Modelling of Complex Real-World Systems
Part C. Applications in Physics

C1. Spin Glasses

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Spin Glasses

Magnetic materials

$N \sim 10^{24}$ interacting
atomic magnetic moments (spins),

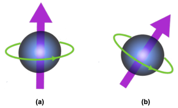
stochastic dynamics
with detailed balance

$$H = -\sum_{i<j} J_{ij} \vec{S}_i \cdot \vec{S}_j - \vec{H} \cdot \sum_i \vec{S}_i$$

if temperature T low (H-theorem!):
go to state with *low energy H*

$J_{ij} > 0$: interactions prefer $\vec{S}_i = \vec{S}_j$
(ferromagnetic order)

$J_{ij} < 0$: interactions prefer $\vec{S}_i = -\vec{S}_j$
(anti-ferromagnetic order)

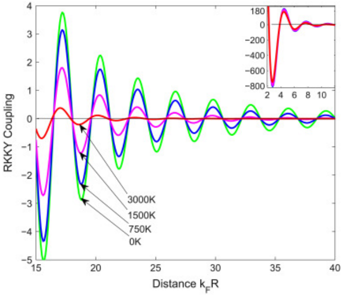


Ferromagnetic ↑ ↑ ↑ ↑ ↑ ↑ ↑	Below T_C, spins are aligned parallel in magnetic domains
Antiferromagnetic ↑ ↓ ↑ ↓ ↑ ↓ ↑	Below T_N, spins are aligned antiparallel in magnetic domains
Ferrimagnetic ↑ ↓ ↑ ↓ ↑ ↓ ↑	Below T_C, spins are aligned antiparallel but do not cancel
Paramagnetic ↑ ↓ ↑ ↑ ↓ ↑ ↓	Spins are randomly oriented (any of the others above T_C or T_N)

Spin glasses

alloys of conducting host metal with magnetic impurities in random locations (CuMn, AuFe, ...)

resulting 'RKKY' interactions J_{ij}:
oscillate with distance between $\vec{S}_i$ and $\vec{S}_j$



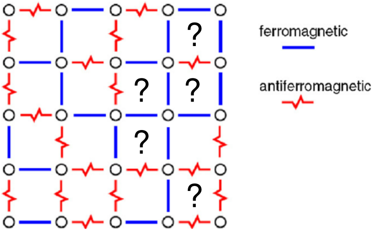
- result: <u>frustration</u>

  impossible to achieve
  $\vec{S}_i = \mathrm{sgn}(J_{ij})\vec{S}_j$ everywhere ...

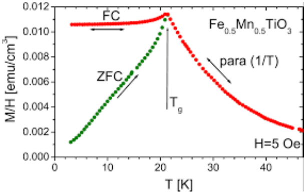
- many local minima of energy

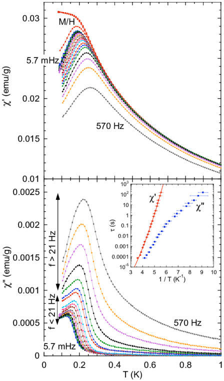
  $H = -\sum_{i<j} J_{ij}\vec{S}_i \cdot \vec{S}_j$

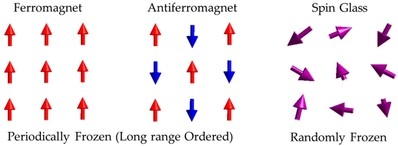
- very slow & complex dynamics, system remembers earlier states ('ageing')



- clear evidence for phase transitions but not visible in $\vec{m} = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle$ ...







$$\text{ferromagnetic order}: \quad \vec{m} = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle \neq \mathbf{0}, \qquad q = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle^2 > 0$$

$$\text{no order}: \quad \vec{m} = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle = \mathbf{0}, \qquad q = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle^2 = 0$$

$$\text{spin glass order}: \quad \vec{m} = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle = \mathbf{0}, \qquad q = \frac{1}{N} \sum_i \langle \vec{S}_i \rangle^2 > 0$$

