Replica methods for loopy sparse random graphs

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ACC Coolen

King's College London

Outline

Tailoring random graph ensembles

Motivation:

stat mechanics of process on complex network $\mathbf{c}^\star$,
use *random graph* $\mathbf{c}$ as proxy

- max entropy ensemble Ω_L, constrained by values of $\omega_1(\mathbf{c})\ldots\omega_L(\mathbf{c})$

hard constraints: $p(\mathbf{c}) \propto \prod_{\ell\leq L} \delta_{\omega_\ell(\mathbf{c}),\omega_\ell(\mathbf{c}^\star)}$

soft constraints: $p(\mathbf{c}) \propto \mathrm{e}^{\sum_{\ell=1}^{L}\hat{\omega}_\ell \omega_\ell(\mathbf{c})}, \quad \langle \omega_\ell(\mathbf{c})\rangle = \omega_\ell(\mathbf{c}^\star)\ \ \forall \ell$

- approximate process on $\mathbf{c}^\star$:

  average generating function of process over graphs in Ω_L

  larger $L \to$ better approxim

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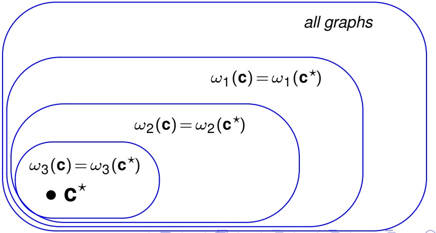

Ising models on tailored random graphs

transition temperature T_c

Ω_A: correct $\langle k \rangle$
Ω_B: correct $p(k)$
Ω_C: correct $p(k)$ and $W(k,k')$

- $\mathbf{c}^\star$ = d-dim cubic lattice
  $p(k) = \delta_{k,2^d}$

- $\mathbf{c}^\star$ = 'small world' lattice
  $p(k \geq 2) = e^{-q} q^{k-2}/(k-2)!$

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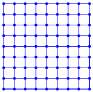

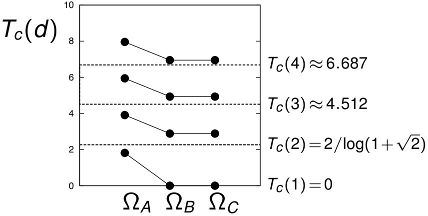

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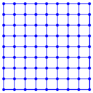

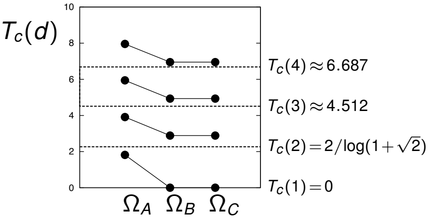

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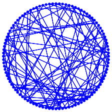
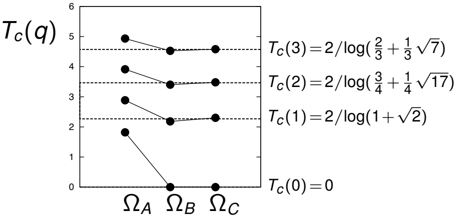




The problem

- biological networks,
  physical lattices,
  communication networks,
  distribution networks,
  socio-ecomomic networks, ...

  → *sparse graphs,*
  → *many short loops*

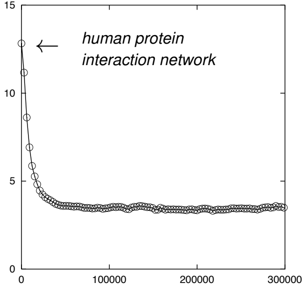
- max entropy graph ensembles
  with prescribed $p(k)$,$W(k,k')$:

  → *sparse graphs,*
  → *locally tree-like*

- realistic tailoring of graphs requires
  adding $\omega(\mathbf{c})$ that enforces short loops

- available analysis methods,
  e.g. replicas, GFA, cavity, belief propagation ...
  work only for locally tree-like graphs

$\frac{1}{N}\mathrm{Tr}(\mathbf{c}^3)$



randomisation with conserved $\{p, W\}$

exceptions:
cubic lattices $d < 3$
spherical models
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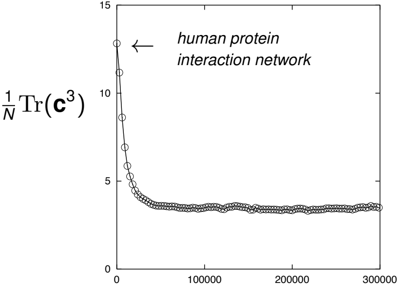

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Loopy random graph ensembles

