Approximate inference on planar graphs using Loop Calculus and Belief Propagation

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Abstract

- Exact inference on a **binary** graphical model defined on a **planar** graph is easy for graphs **without external fields** [6, 3], otherwise is intractable [1].
- We introduce novel results for **approximate inference on general planar** graphical models using the **loop calculus framework** [2].
- The loop calculus allows to express the exact partition function Z of a graphical model as a **finite sum of terms** that can be evaluated once the belief propagation (BP) solution is known.
- We develop an algorithm for the approach presented in [2] which represents an efficient truncation scheme on planar graphs and a new representation of the series in terms of Pfaffians of matrices.
- We show that the first term of the Pfaffian series can provide very accurate approximations.
- The algorithm **outperforms previous truncation schemes** of the loop series and is **competitive with other state-of-the-art methods** for approximate inference.

Belief Propagation and loop Series

We use a Forney graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ representation of a probabilistic model defined on binary variables:

- $\mathcal{V}$ is a set of nodes, where each node $a \in \mathcal{V}$ represents an interaction.
- Each edge $(a, b) \in \mathcal{E}$ represents a binary variable $\sigma_{ab} := \{\pm 1\}$.

The joint probability distribution of such a model factorizes as:

$$p(\boldsymbol{\sigma}) = Z^{-1} \prod_{a \in \mathcal{V}} f_a(\boldsymbol{\sigma}_a), \qquad Z = \sum_{\boldsymbol{\sigma}} \prod_{a \in \mathcal{V}} f_a(\boldsymbol{\sigma}_a),$$

where Z is the partition function.

- *A* **generalized loop** *or "loop" in a graph* $\mathcal{G}$ *is any subgraph* C *such that each node in* C *has degree 2 or larger.*

Given the partition function Z^{BP} obtained at a fixed point of the BP algorithm, the exact Z is related with Z^{BP} via the **loop series expansion**:

$$Z = Z^{BP} \cdot z, \qquad z = \left(1 + \sum_{C \in \mathcal{C}} r_C\right), \qquad r_C = \prod_{a \in C} \mu_{a;\bar{a}_C}, \qquad (1)$$

where $\mathcal{C}$ is the set of all the generalized loops within the graph. Each loop term r_C is a product of terms $\mu_{a,\bar{a}_C}$ associated with every node a of the loop:

$$\mu_{a;\bar{a}_C} = \frac{\sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \prod_{b \in \bar{a}_C} (\sigma_{ab} - m_{ab})}{\prod_{b \in \bar{a}_C} \sqrt{1 - m_{ab}^2}}, \qquad m_{ab} = \sum_{\sigma_{ab}} \sigma_{ab} b_{ab}(\sigma_{ab}).$$

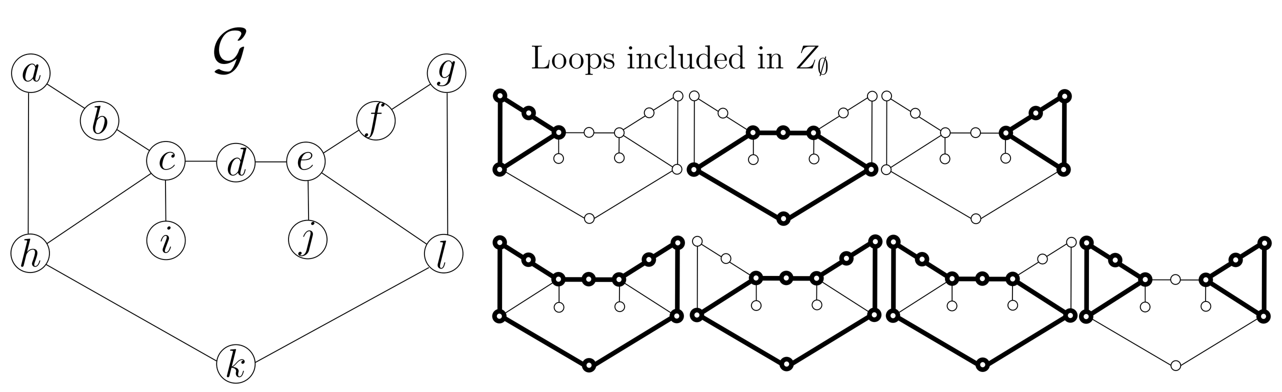
$b_a(\cdot)$ and $b_{ab}(\cdot)$ denote the BP "beliefs". $\bar{a}_C$ denotes the set of neighbors of a within C.