Spin glass models

common simplifications	desired features
Ising spins: $\sigma_i = \pm 1$	large T: P phase, $m=0$, $q=0$
$H = -\sum_{i<j} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i$	large h: F phase, $m\neq 0$, $q>0$
$m = N^{-1}\sum_i \langle\sigma_i\rangle,\ q = N^{-1}\sum_i \langle\sigma_i\rangle^2$	small T, $h=0$: SG phase, $m=0$, $q>0$
random bonds, $J_{ij}>0$ and $J_{ij}<0$	T small: slow dynamics, ageing
long range, no lattice	T small: many locally stable states

- Sherrington-Kirkpatrick model (1975):

$$J_{ij} = \frac{J_0}{N} + \frac{J}{\sqrt{N}} z_{ij}, \qquad z_{ij} : \ \frac{1}{2}N(N-1) \text{ i.i.d.r.v.}, \ \ p(z) = \frac{\mathrm{e}^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$$

- Van Hemmen model (1983):
  (no slow dynamics, no ageing ...)

$$J_{ij} = \frac{1}{N}(\xi_i\eta_j + \eta_i\xi_j), \qquad \xi_i, \eta_i : \ 2N \text{ i.i.d.r.v}, \ \ p(\xi) = \frac{\mathrm{e}^{-\frac{1}{2}\xi^2}}{\sqrt{2\pi}}, \ p(\eta) = \frac{\mathrm{e}^{-\frac{1}{2}\eta^2}}{\sqrt{2\pi}}$$

- Droplet model (Fisher & Huse, 1986):
  very different setup, will not discuss here ...

Sherrington Kirkpatrick model

Equilibrium solution using replicas

$$\mathrm{D}z = (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}\mathrm{d}z$$
$$\int \mathrm{D}z\; \mathrm{e}^{xz} = \mathrm{e}^{\frac{1}{2}x^2}$$

$$\boldsymbol{\sigma} \in \{-1,1\}^N, \qquad H(\boldsymbol{\sigma}) = -\sum_{i<j}\left(\frac{J_0}{N} + \frac{J}{\sqrt{N}} z_{ij}\right)\sigma_i\sigma_j - h\sum_i \sigma_i$$

- disorder-averaged free energy density

$$\begin{aligned}
\bar{f}_N &= -\frac{1}{\beta N}\overline{\log\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]} = -\lim_{n\to 0}\frac{1}{n\beta N}\log\overline{\left[\sum_{\boldsymbol{\sigma}} \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^n} \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \overline{\mathrm{e}^{-\beta\sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}}\right] \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\frac{\beta J_0}{N}\sum_{i<j}\sum_\alpha \sigma_i^\alpha\sigma_j^\alpha + \beta h\sum_i\sum_\alpha \sigma_i^\alpha}\,\overline{\mathrm{e}^{\frac{\beta J}{\sqrt{N}}\sum_{i<j} z_{ij}\sum_\alpha \sigma_i^\alpha\sigma_j^\alpha}}\right] \\
&= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left[\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\frac{\beta J_0}{N}\sum_{i<j}\sum_\alpha \sigma_i^\alpha\sigma_j^\alpha + \beta h\sum_i\sum_\alpha \sigma_i^\alpha}\prod_{i<j}\mathrm{e}^{\frac{(\beta J)^2}{2N}(\sum_\alpha \sigma_i^\alpha\sigma_j^\alpha)^2}\right]
\end{aligned}$$