Simplest loopy ensemble

control average degree $\langle k\rangle$
and density of triangles $\langle m\rangle$
(Strauss '86, Jonasson '99)

$$p(\mathbf{c}) \propto \mathrm{e}^{u\sum_{ij} c_{ij} + v\sum_{ijk} c_{ij}c_{jk}c_{ki}}$$

- to calculate:

$$\langle k\rangle = \langle \frac{1}{N}\sum_{ij} c_{ij}\rangle, \qquad \langle m\rangle = \langle \frac{1}{N}\sum_{ijk} c_{ij}c_{jk}c_{ki}\rangle, \qquad S = -\frac{1}{N}\sum_{\mathbf{c}} p(\mathbf{c})\log p(\mathbf{c})$$

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- control degrees and closed paths of **all** lengths

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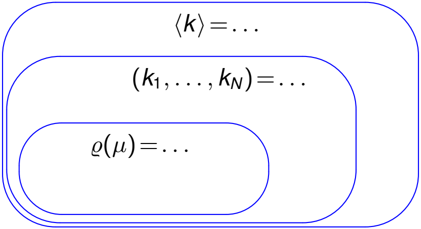

How informative are spectra of sparse graphs?

- How many non-isomorphic graphs are there with given degrees $(k_1, \ldots, k_N)$ and a given spectrum $\varrho(\mu)$?
- How similar are processes on non-isomorphic graphs with given degrees $(k_1, \ldots, k_N)$ and spectrum $\varrho(\mu)$?

  spherical spins: free energies identical (Berlin, Kac)
  zero field Ising models above T_c: free energies identical (Parisi)

Possible analytical route

graph ensemble : $$p(\mathbf{c}) = Z^{-1}[\hat{\varrho}]\, \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} \prod_i \delta_{k_i,\sum_j c_{ij}}$$

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- replica method, steepest descent for $N\to\infty$, analytical continuation to *imaginary* (n_μ, m_μ), limits $\epsilon, \Delta\downarrow 0$
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Origin of the identity

spectral ensemble constraints,
use Edwards-Jones ('76):

$$\varrho(\mu|\mathbf{c}) = \frac{2}{N\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im} \frac{\partial}{\partial\mu} \log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}), \qquad Z(\mu|\mathbf{c}) = \int_{\mathrm{I\!R}^N} \mathrm{d}\boldsymbol{\phi}\; \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{1}]\boldsymbol{\phi}}$$

integrate by parts,
discretise integral,

$$\begin{aligned} \mathrm{e}^{N\int \mathrm{d}\mu\; \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} &= \mathrm{e}^{N\int \mathrm{d}\mu\; \hat{\varrho}(\mu)\frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})} \\ &= \lim_{\varepsilon,\Delta\downarrow 0} \prod_{\mu} \mathrm{e}^{-2\mathrm{Im}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}).\,\frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)} \end{aligned}$$

$$\mathrm{e}^{-2\,\mathrm{Im}\log z} = z^{\mathrm{i}}.\overline{z}^{\,-\mathrm{i}}$$

$$\Phi[\hat{\varrho}] = \lim_{\varepsilon,\Delta\downarrow 0} \frac{1}{N} \log \sum_{\mathbf{c}} \Big[\prod_i \delta_{k_i,\sum_j c_{ij}}\Big] \prod_{\mu} \Big[Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})^{\mathrm{i}}\; \overline{Z(\mu+\mathrm{i}\varepsilon|\mathbf{c})}^{-\mathrm{i}}\Big]^{\frac{\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}$$

Origin of the identity

spectral ensemble constraints,
use Edwards-Jones ('76):

$$\varrho(\mu|\mathbf{c}) = \frac{2}{N\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\frac{\partial}{\partial\mu}\log Z(\mu+\mathrm{i}\varepsilon|\mathbf{c}), \qquad Z(\mu|\mathbf{c}) = \int_{\mathrm{I\!R}^N}\mathrm{d}\boldsymbol{\phi}\ \mathrm{e}^{-\frac{1}{2}\mathrm{i}\boldsymbol{\phi}\cdot[\mathbf{c}-\mu\mathbf{1}]\boldsymbol{\phi}}$$

