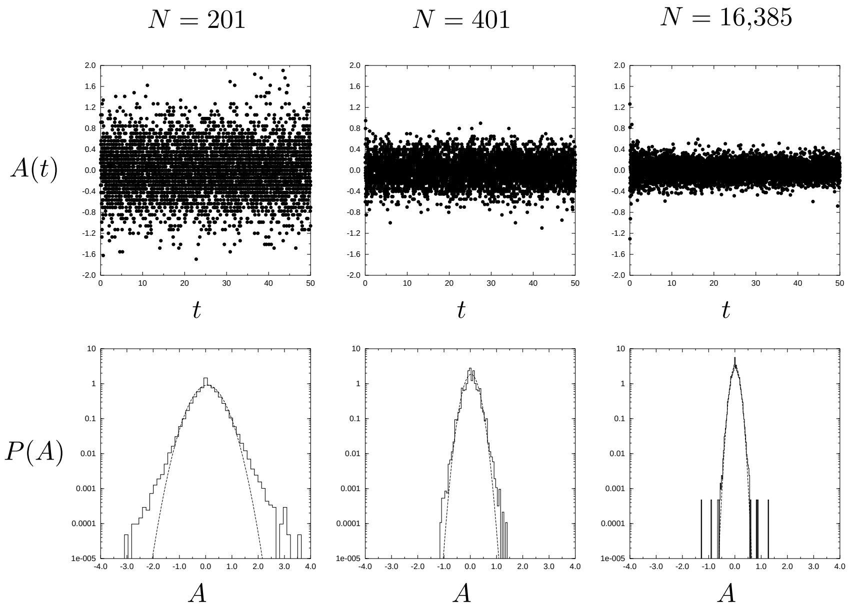
- Are stylized facts as seen in MGs just describing finite size effects at criticality?

- If so, don't we need MG models that are permanently critical?

- Why are grand canonical MGs not permanently critical?

  if $\alpha > \alpha_c$
  $\rightarrow$ predictable market
  $\rightarrow$ increase in N
  $\rightarrow$ reduction of $\alpha = p/N$
  until $\alpha = \alpha_c$ ...

- Perhaps inner product MGs?



Bids & bid statistics of MGs with speculators (variable number of agents), at criticality

4. Applicability - are real markets really like MGs?

- What is the essence of the MG's definition?
  – strategy valuations reward minority decisions
  – does minority rule emerge from valuations based on wealth production?
  (or do we need to introduce hard-core contrarians to get it)

- What is the essence of the MG's behaviour?
  – ergodic & (partly) predictable phase, vs non-ergodic & non-predictable phase
  – frozen & fickle agents, market-efficient phase $\sigma < \sigma_{\rm random}$
  – effective single-agent eqn, retarded self-interaction & non-white noise

required: models with

(i) evolving capitals and stock portfolios
(ii) price dynamics

example: the $-game

$$p_{ia}(\ell+1) = p_i(\ell) - A(\ell)b_i(\ell) \quad \rightarrow \quad p_{ia}(\ell+1) = p_i(\ell) + A(\ell)b_i(\ell-1)$$

GFA theory of the \$-game

easily incorporated:

$$\begin{aligned} p_{ia}(\ell+1) &= p_{ia}(\ell) + \frac{\varepsilon\tilde{\eta}}{\sqrt{N}} A(\ell) \sum_{\boldsymbol{\lambda}} R_{ia}^{\boldsymbol{\lambda}[\ell-\kappa,A,Z]} \\ A(\ell) &= A_e(\ell) + \frac{1}{\sqrt{N}} \sum_i \sum_{\boldsymbol{\lambda}} R_{ia_i(\ell)}^{\boldsymbol{\lambda}[\ell,A,Z]}, \qquad a_i(\ell) = \arg \max_{a\in\{1,\ldots,S\}} \{p_{ia}(\ell)\} \end{aligned}$$

e.g.

$$\text{MG:}\;\; (\varepsilon,\kappa) = (-1,0), \qquad \$-\text{Game:}\; (\varepsilon,\kappa) = (1,1)$$