- rewrite to identify order parameters

$$\overline{f}_N = -\lim_{n\to 0}\frac{1}{n\beta N}\log\left\{\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{\frac{\beta J_0}{2N}\sum_{ij}\sum_\alpha \sigma_i^\alpha\sigma_j^\alpha+\beta h\sum_i\sum_\alpha\sigma_i^\alpha+\frac{(\beta J)^2}{4N}\sum_{ij}(\sum_\alpha\sigma_i^\alpha\sigma_j^\alpha)^2}\right\} + \lim_{n\to 0}\frac{1}{n\beta N}\left(\frac{1}{2}n\beta J_0+\frac{1}{4}n^2(\beta J)^2\right)$$

$$= -\lim_{n\to 0}\frac{1}{n\beta N}\log\left\{\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n}\mathrm{e}^{N\left[\frac{1}{2}\beta J_0\sum_\alpha(\frac{1}{N}\sum_i\sigma_i^\alpha)^2+\beta h\sum_\alpha(\frac{1}{N}\sum_i\sigma_i^\alpha)+\frac{(\beta J)^2}{4}\sum_{\alpha\beta}(\frac{1}{N}\sum_i\sigma_i^\alpha\sigma_i^\beta)^2\right]}\right\} + \mathcal{O}(N^{-1})$$

- insert for all $\alpha,\beta=1\ldots n$:

$$1=\int \mathrm{d}m_\alpha\;\delta\Big[m_\alpha-\frac{1}{N}\sum_i\sigma_i^\alpha\Big] = \int\frac{\mathrm{d}m_\alpha\mathrm{d}\hat{m}_\alpha}{2\pi/N}\;\mathrm{e}^{\mathrm{i}N\hat{m}_\alpha[m_\alpha-\frac{1}{N}\sum_i\sigma_i^\alpha]}$$

$$1=\int \mathrm{d}q_{\alpha\beta}\;\delta\Big[q_{\alpha\beta}-\frac{1}{N}\sum_i\sigma_i^\alpha\sigma_i^\beta\Big] = \int\frac{\mathrm{d}q_{\alpha\beta}\mathrm{d}\hat{q}_{\alpha\beta}}{2\pi/N}\;\mathrm{e}^{\mathrm{i}N\hat{q}_{\alpha\beta}[q_{\alpha\beta}-\frac{1}{N}\sum_i\sigma_i^\alpha\sigma_i^\beta]}$$

short-hands: $\mathbf{m}=\{m_\alpha\}$, $\hat{\mathbf{m}}=\{\hat{m}_\alpha\}$, n-dim vectors

$\mathbf{q}=\{q_{\alpha\beta}\}$, $\hat{\mathbf{q}}=\{\hat{q}_{\alpha\beta}\}$, $n\times n$ matrices

- swap order of limits $n\to 0$ and $N\to\infty$, work towards steepest descent expression

$$\lim_{N\to\infty}\bar{f}_N = -\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{n\beta N}\log\int \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{q}}\mathrm{d}\hat{\mathbf{m}}\sum_{\boldsymbol{\sigma}^1\ldots\boldsymbol{\sigma}^n} \mathrm{e}^{-\mathrm{i}\sum_i\sum_\alpha \hat{m}_\alpha\sigma_i^\alpha-\mathrm{i}\sum_i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sigma_i^\alpha\sigma_i^\beta}$$
$$\times\, \mathrm{e}^{N\left[\mathrm{i}\sum_\alpha\hat{m}_\alpha m_\alpha+\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta}+\frac{1}{2}\beta J_0\sum_\alpha m_\alpha^2+\beta h\sum_\alpha m_\alpha+\frac{(\beta J)^2}{4}\sum_{\alpha\beta}q_{\alpha\beta}^2\right]}$$
$$= -\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{n\beta N}\log\int \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{q}}\mathrm{d}\hat{\mathbf{m}}\Big(\sum_{\sigma^1\ldots\sigma^n}\mathrm{e}^{-\mathrm{i}\sum_\alpha\hat{m}_\alpha\sigma^\alpha-\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sigma^\alpha\sigma^\beta}\Big)^N$$
$$\times\, \mathrm{e}^{N\left[\mathrm{i}\sum_\alpha\hat{m}_\alpha m_\alpha+\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta}+\frac{1}{2}\beta J_0\sum_\alpha m_\alpha^2+\beta h\sum_\alpha m_\alpha+\frac{(\beta J)^2}{4}\sum_{\alpha\beta}q_{\alpha\beta}^2\right]}$$
$$= -\lim_{n\to 0}\lim_{N\to\infty}\frac{1}{n\beta N}\log\int \mathrm{d}\mathbf{q}\mathrm{d}\mathbf{m}\mathrm{d}\hat{\mathbf{q}}\mathrm{d}\hat{\mathbf{m}}\,\mathrm{e}^{N\Psi[\mathbf{m},\hat{\mathbf{m}},\mathbf{q},\hat{\mathbf{q}}]}$$
$$= -\lim_{n\to 0}\frac{1}{n\beta}\mathrm{extr}_{\mathbf{m},\mathbf{q},\hat{\mathbf{m}},\hat{\mathbf{q}}}\Psi[\mathbf{m},\hat{\mathbf{m}},\mathbf{q},\hat{\mathbf{q}}]$$