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Outline

Replica analysis of generating function

path integral representation:

$$\Phi[\hat{\varrho}] = \frac{1}{2}\langle k\rangle \log\Big(\frac{N}{\langle k\rangle}\Big) + \mathcal{O}\Big(\frac{1}{N}\Big)$$

$$+ \lim_{\Delta,\varepsilon\downarrow 0}\; \lim_{n_\mu \to \frac{\mathrm{i}\Delta}{\pi}\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)}\; \lim_{m_\mu\to -n_\mu} \frac{1}{N}\log\int\{\mathrm{d}\mathcal{P}\mathrm{d}\hat{\mathcal{P}}\}\; \mathrm{e}^{N\left[\Psi[\mathcal{P},\hat{\mathcal{P}}]+\epsilon_N\right]}$$

$$\begin{aligned}\Psi[\mathcal{P},\hat{\mathcal{P}}] &= \mathrm{i}\int \mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\omega\; \hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\\ &+\frac{1}{2}\langle k\rangle\int \mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\mathrm{d}\omega\mathrm{d}\boldsymbol{\phi}'\mathrm{d}\boldsymbol{\psi}'\mathrm{d}\omega'\; \mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)\mathcal{P}(\boldsymbol{\phi}',\boldsymbol{\psi}',\omega')\mathrm{e}^{-\mathrm{i}(\omega+\omega')+\mathrm{i}(\boldsymbol{\psi}\cdot\boldsymbol{\psi}'-\boldsymbol{\phi}\cdot\boldsymbol{\phi}')}\\ &+\sum_k p(k)\log\int_{-\pi}^{\pi}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}k\omega}\int\mathrm{d}\boldsymbol{\phi}\mathrm{d}\boldsymbol{\psi}\;\mathrm{e}^{-\frac{1}{2}\boldsymbol{\phi}\cdot(\varepsilon\mathbf{I}-\mathrm{i}\mathbf{M})\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\psi}\cdot(\varepsilon\mathbf{I}+\mathrm{i}\mathbf{M})\boldsymbol{\psi}-\mathrm{i}\hat{\mathcal{P}}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)}\end{aligned}$$

$$\boldsymbol{\phi}=\{\phi_{\mu,\alpha_\mu}\},\quad \boldsymbol{\psi}=\{\psi_{\mu,\beta_\mu}\}$$

$$M_{\mu,\alpha_\mu;\mu'\alpha'_\mu} = \mu\delta_{\alpha_\mu,\alpha'_\mu}$$

$$\lim_{N\to\infty}\epsilon_N = 0$$

order parameter:

$$\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega) = \frac{1}{N}\sum_i \delta(\boldsymbol{\phi}-\boldsymbol{\phi}^i)\delta(\boldsymbol{\psi}-\boldsymbol{\psi}^i)\delta(\omega-\omega_i)\Big]$$

Replica analysis of generating function

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Replica symmetry ansatz

Closed eqns for $\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi}) = \int_{-\pi}^{\pi} \mathrm{d}\omega\ \mathrm{e}^{-\mathrm{i}\omega} \mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)$,

$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi})$ symmetric under permutations of $\{\phi_{\mu,1},\ldots,\phi_{\mu,n_\mu}\}$ and $\{\psi_{\mu,1},\ldots,\psi_{\mu,m_\mu}\}$

De Finetti:

$$\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi}) = \mathcal{C}\int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}]\Big[\prod_\mu \prod_{\alpha_\mu=1}^{n_\mu} \pi(\phi_{\mu,\alpha_\mu}|\mu)\Big]\Big[\prod_\mu \prod_{\beta_\mu=1}^{m_\mu} \overline{\pi(\psi_{\mu,\beta_\mu}|\mu)}\Big]$$

$\int\{\mathrm{d}\pi\}\mathcal{W}[\{\pi\}] = 1$,

$\mathcal{W}[\{\pi\}] > 0$ only if $\int \mathrm{d}x\ \pi(x|\mu) = 1$

– insert ansatz into saddle-point eqn
– derive closed eqns for $\mathcal{C}$ and $\mathcal{W}[\{\pi\}]$
– insert into generation function $\Phi[\hat{\varrho}]$

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Closed eqns for $\mathcal{W}(\boldsymbol{\phi},\boldsymbol{\psi})=\int_{-\pi}^{\pi}\mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega}\mathcal{P}(\boldsymbol{\phi},\boldsymbol{\psi},\omega)$,

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result:

$$\mathcal{W}[\{\pi\}] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle} \times \frac{\Big[\prod_{\ell<k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_{k-1}\}]\delta_{\mathrm{F}}\Big[\pi(.|\mu)-\pi(.|\mu,\pi_1,\ldots,\pi_{k-1})\Big]}{\Big[\prod_{\ell\leq k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_k\}]}$$

with

$$\pi(\phi|\mu,\pi_1,\ldots,\pi_k) = \frac{\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)}{\int\mathrm{d}\phi'\,\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi'^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi'|\mu)}$$

$$\mathcal{A}[\{\pi_1,\ldots,\pi_k\}] = \prod_\mu\Big[\Big(\int\mathrm{d}\phi\,\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)\Big)^{n_\mu} \times\overline{\Big(\int\mathrm{d}\phi\,\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)\Big)}^{m_\mu}\Big]$$

$$\hat{\pi}(\phi|\mu) = \int\mathrm{d}\phi'\,\mathrm{e}^{-\mathrm{i}\phi\phi'}\pi(\phi'|\mu),$$

normalisation:

$$\mathcal{C}^2 = \sum_{k>0}p(k)\frac{k}{\langle k\rangle}\frac{\Big[\prod_{\ell<k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_{k-1}\}]}{\Big[\prod_{\ell\leq k}\int\{\mathrm{d}\pi_\ell\}\mathcal{W}[\{\pi_\ell\}]\Big]\mathcal{A}[\{\pi_1,\ldots,\pi_k\}]}$$