We consider planar graphs with all nodes of degree not larger than 3 and we denote by *triplet* a node with degree 3 in $\mathcal{G}$.

- *A* **2-regular loop** *is a loop in which all nodes have degree* exactly *2.*
- *The* **2-regular partition function** $Z_\emptyset$ *is the* **truncated** *form of* (1) *which sums all* 2*-regular loops* **only***:*

$$Z_\emptyset = Z^{BP} \cdot z_\emptyset, \qquad z_\emptyset = 1 + \sum_{C \in \mathcal{C} s.t. |\bar{a}_C| = 2, \forall a \in C} r_C.$$

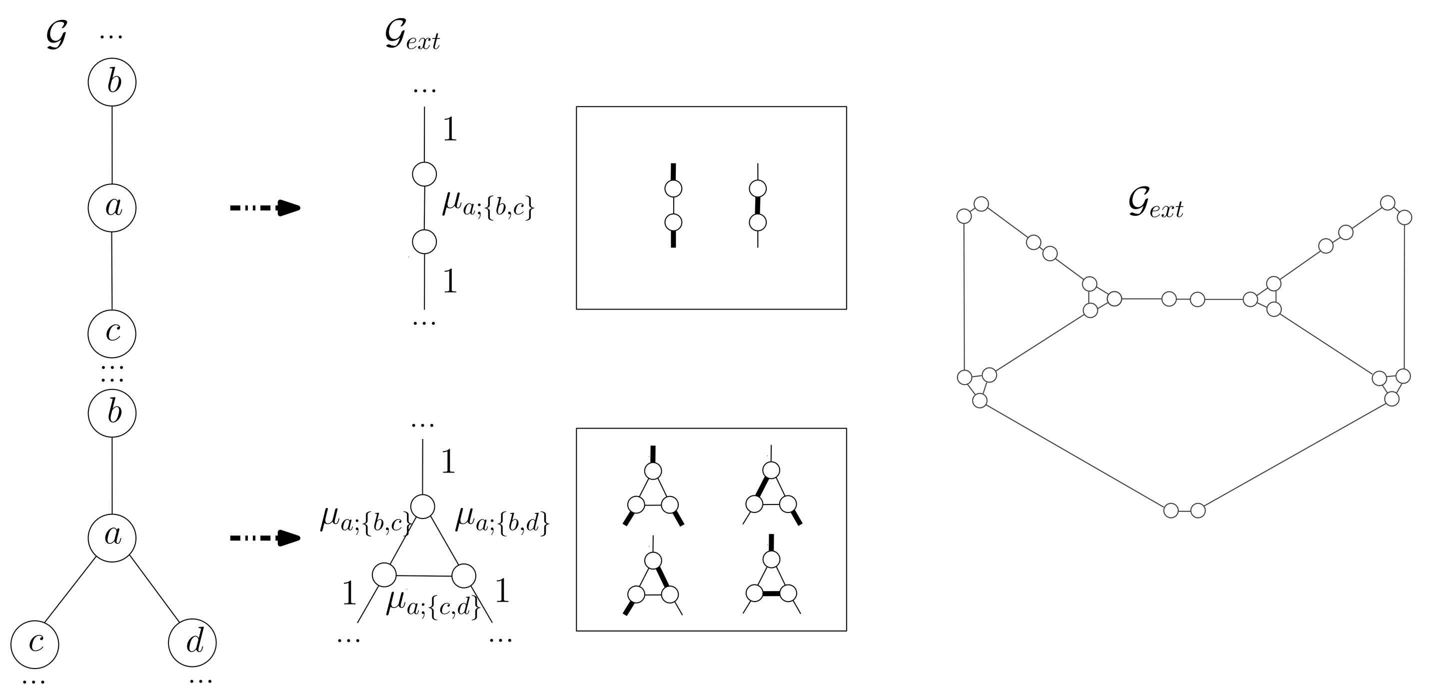
Example:



Loop series as Pfaffian series

First term: The 2-regular partition function $Z_\emptyset$. Efficient computation using perfect matchings

- The original graph $\mathcal{G}$ is extended $\mathcal{G}_{ext}$ applying Fisher's rules [3]:



- **Perfect matchings** in $\mathcal{G}_{ext}$ correspond to 2-**regular loops** in $\mathcal{G}$.
- Therefore $z_\emptyset = \sum$ weighted perfect matchings in $\mathcal{G}_{ext}$. If $\mathcal{G}$ is **planar** $\rightarrow z_\emptyset$ can be computed in time $\mathcal{O}(N^3_{\mathcal{G}_{ext}})$ using **Kasteleyn algorithm** [6] (requires the evaluation of a Determinant/Pfaffian of a matrix).
- Exact inference for the **zero exernal field** case: $Z_\emptyset = Z$.

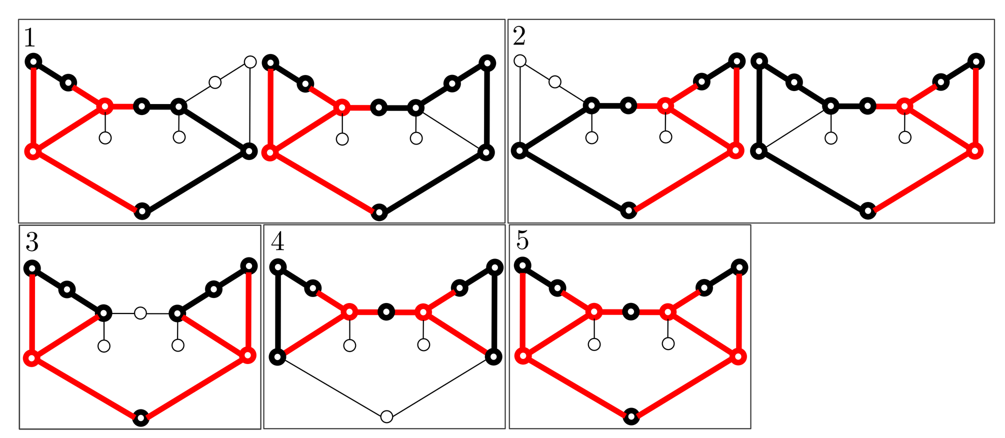
Higer order terms

- For each possible set Ψ including an **even number of triplets**, there exists a unique correspondence between **loops in** $\mathcal{G}$ including the triplets in Ψ and **perfect matchings** in another extended graph $\mathcal{G}_{ext_\Psi}$ constructed **after removal of the triplets** Ψ in $\mathcal{G}$.
- Full loop series is represented as Pfaffian series and each term Z_Ψ is tractable (requires the evaluation of a Pfaffian of a matrix):

$$z = \sum_\Psi Z_\Psi, \quad Z_\Psi = z_\Psi \prod_{a \in \Psi} \mu_{a;\bar{a}}, \quad |z_\Psi| = |\sum \text{weighted perf. matchings } \mathcal{G}_{ext_\Psi}|$$

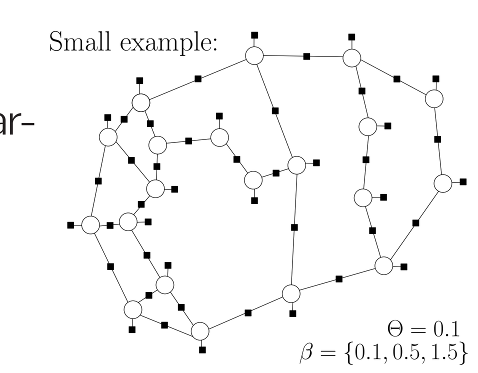
Example:

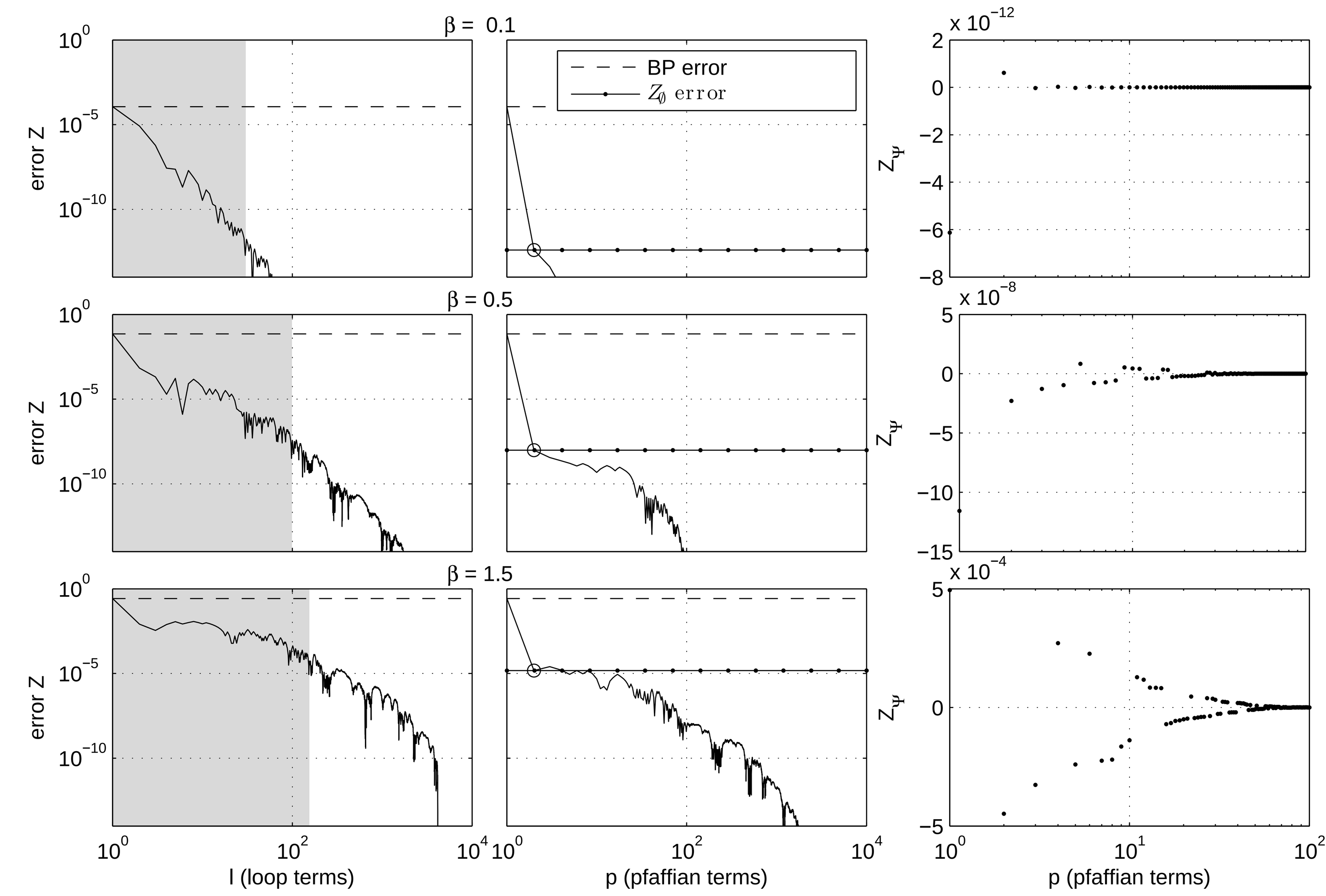
1. $\Psi = \{c, h\}$
2. $\Psi = \{e, l\}$
3. $\Psi = \{h, l\}$
4. $\Psi = \{c, e\}$
5. $\Psi = \{c, e, h, l\}$



Results (Ising model with mixed interactions)

- Loop terms can be **positive or negative**.
- Nonzero external field $\rightarrow$ The model is planar-intractable
- Pairwise interactions $\sim \mathcal{N}(0, \beta/2)$.
- Local fields $\sim \mathcal{N}(0, \beta\Theta)$.