GFA eqns:

$$\frac{d}{dt} q(t) = \theta(t) - \alpha \int_0^t dt'\; R(t,t')\; \sigma[q(t')] + \sqrt{\alpha}\; \eta(t)$$

$$C(t,t') = \langle \sigma[q(t)] \sigma[q(t')] \rangle_\star \qquad G(t,t') = \frac{\delta}{\delta\theta(t')} \langle \sigma[q(t)] \rangle_\star$$

$$\begin{aligned} R(t,t') &= -\lim_{\delta_N\to 0} \frac{\varepsilon\tilde{\eta}}{2p\delta_N} \frac{\delta}{\delta A_e(t')} \;\; \left\langle\!\left\langle A(\ell) \delta_{\boldsymbol{\lambda}(\ell-\kappa,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \right\rangle\!\right\rangle_{\{A,Z\}} \Big|_{\ell=t/\delta_N, \ell'=t'/\delta_N} \\ \Sigma(t,t') &= \lim_{\delta_N\to 0} \frac{\tilde{\eta}^2}{2p{\delta_N}^2} \left\langle\!\left\langle A(\ell) A(\ell') \delta_{\boldsymbol{\lambda}(\ell-\kappa,A,Z),\boldsymbol{\lambda}(\ell'-\kappa,A,Z)} \right\rangle\!\right\rangle_{\{A,Z\}} \Big|_{\ell=t/\delta_N, \ell'=t'/\delta_N} \end{aligned}$$

Detailed models: agent capital, portfolio & price dynamics

agents:

$$\begin{aligned} c_i(\ell) \in \mathbb{R}_0^+ : &\quad \text{capital held by agent } i \\ s_i(\ell) \in \mathbb{R}_0^+ : &\quad \text{number of shares held by agent } i \\ v_i(\ell) \in \mathbb{R} : &\quad \text{strategy valuation of agent } i \end{aligned}$$

market:

$$\begin{aligned} X(\ell) \in \mathbb{R}_0^+ : &\quad \text{share price of the asset} \\ D(\ell) \in \mathbb{R}_0^+ : &\quad \text{average demand for shares} \\ O(\ell) \in \mathbb{R}_0^+ : &\quad \text{average supply of shares} \\ H(\ell) \in \mathbb{R} : &\quad \text{recent market (price) history} \end{aligned}$$

relations:

$$\begin{aligned} d_i(\ell) &= \frac{1}{2}\tau_i\theta[v_i(\ell)-\xi_i][1+R_i(H(\ell))]c_i(\ell), \qquad D(\ell) = \frac{1}{N}\sum_i d_i(\ell) \\ o_i(\ell) &= \frac{1}{2}\tau_i\theta[v_i(\ell)-\xi_i][1-R_i(H(\ell))]s_i(\ell), \qquad O(\ell) = \frac{1}{N}\sum_i o_i(\ell) \end{aligned}$$

$R_i(H) \in \{-1, 1\}$: strategies
$R_i(H) = 1$: buy order for fraction τ_i of agent's capital
$R_i(H) = -1$: sell order for fraction τ_i of agent's shares

trading (R: interest rate)

$$\begin{aligned} c_i(\ell{+}1) &= c_i(\ell)(1+R) - d_i(\ell) + o_i(\ell)X(\ell{+}1) \\ s_i(\ell{+}1) &= s_i(\ell) + d_i(\ell)/X(\ell{+}1) - o_i(\ell) \end{aligned}$$

strategy valuations based on wealth creation:

$$\begin{aligned} v_i(\ell{+}1) &= v_i(\ell) + \frac{1}{2}\eta\tau_i\Big[X(\ell{+}1)/X(\ell) - 1 - R\Big] \\ &\qquad\qquad \times\Big\{c_i(\ell{-}1)[1{+}R_i(H(\ell{-}1))] - s_i(\ell{-}1)X(\ell)[1{-}R_i(H(\ell{-}1))]\Big\} \end{aligned}$$

price dynamics:

$$X(\ell+1) = X(\ell) + \frac{1}{L}\Big[X_0(\ell{+}1){-}X(\ell) + Tz(\ell)X(\ell)\Big], \qquad X_0(\ell+1) = \frac{D(\ell)}{O(\ell)}$$

(stochastic motion towards clearing price, liquidity L)

Preliminary results:
(i) GFA theory complicated and *qualitatively different* from MG
(ii) generally no minority mechanism, one needs psychological factors?
(everyone happy to be trend-follower when everyone increases wealth ...)

SUMMARY - generating functional analysis of MGs

- **Nearly finished**: theory of Inner Product MGs

  GFA for generalized MGs, real histories
  history covariance eigenvalue spectrum
  inner product MGs have new features: universality, criticality ...

- **Ongoing & only just begun**:

  MGs interacting with tycoons & regulators (ordinary, spherical)
  Application of MG tools to more realistic market models

- **Continuing in the background**:

  refining the mathematical tools ...