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Exploiting the nature of the propagation

order parameter $\mathcal{W}[\{\pi\}]$:
stationary state of stochastic propagation
of complex distributions:

$$\pi(\phi|\mu) \;\;\rightarrow\;\; \pi(\phi|\mu,\pi_1,\ldots,\pi_{k-1}) = \frac{\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi|\mu)}{\int\!\mathrm{d}\phi'\;\mathrm{e}^{-\frac{1}{2}(\varepsilon-\mathrm{i}\mu)\phi'^2}\prod_{\ell\leq k}\hat{\pi}_\ell(\phi'|\mu)}$$

shape-preserving
for $\pi(\phi|\mu)$ of the form

$$\pi(\phi|x,u) = \frac{\mathrm{e}^{-\frac{1}{2}\mathrm{i}x\phi^2+\mathrm{i}u\phi}}{\int\!\mathrm{d}\phi'\;\mathrm{e}^{-\frac{1}{2}\mathrm{i}x\phi'^2+\mathrm{i}u\phi'}}$$

$x(\mu)$, $u(\mu)$: complex functions,
$\mathrm{Im}\,x(\mu)<0$ for all $\mu\in\mathbb{R}$

$$x'(\mu) = -\mathrm{i}\varepsilon-\mu-\sum_{\ell<k}\frac{1}{x_\ell(\mu)}, \qquad u'(\mu) = -\sum_{\ell<k}\frac{u_\ell(\mu)}{x_\ell(\mu)}$$

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simplest case, $u(\mu)=0$:

$$\mathcal{W}[\{x\}] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\frac{\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\mathcal{F}_k[\{x\}]}$$

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$$\mathcal{F}_k[\{x\}] = \Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\}\;\mathcal{W}[\{x_\ell\}]\Big]\delta_{\mathrm{F}}\big[x-F[x_1,\ldots,x_k]\big]$$

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spectrum:

$$\varrho(\mu) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\mu}\Big\{\sum_k p(k)\frac{\int\{\mathrm{d}x\}\;\mathcal{A}[\{x\}]\mathcal{F}_k[\{x\}]\mathrm{sgn}\big[x(\mu)\big]}{\int\{\mathrm{d}x\}\;\mathcal{A}[\{x\}]\mathcal{F}_k[\{x\}]} + \langle k\rangle\mathcal{C}^2\int\{\mathrm{d}x\mathrm{d}x'\}\mathcal{W}[\{x\}]\mathcal{W}[\{x'\}]\mathcal{B}[\{x\},\{x'\}] \times\theta[x(\mu)x'(\mu)]\theta[1-x(\mu)x'(\mu)]\mathrm{sgn}[x(\mu)+x'(\mu)]\Big\}$$

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loopy graph ensembles

$$\mathcal{W}[\{x\}] = \frac{\sum_{k>0} p(k) \frac{k}{\langle k \rangle} \frac{\mathcal{A}[\{x\}]\mathcal{F}_{k-1}[\{x\}]}{\int \{\mathrm{d}x'\} \mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}{\sum_{k>0} p(k) \frac{k}{\langle k \rangle} \frac{\int \{\mathrm{d}x'\} \mathcal{A}[\{x'\}]\mathcal{F}_{k-1}[\{x'\}]}{\int \{\mathrm{d}x'\} \mathcal{A}[\{x'\}]\mathcal{F}_k[\{x'\}]}}$$

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tree-like limit:

$\hat{\varrho}(\mu) \to 0$:

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structure of message-passing algorithms,
e.g. belief propagation, cavity method

meaning of $\mathcal{A}[\{x\}] \neq 1$?
nontrivial acceptance probabilities

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Outline

Processes on loopy graphs

$$H(\sigma_1,\ldots,\sigma_N|\mathbf{c}) = -J\sum_{i<j} c_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i$$

$$f(\mathbf{c}) = -\frac{1}{\beta N}\log\sum_{\sigma_1\ldots\sigma_N}\exp[-\beta H(\sigma_1,\ldots,\sigma_N|\mathbf{c})]$$

average free energy density,
use $\overline{\log Z} = \lim_{n\to 0} n^{-1}\log\overline{Z^n}$

$$\bar{f} = -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta Nn}\log\sum_{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_N} \mathrm{e}^{\beta h\sum_{\alpha=1}^n\sum_i\sigma_i^\alpha - \beta N E_{\rm eff}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N)}$$

effective interaction energy for replicated spins $\boldsymbol{\sigma}_i = (\sigma_i^1,\ldots,\sigma_i^n)$

$$E_{\rm eff}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N) = -\frac{1}{\beta N}\log\sum_{\mathbf{c}} p(\mathbf{c})\mathrm{e}^{\beta J\sum_{i<j}c_{ij}\sum_{\alpha=1}^n\sigma_i^\alpha\sigma_j^\alpha}$$

$$p(\mathbf{c}) \propto \mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})}\prod_i \delta_{k_i,\sum_j c_{ij}}$$

(extra layer of replicas, but with $n\in\mathbb{R}$, $\downarrow 0$)

Processes on loopy graphs

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average free energy density,

use $\overline{\log Z} = \lim_{n\to 0} n^{-1}\log\overline{Z^n}$

$$\bar{f} = -\lim_{N\to\infty}\lim_{n\to 0}\frac{1}{\beta N n}\log\sum_{\boldsymbol{\sigma}_1\ldots\boldsymbol{\sigma}_N} \mathrm{e}^{\beta h\sum_{\alpha=1}^n\sum_i \sigma_i^\alpha - \beta N E_{\mathrm{eff}}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N)}$$

effective interaction energy for replicated spins $\boldsymbol{\sigma}_i = (\sigma_i^1,\ldots,\sigma_i^n)$

$$E_{\mathrm{eff}}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_N) = -\frac{1}{\beta N}\log\sum_{\mathbf{c}} p(\mathbf{c})\mathrm{e}^{\beta J\sum_{i<j} c_{ij}\sum_{\alpha=1}^n \sigma_i^\alpha\sigma_j^\alpha}$$

$$p(\mathbf{c}) \propto \mathrm{e}^{N\int \mathrm{d}\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})}\prod_i \delta_{k_i,\sum_j c_{ij}}$$

(extra layer of replicas, but with $n\in\mathbb{R}, \downarrow 0$)

new generating function

$$\Phi_K[\hat{\varrho},\{\boldsymbol{\sigma}\}]=\frac{1}{N}\log\sum_{\mathbf{c}}\mathrm{e}^{N\int\mathrm{d}\mu\;\hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})+K\sum_{i<j}c_{ij}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j}\prod_{i=1}^{N}\delta_{k_i,\sum_j c_{ij}}$$

- $N\to\infty$:
  dependence of Φ_K
  on spins only via

$$\mathcal{D}(\boldsymbol{\sigma},k)=\frac{1}{N}\sum_i\delta_{k,k_i}\delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}_i},\qquad\boldsymbol{\sigma}\in\{-1,1\}^n$$

- graph problem
  coupled to spin problem:

$$-\beta\bar{f}=\lim_{n\to0}\mathrm{extr}_{\{\mathcal{D},\hat{\mathcal{D}}\}}\frac{1}{n}\Big\{\mathrm{i}\sum_{\boldsymbol{\sigma},k}\hat{\mathcal{D}}(\boldsymbol{\sigma},k)\mathcal{D}(\boldsymbol{\sigma},k)-\beta E_{\mathrm{eff}}[\mathcal{D}]+\sum_k p(k)\log\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\beta h\sum_\alpha\sigma_\alpha-\mathrm{i}\hat{\mathcal{D}}(\boldsymbol{\sigma},k)}\Big\}$$

$$-\beta E_{\mathrm{eff}}[\mathcal{D}]=\Phi_K[\hat{\varrho},\mathcal{D}]-\Phi[\hat{\varrho}]$$

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resulting RS theory

explicit formula for $\bar{f}$
in terms of order parameter $\mathcal{W}_K[\{x\},v]$

$$\mathcal{W}_K[\{x\},v] = \frac{1}{\mathcal{C}^2}\sum_{k>0} p(k)\frac{k}{\langle k\rangle}\int \mathrm{d}u\; W_k(u)\int \frac{\mathrm{d}\hat{v}}{2\pi}\;\mathrm{e}^{\mathrm{i}\hat{v}(v-u)}$$