$$\Psi[\mathbf{m},\hat{\mathbf{m}},\mathbf{q},\hat{\mathbf{q}}] = \mathrm{i}\sum_{\alpha=1}^n\hat{m}_\alpha m_\alpha+\mathrm{i}\sum_{\alpha\beta=1}^n\hat{q}_{\alpha\beta}q_{\alpha\beta}+\beta\sum_{\alpha=1}^n(\frac{1}{2}J_0m_\alpha^2+hm_\alpha)$$
$$+\frac{1}{4}(\beta J)^2\sum_{\alpha\beta=1}^n q_{\alpha\beta}^2+\log\Big(\sum_{\sigma^1\ldots\sigma^n}\mathrm{e}^{-\mathrm{i}\sum_\alpha\hat{m}_\alpha\sigma^\alpha-\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sigma^\alpha\sigma^\beta}\Big)$$

Analysis of replica saddle point equations

new notation: $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)$

$$\Psi[\ldots] = \mathrm{i}\sum_{\alpha}\hat{m}_\alpha m_\alpha + \mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta} + \beta\sum_{\alpha}(\frac{1}{2}J_0 m_\alpha^2 + hm_\alpha) + \frac{1}{4}(\beta J)^2\sum_{\alpha\beta}q_{\alpha\beta}^2 + \log\sum_{\boldsymbol{\sigma}}\mathrm{e}^{-\mathrm{i}\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}}$$

- saddle point eqns

$$\frac{\partial\Psi}{\partial m_\alpha}=0,\ \frac{\partial\Psi}{\partial q_{\alpha\gamma}}=0: \quad \beta J_0 m_\alpha + \beta h + \mathrm{i}\hat{m}_\alpha = 0, \quad \frac{1}{2}(\beta J)^2 q_{\alpha\gamma} + \mathrm{i}\hat{q}_{\alpha\gamma} = 0$$

$$\frac{\partial\Psi}{\partial\hat{m}_\alpha}=0: \quad \mathrm{i}m_\alpha - \mathrm{i}\frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\,\mathrm{e}^{-\mathrm{i}\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{-\mathrm{i}\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}}} = 0$$

$$\frac{\partial\Psi}{\partial\hat{q}_{\alpha\gamma}}=0: \quad \mathrm{i}q_{\alpha\gamma} - \mathrm{i}\frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\sigma_\gamma\,\mathrm{e}^{-\mathrm{i}\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{-\mathrm{i}\hat{\mathbf{m}}\cdot\boldsymbol{\sigma}-\mathrm{i}\boldsymbol{\sigma}\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}}} = 0$$

- eliminate $\hat{\mathbf{m}}, \hat{\mathbf{q}}$

$$m_\alpha = \frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\,\mathrm{e}^{\beta\sum_\lambda(J_0 m_\lambda + h)\sigma_\lambda + \frac{1}{2}(\beta J)^2\sum_{\lambda\neq\zeta}\sigma_\lambda q_{\lambda\zeta}\sigma_\zeta}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\beta\sum_\lambda(J_0 m_\lambda + h)\sigma_\lambda + \frac{1}{2}(\beta J)^2\sum_{\lambda\neq\zeta}\sigma_\lambda q_{\lambda\zeta}\sigma_\zeta}}$$