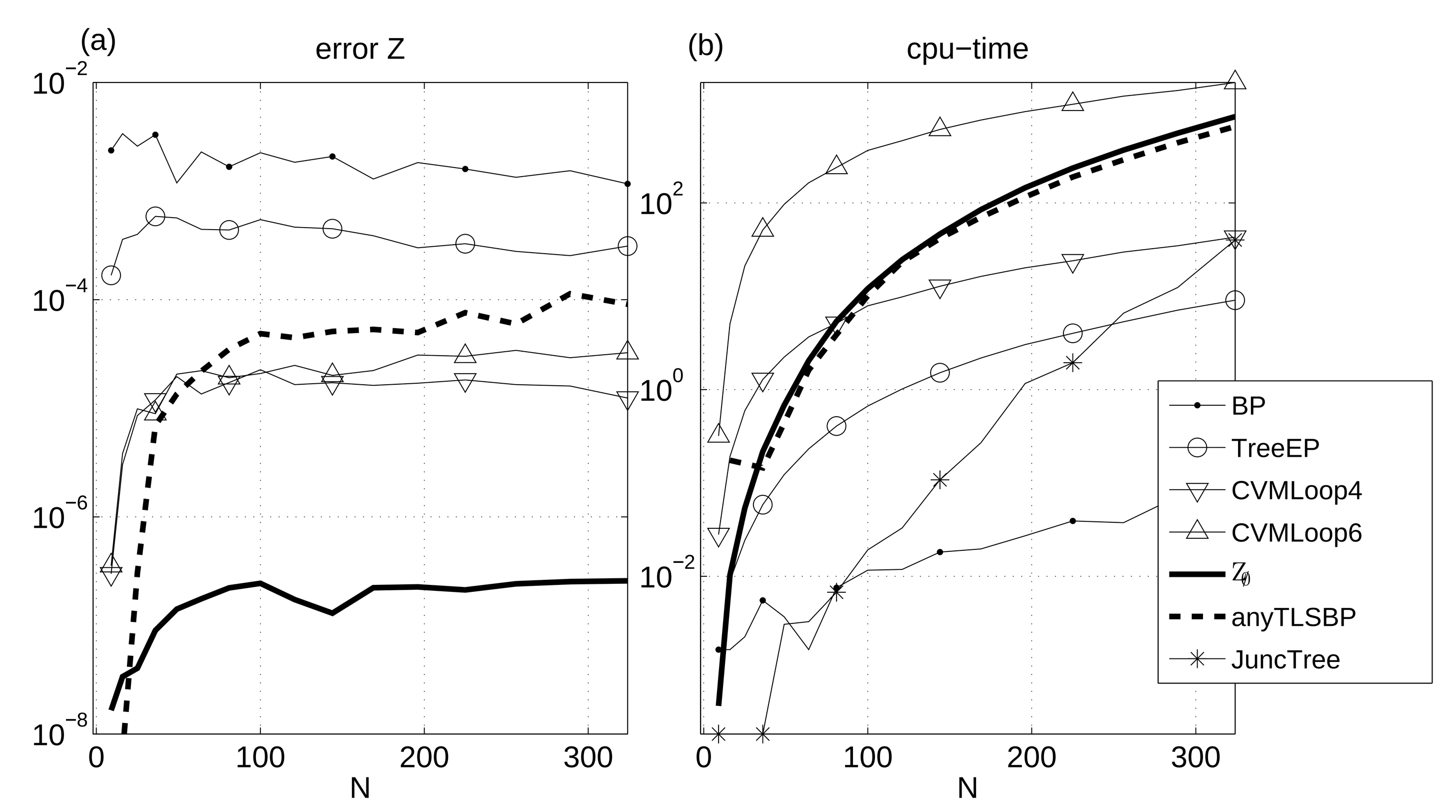
- **Without explicit search of loops**, the $z_\emptyset$ correction can give a **significant improvement** on the BP solution, even in hard problems.

Scalability on Ising grids with mixed interactions

We compare the $Z_\emptyset$ approximation with BP and the following algorithms:

- **Tree-Structured Expectation Propagation** (**TreeEP**) [7].
- **Cluster Variation Method** (**CVM-Loopk**) [5].
- **anyTLSBP** : Another truncation algorithm for the loop series [4].

Results as a function of the grid size for **strong couplings** $\beta = 1$ and **very weak local fields** $\Theta = 0.01$.



- $Z_\emptyset$ is the best approximate method at the cost of more cpu-time.
- Results are independent of the network size.

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