$$\times \frac{\mathcal{A}[\{x\}]\Big[\prod_{\ell<k}\int\{\mathrm{d}x_\ell\}\mathrm{d}v_\ell\; \mathcal{W}_K[\{x_\ell\},v_\ell]\mathrm{e}^{-\mathrm{i}\hat{v}H(v_\ell)}\Big]\delta_{\mathrm{F}}\big[x-F[x_1,\ldots,x_{k-1}]\big]}{\int\{\mathrm{d}x'\}\mathcal{A}[\{x'\}]\Big[\prod_{\ell\leq k}\int\{\mathrm{d}x_\ell\}\mathrm{d}v_\ell\; \mathcal{W}_K[\{x_\ell\},v_\ell]\mathrm{e}^{-\mathrm{i}\hat{v}H(v_\ell)}\Big]\delta_{\mathrm{F}}\big[x'-F[x_1,\ldots,x_k]\big]}$$

$$H(v) = \frac{1}{\beta}\mathrm{atanh}[\tanh(\beta v)\tanh(K)]$$

- $\int \mathrm{d}v\; \mathcal{W}_K[\{x\},v] = \mathcal{W}[\{x\}]$
  v: effective field at site characterised by $\{x\}$,
  $\mathcal{W}_K[v|\{x\}]$: impact of local topology on local order
- setting $\mathcal{W}_K[\{x\},v] = \mathcal{W}_K[\{x\}]\mathcal{W}_K(v)$:
  Bethe lattice result

  $T_c = 2J/\log[d/(d-1)]$ ✓

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Limit of locally tree-like graphs

$\hat{\varrho}(\mu) \to 0$ for all μ: $\quad p(\mathbf{c}) \propto \prod_i \delta_{k_i,\sum_j c_{ij}}$

$\mathcal{A}[\{x\}] = \mathcal{B}[\{x\},\{x'\}] = \mathcal{C} = 1$

- entropy per node:

$$S = \frac{1}{2}\langle k\rangle\left[\log\left(\frac{N}{\langle k\rangle}\right)+1\right] + \sum_k p(k)\log\tilde{p}(k) + \epsilon_N \qquad \checkmark$$

- spectrum for regular graphs, $k>1$:
  McKay's '81 formula:

$$\varrho(\mu) = \theta\left[2\sqrt{k-1}-|\mu|\right]\,\frac{k\sqrt{4(k-1)-\mu^2}}{2\pi(k^2-\mu^2)} \qquad \checkmark$$

- spectrum for arbitrary $p(k)$:
  Dorogovtsev et al '03 formula
  (Poisson $p(k)$: Rodgers-Bray '88)

$$G(z|\mu) = 1-\sqrt{z}\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\,\mathrm{e}^{\mathrm{i}y\mu}\,\Phi\big(G(y|\mu)\big)J_1(2\sqrt{yz}) \qquad \checkmark$$

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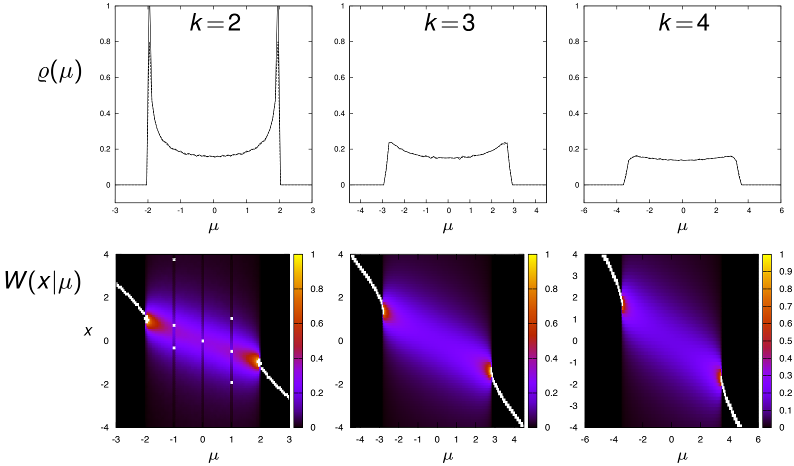
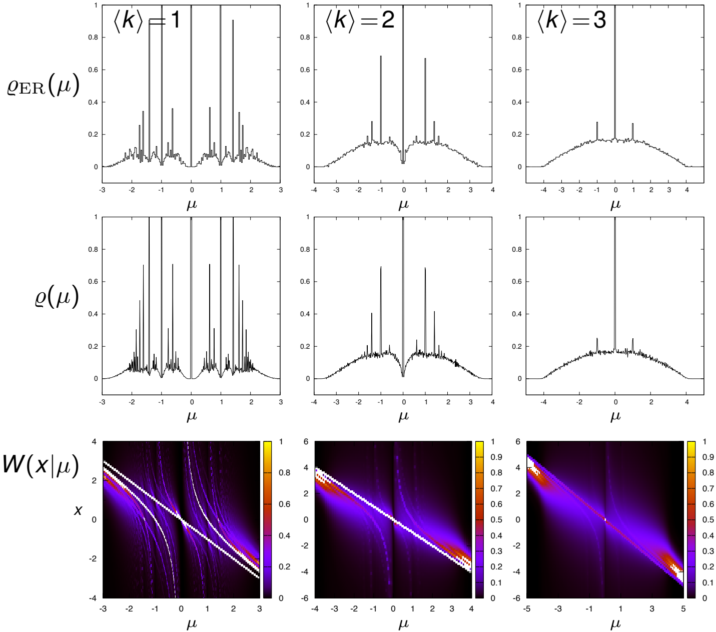
Regular treelike graphs:



Erdös-Rényi graphs:



Spectra of loopy graphs

– no exact solutions available ...
– population dynamics tricky and slow to equilibrate ...

– loopy graph ensembles tricky to simulate ...
(slow MCMC equilibration, phase transitions, finite size effects)

$p(\mathbf{c}) \propto \mathrm{e}^{\alpha \sum_{ij} c_{ij} + \beta \sum_i [\sum_j c_{ij}]^2}$

$N = 300,\ \textit{timescales} \sim N^3?$

still relatively simple dynamics,
elementary moves: $c_{ij} \to 1 - c_{ij}$

present problem:
edge−swap MCMC...
more complex landscape...

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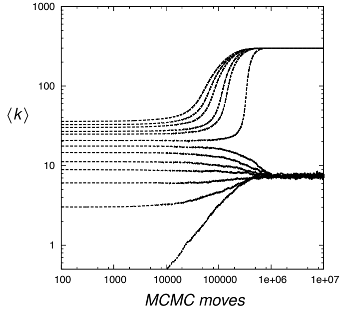

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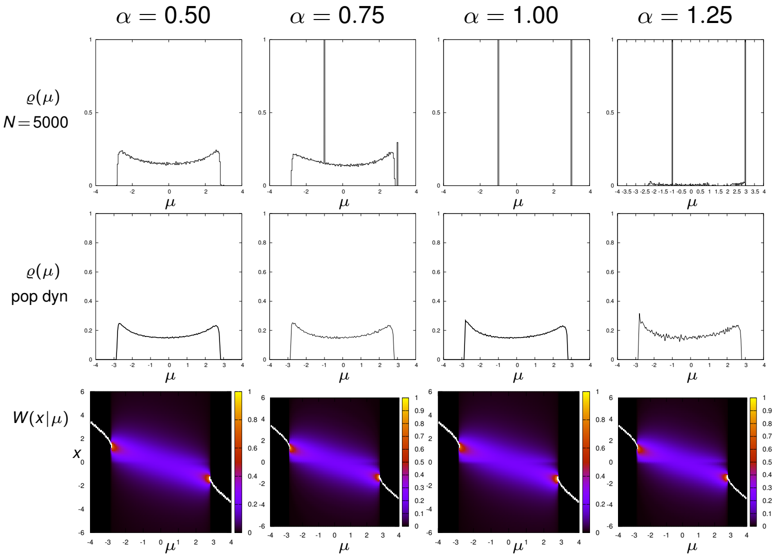

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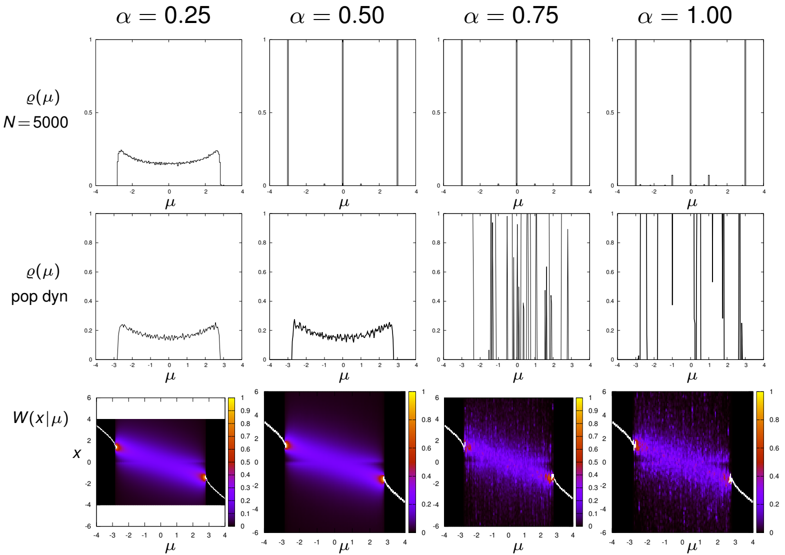