$$q_{\alpha\gamma} = \frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\sigma_\gamma\,\mathrm{e}^{\beta\sum_\lambda(J_0 m_\lambda + h)\sigma_\lambda + \frac{1}{2}(\beta J)^2\sum_{\lambda\neq\zeta}\sigma_\lambda q_{\lambda\zeta}\sigma_\zeta}}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^{\beta\sum_\lambda(J_0 m_\lambda + h)\sigma_\lambda + \frac{1}{2}(\beta J)^2\sum_{\lambda\neq\zeta}\sigma_\lambda q_{\lambda\zeta}\sigma_\zeta}}$$

Replica symmetry

Intuition for the form of the solution

- for all (α,β): $q_{\alpha\alpha}=1,\ \ q_{\alpha\beta}=q_{\beta\alpha}$

- $\beta=0$,
  infinite noise
  $$m_\alpha=\frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha \mathrm{e}^0}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^0}=0,\qquad q_{\alpha\gamma}=\frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\sigma_\gamma \mathrm{e}^0}{\sum_{\boldsymbol{\sigma}}\mathrm{e}^0}=\delta_{\alpha\gamma}$$

- $0<\beta\ll 1$, high noise,
  equation for m_α:

$$\begin{aligned} m_\alpha &= \frac{\sum_{\boldsymbol{\sigma}}\sigma_\alpha\left[1+\beta\sum_\lambda(J_0m_\lambda+h)\sigma_\lambda+\mathcal{O}(\beta^2)\right]}{\sum_{\boldsymbol{\sigma}}\left[1+\beta\sum_\lambda(J_0m_\lambda+h)\sigma_\lambda+\mathcal{O}(\beta^2)\right]} \\ &= \frac{\beta\sum_\lambda(J_0m_\lambda+h)\sum_{\boldsymbol{\sigma}}\sigma_\alpha\sigma_\lambda+\mathcal{O}(\beta^2)}{2^n+\mathcal{O}(\beta^2)}=\beta(J_0m_\alpha+h)+\mathcal{O}(\beta^2) \end{aligned}$$

so: $m_\alpha=\beta h+\mathcal{O}(\beta^2)$ for all α

- $0 < \beta \ll 1$, high noise,
  equation for $q_{\alpha\gamma}$, $\alpha \neq \gamma$:

$$q_{\alpha\gamma} = \frac{\sum_{\boldsymbol{\sigma}} \sigma_\alpha \sigma_\gamma \left[1 + \beta \sum_\lambda (J_0 m_\lambda + h)\sigma_\lambda + \mathcal{O}(\beta^2)\right]}{\sum_{\boldsymbol{\sigma}} \left[1 + \beta \sum_\lambda (J_0 m_\lambda + h)\sigma_\lambda + \mathcal{O}(\beta^2)\right]} = \delta_{\alpha\gamma} + \mathcal{O}(\beta^2)$$

need $\mathcal{O}(\beta^3) \ldots$

next order, use $m_\alpha = \beta h + \mathcal{O}(\beta^2)$ ...

$$\begin{aligned} q_{\alpha\gamma} &= \frac{\sum_{\boldsymbol{\sigma}} \sigma_\alpha \sigma_\gamma \left[1 + \frac{1}{2}(\beta h)^2 \sum_{\lambda\neq\zeta} \sigma_\lambda \sigma_\zeta + \frac{1}{2}(\beta J)^2 \sum_{\lambda \neq \zeta} q_{\lambda\zeta} \sigma_\lambda \sigma_\zeta + \mathcal{O}(\beta^3)\right]}{\sum_{\boldsymbol{\sigma}} \left[1 + \frac{1}{2} n (\beta h)^2 + \mathcal{O}(\beta^3)\right]} \\ &= \frac{1}{2}\beta^2 \sum_{\lambda\neq\zeta} (h^2 + J^2 q_{\lambda\zeta})\, 2^{-n} \sum_{\boldsymbol{\sigma}} \sigma_\alpha \sigma_\gamma \sigma_\lambda \sigma_\zeta + \mathcal{O}(\beta^3) \\ &= \frac{1}{2}\beta^2 \sum_{\lambda\neq\zeta} (h^2 + J^2 q_{\lambda\zeta})\, (\delta_{\alpha\lambda}\delta_{\gamma\zeta} + \delta_{\alpha\zeta}\delta_{\gamma\lambda}) + \mathcal{O}(\beta^3) \\ &= \beta^2 (h^2 + J^2 q_{\alpha\gamma}) + \mathcal{O}(\beta^3) \end{aligned}$$

so: $q_{\alpha\beta} = \delta_{\alpha\beta} + (1 - \delta_{\alpha\beta})\beta^2 h^2 + \mathcal{O}(\beta^3)$ for all (α, β)

conclusion: for high temperatures
solution is of the following form

for all $\alpha,\beta=1\ldots n$: $\quad m_\alpha=m, \quad q_{\alpha\beta}=\delta_{\alpha\beta}+q(1-\delta_{\alpha\beta})$

symmetric under <u>all</u> permutations of $(1,\ldots,n)$,
$\rightarrow$ replica symmetric solution (RS)

- what does RS mean intuitively?
  are solutions $\{m_\alpha, q_{\alpha\beta}\}$ always of RS form?
  if no, how to find solutions with broken replica symmetry (RSB)?

- need physical interpretation!
  use alternative form(s) of replica identity:

$$\overline{\langle g(\boldsymbol{\sigma})\rangle} = \lim_{n\to 0}\frac{1}{n}\sum_{\gamma=1}^{n}\sum_{\boldsymbol{\sigma}^1}\ldots\sum_{\boldsymbol{\sigma}^n}\overline{g(\boldsymbol{\sigma}^\gamma)\mathrm{e}^{-\beta\sum_{\alpha=1}^{n}H(\boldsymbol{\sigma}^\alpha)}}$$

$$\overline{\langle\langle g(\boldsymbol{\sigma},\boldsymbol{\sigma}')\rangle\rangle} = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\alpha\neq\gamma=1}^{n}\sum_{\boldsymbol{\sigma}^1}\ldots\sum_{\boldsymbol{\sigma}^n}\overline{g(\boldsymbol{\sigma}^\alpha,\boldsymbol{\sigma}^\gamma)\mathrm{e}^{-\beta\sum_{\alpha=1}^{n}H(\boldsymbol{\sigma}^\alpha)}}$$

apply to $\quad P(m|\boldsymbol{\sigma})=\delta\Big[m-\frac{1}{N}\sum_{i=1}^{N}\sigma_i\Big], \quad P(q|\boldsymbol{\sigma},\boldsymbol{\sigma}')=\delta\Big[q-\frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'\Big]$

- repeat steps of calculation of $\overline{f}$, gives expressions in terms of saddle-point soln $\{m_\alpha, q_{\alpha\gamma}\}$:

$$\lim_{N\to\infty} \overline{\langle P(m|\boldsymbol{\sigma})\rangle} = \lim_{n\to 0} \frac{1}{n}\sum_{\alpha=1}^{n} \delta[m-m_\alpha]$$

$$\lim_{N\to\infty} \overline{\langle\langle P(q|\boldsymbol{\sigma},\boldsymbol{\sigma}')\rangle\rangle} = \lim_{n\to 0} \frac{1}{n(n-1)}\sum_{\alpha\neq\gamma=1}^{n} \delta[q-q_{\alpha\gamma}]$$

(see exercises)

- ergodic systems in equilibrium: fluctuations in quantities like $N^{-1}\sum_i \sigma_i$ or $N^{-1}\sum_i \sigma_i\sigma_i'$ scale as $\mathcal{O}(N^{-1/2})$

$$\lim_{N\to\infty} \langle P(m|\boldsymbol{\sigma})\rangle = \lim_{N\to\infty} \left\langle \delta\left[m - \frac{1}{N}\sum_{i=1}^{N}\sigma_i\right]\right\rangle = \delta\left[m - \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i\rangle\right]$$

$$\lim_{N\to\infty} \langle\langle P(q|\boldsymbol{\sigma},\boldsymbol{\sigma}')\rangle\rangle = \lim_{N\to\infty} \left\langle\left\langle \delta\left[q - \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma_i'\right]\right\rangle\right\rangle = \delta\left[q - \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i\rangle^2\right]$$