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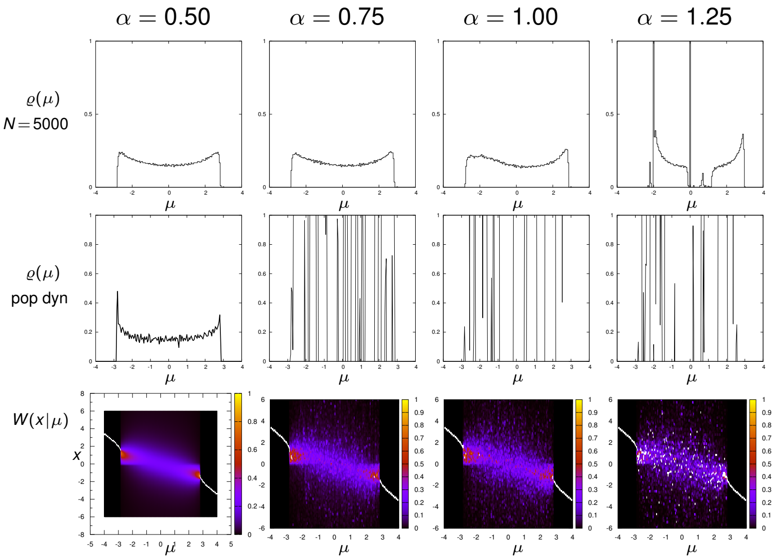

To be investigated

- Numerical precision:

  Extend simulation times of graph ensembles to MCMC steps $\sim N^3$,
  relation between μ-resolution in $x(\mu)$ and pop dynamics convergence

- Analytical technicalities:

  issues related to cut in complex plane of $\log z$,
  other saddle-point types,
  order or limits $\epsilon\downarrow 0$, $\Delta\downarrow 0$, $n_\mu \to \mathrm{i}(\Delta/\pi)\frac{\mathrm{d}}{\mathrm{d}\mu}\hat{\varrho}(\mu)$, ...

- Test cases:

  Exactly solvable models? e.g. '$p(\mathbf{c}) \propto \mathrm{e}^{\alpha \mathrm{Tr}(\mathbf{c}^3)} \prod_i \delta_{2,\sum_j c_{ij}}$
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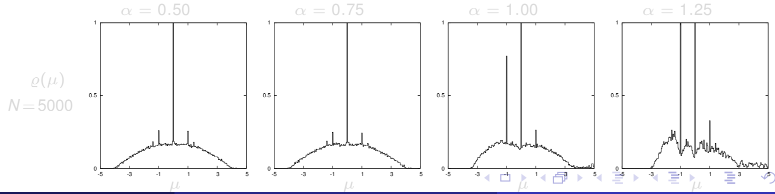
$$p(\mathbf{c}) \propto \mathrm{e}^{\alpha \mathrm{Tr}(\mathbf{c}^3)} \prod_i \delta_{k_i,\sum_j c_{ij}} \qquad p(k) = \mathrm{e}^{-\langle k\rangle} \langle k\rangle^k / k!$$



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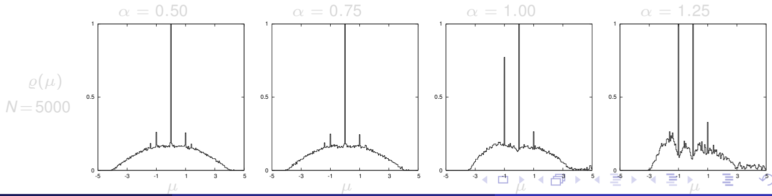

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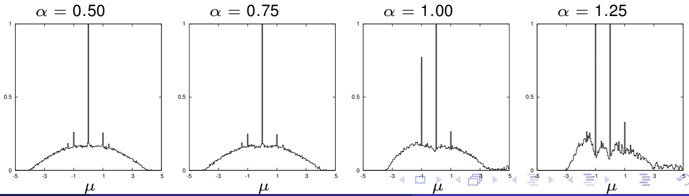
$\varrho(\mu)$
$N = 5000$



Outline

Summary

- new analytical approach to (processes on) loopy networks, based on max entropy graph ensembles characterised by *degrees and spectrum*

- replica formula for tricky constraint allows sum over graphs to be done:

$$e^{N\int d\mu\ \hat{\varrho}(\mu)\varrho(\mu|\mathbf{c})} = \lim_{\varepsilon,\Delta\downarrow 0}\prod_{\mu}\Big[Z(\mu+i\varepsilon|\mathbf{c})^{in(\mu)}\ \overline{Z(\mu+i\varepsilon|\mathbf{c})}^{\,-in(\mu)}\Big]$$

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- RS order parameter equations for *loopy* graphs interpreted as stationary state of message passing with nontrivial acceptance probabilities

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Technicalities related to cut of log Z in complex plane?
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