- meaning of replica symmetry (RS):
  ergodic equilibrium state with

$$\lim_{N\to\infty} \overline{\langle P(m|\boldsymbol{\sigma})\rangle} = \delta\Big[m - \frac{1}{N}\sum_{i=1}^{N}\overline{\langle\sigma_i\rangle}\Big]$$

$$\lim_{N\to\infty} \overline{\langle\langle P(q|\boldsymbol{\sigma},\boldsymbol{\sigma}')\rangle\rangle} = \delta\Big[q - \frac{1}{N}\sum_{i=1}^{N}\overline{\langle\sigma_i\rangle^2}\Big]$$

$$\forall\alpha: \quad m_\alpha = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\overline{\langle\sigma_i\rangle}, \qquad \forall\alpha\neq\gamma: \quad q_{\alpha\beta} = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\overline{\langle\sigma_i\rangle^2} = q_{\mathrm{EA}}$$

- RSB can occur !
  nontrivial breaking of ergodicity at low temperatures ...
  (De Almeida & Thouless 1978, Parisi 1978)
  (see also exercises)

Replica symmetric solution of the SK model

$m_\alpha = m$, $q_{\alpha\neq\beta} = q$, now find m and q ...

- RS saddle-point eqns for m_α,
  insert RS form and use $\exp(\frac{1}{2}x^2) = \int \mathrm{D}z\, \mathrm{e}^{xz}$

$$
\begin{aligned}
m &= \frac{\sum_{\boldsymbol{\sigma}} \sigma_\alpha \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q \sum_{\lambda\neq\zeta} \sigma_\lambda \sigma_\zeta}}{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q \sum_{\lambda\neq\zeta} \sigma_\lambda \sigma_\zeta}} \\
&= \frac{\sum_{\boldsymbol{\sigma}} \sigma_\alpha \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q [\sum_\lambda \sigma_\lambda]^2}}{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q [\sum_\lambda \sigma_\lambda]^2}} \\
&= \frac{\int \mathrm{D}z \sum_{\boldsymbol{\sigma}} \sigma_\alpha \prod_{\lambda=1}^n \mathrm{e}^{\beta(J_0 m+h+Jz\sqrt{q})\sigma_\lambda}}{\int \mathrm{D}z \sum_{\boldsymbol{\sigma}} \prod_{\lambda=1}^n \mathrm{e}^{\beta(J_0 m+h+Jz\sqrt{q})\sigma_\lambda}} \\
&= \frac{\int \mathrm{D}z\ \sinh[\beta(J_0 m+h+Jz\sqrt{q})] \cosh^{n-1}[\beta(J_0 m+h+Jz\sqrt{q})]}{\int \mathrm{D}z\ \cosh^n[\beta(J_0 m+h+Jz\sqrt{q})]}
\end{aligned}
$$

- RS saddle-point eqns for $q_{\alpha\neq\gamma}$,
  insert RS form and use $\exp(\frac{1}{2}x^2) = \int \mathrm{D}z\, \mathrm{e}^{xz}$

$$
\begin{aligned}
q &= \frac{\sum_{\boldsymbol{\sigma}} \sigma_\alpha \sigma_\gamma \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q \sum_{\lambda\neq\zeta} \sigma_\lambda\sigma_\zeta}}{\sum_{\boldsymbol{\sigma}} \mathrm{e}^{\beta(J_0 m+h)\sum_\lambda \sigma_\lambda + \frac{1}{2}(\beta J)^2 q \sum_{\lambda\neq\zeta} \sigma_\lambda\sigma_\zeta}} \\
&= \frac{\int \mathrm{D}z \sum_{\boldsymbol{\sigma}} \sigma_\alpha\sigma_\gamma \prod_{\lambda=1}^n \mathrm{e}^{\beta(J_0 m+h+Jz\sqrt{q})\sigma_\lambda}}{\int \mathrm{D}z \sum_{\boldsymbol{\sigma}} \prod_{\lambda=1}^n \mathrm{e}^{\beta(J_0 m+h+Jz\sqrt{q})\sigma_\lambda}} \\
&= \frac{\int \mathrm{D}z\ \sinh^2[\beta(J_0 m+h+Jz\sqrt{q})]\cosh^{n-2}[\beta(J_0 m+h+Jz\sqrt{q})]}{\int \mathrm{D}z\ \cosh^n[\beta(J_0 m+h+Jz\sqrt{q})]}
\end{aligned}
$$

- two coupled eqns for (m, q)

$$
\begin{aligned}
m &= \frac{\int \mathrm{D}z\ \sinh[\beta(J_0 m+h+Jz\sqrt{q})]\cosh^{n-1}[\beta(J_0 m+h+Jz\sqrt{q})]}{\int \mathrm{D}z\ \cosh^n[\beta(J_0 m+h+Jz\sqrt{q})]} \\
q &= \frac{\int \mathrm{D}z\ \sinh^2[\beta(J_0 m+h+Jz\sqrt{q})]\cosh^{n-2}[\beta(J_0 m+h+Jz\sqrt{q})]}{\int \mathrm{D}z\ \cosh^n[\beta(J_0 m+h+Jz\sqrt{q})]}
\end{aligned}
$$

$n \to 0$:

$$m = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle}, \qquad q = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i \rangle^2}$$

$$m = \int \mathrm{D}z \ \tanh[\beta(J_0 m + h + Jz\sqrt{q})], \qquad q = \int \mathrm{D}z \ \tanh^2[\beta(J_0 m + h + Jz\sqrt{q})]$$

- phases of the system for $h = 0$

paramagnetic (P) :	$m=0,\ q=0$	(no order, $\langle\sigma_i\rangle = 0$ for all i)
spin glass (SG) :	$m=0,\ q>0$	$q = \int \mathrm{D}z \ \tanh^2[\beta J z\sqrt{q})]$
ferromagnetic (F) :	$m\neq 0,\ q>0$	

- bifurcations away from $(m, q) = (0, 0)$:

$$m = \int \mathrm{D}z \ \left[\beta J_0 m + \beta J z\sqrt{q} + \mathcal{O}(m, \sqrt{q})^3\right] = \beta J_0 m + \ldots$$

$$q = \int \mathrm{D}z \ \left[\beta J_0 m + \beta J z\sqrt{q} + \mathcal{O}(m, \sqrt{q})^3\right]^2 = (\beta J_0)^2 m^2 + (\beta J)^2 q + \ldots$$

continuous P$\to$F transition: $\beta J_0 = 1$
continuous P$\to$SG transition: $\beta J = 1$

- bifurcations SG to F, i.e. of $m \neq 0$

$$q = \int \mathrm{D}z \ \tanh^2[\beta J z \sqrt{q}], \qquad \mathrm{soln}: q_{\mathrm{EA}}$$

$$m = \int \mathrm{D}z \ \tanh[\beta J z \sqrt{q_{\mathrm{EA}}}] + \beta J_0 m \int \mathrm{D}z \ [1 - \tanh^2[\beta J z \sqrt{q_{\mathrm{EA}}}] + \mathcal{O}(m^2)$$

$$= \beta J_0 m (1 - q_{\mathrm{EA}}) + \mathcal{O}(m^2) \qquad \text{SG}\to\text{F transition:} \quad \beta J_0 = (1 - q_{\mathrm{EA}})^{-1}$$

RS phase diagram of SK model

P→F: $\beta J_0 = 1$, i.e. $J_0/J = T/J$

P→SG: $\beta J = 1$, i.e. $T/J = 1$

F→SG: $\beta J_0 = (1 - q_{\mathrm{EA}})^{-1}$