

Modelling of Complex Real-World Systems

Part C. Applications in Physics

C2. Spins on Graphs

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- 1 Topological heterogeneity
- 2 Replica analysis
- 3 Saddle point equations
- 4 RS solution
- 5 Phase transitions

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Ising spins on graphs

Models so far:

force heterogeneity

all pairs of dynamical variables
have mutual interactions J_{ij}

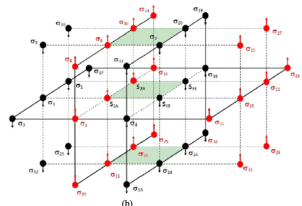
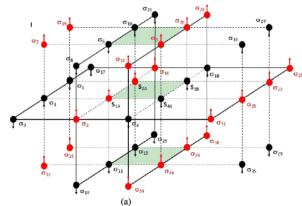
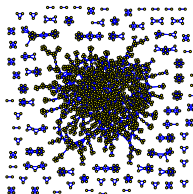
$$h_i(\sigma) = \sum_j \textcolor{red}{J}_{ij} \sigma_j + \theta_i$$

Most real world systems:
topological heterogeneity

dynamical variables interact
with *specific partners only*

$$h_i(\sigma) = \sum_j \textcolor{red}{A}_{ij} \textcolor{red}{J}_{ij} \sigma_j + \theta_i$$

$\{\textcolor{red}{A}_{ij}\}$: graph adjacency matrix



Definitions

- N Ising spins $\sigma_i \in \{-1, 1\}$,
 $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$

$$H(\boldsymbol{\sigma}) = - \sum_{i < j} A_{ij} J_{ij} \sigma_i \sigma_j$$

interactions: $J_{ij} \in \mathbb{R}$, drawn randomly from $P(J)$

topology: $p(\mathbf{A}) = C^{-1} \delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}$, $k_i(\mathbf{A}) = \sum_j A_{ij}$

$$A_{ij} = A_{ji}, \quad A_{ii} = 0$$

- disorder-averaged free energy density,
 use $\overline{\log Z} = \lim_{n \rightarrow 0} n^{-1} \log \overline{Z^n}$:

$$\begin{aligned} \bar{f} &= - \lim_{N \rightarrow \infty} \frac{1}{\beta N} \overline{\left(\log \sum_{\boldsymbol{\sigma} \in \{-1, 1\}^N} e^{-\beta H(\boldsymbol{\sigma})} \right)} \\ &= - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{\beta n N} \log \sum_{\boldsymbol{\sigma}^1 \dots \boldsymbol{\sigma}^n \in \{-1, 1\}^{nN}} \overline{e^{-\beta \sum_{\alpha=1}^n H(\boldsymbol{\sigma}^\alpha)}} \\ &= - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{\beta n N} \log \sum_{\boldsymbol{\sigma}^1 \dots \boldsymbol{\sigma}^n \in \{-1, 1\}^{nN}} \overline{e^{\beta \sum_{i < j} A_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}} \end{aligned}$$

Average over graphs

change to Erdős-Rényi measure,
to enable expansion for large N

$$p_{\text{ER}}(\mathbf{A}) = \prod_{i < j} \left[\frac{\langle k \rangle}{N} \delta_{A_{ij}, 1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{A_{ij}, 0} \right], \quad \langle k \rangle = \frac{1}{N} \sum_i k_i$$

now

$$\begin{aligned} p(\mathbf{A}) &= \frac{\delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}}{\mathcal{Z}} = \frac{\delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}}{C} \frac{p_{\text{ER}}(\mathbf{A})}{p_{\text{ER}}(\mathbf{A})} = \frac{\delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}}{C} \frac{p_{\text{ER}}(\mathbf{A})}{\left(\frac{\langle k \rangle}{N}\right)^{\sum_{i < j} A_{ij}} \left(1 - \frac{\langle k \rangle}{N}\right)^{\frac{1}{2}N(N-1) - \sum_{i < j} A_{ij}}} \\ &= \frac{\delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}}{C} \frac{p_{\text{ER}}(\mathbf{A})}{\left(\frac{\langle k \rangle}{N}\right)^{\frac{1}{2} \sum_{ij} A_{ij}} \left(1 - \frac{\langle k \rangle}{N}\right)^{\frac{1}{2}N(N-1) - \frac{1}{2} \sum_{ij} A_{ij}}} \\ &= \frac{\delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}}{C} \frac{p_{\text{ER}}(\mathbf{A})}{\left(\frac{\langle k \rangle}{N}\right)^{\frac{1}{2}N\langle k \rangle} \left(1 - \frac{\langle k \rangle}{N}\right)^{\frac{1}{2}N(N-1) - \frac{1}{2}N\langle k \rangle}} \end{aligned}$$

so

$$\sum_{\mathbf{A}} p(\mathbf{A}) \Phi(\mathbf{A}) = \frac{1}{\mathcal{Z}} \sum_{\mathbf{A}} p_{\text{ER}}(\mathbf{A}) \delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})} \Phi(\mathbf{A}) \quad \mathcal{Z} = \sum_{\mathbf{A}} p_{\text{ER}}(\mathbf{A}) \delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})}$$

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Replica analysis

- average over graphs
and over interactions

$$\sigma_i = (\sigma_i^1, \dots, \sigma_i^n), \quad \delta_{k\ell} = \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} e^{i\omega(k-\ell)}$$

$$\begin{aligned}
 & \overline{e^{\beta \sum_{i<j} A_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}} \\
 &= \sum_{\mathbf{A}} \rho(\mathbf{A}) \prod_{i<j} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \\
 &= \frac{1}{\mathcal{Z}} \sum_{\mathbf{A}} \rho_{\text{ER}}(\mathbf{A}) \delta_{\mathbf{k}, \mathbf{k}(\mathbf{A})} \prod_{i<j} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \\
 &= \frac{1}{\mathcal{Z}} \sum_{\mathbf{A}} \rho_{\text{ER}}(\mathbf{A}) \left[\prod_i \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} e^{i\omega(k_i - \sum_j A_{ij})} \right] \prod_{i<j} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \sum_{\mathbf{A}} \rho_{\text{ER}}(\mathbf{A}) e^{-i \sum_{ij} \omega_i A_{ij}} \prod_{i<j} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \sum_{\mathbf{A}} \rho_{\text{ER}}(\mathbf{A}) \prod_{i<j} \left[e^{-i(\omega_i + \omega_j) A_{ij}} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \right]
 \end{aligned}$$

- insert Erdős-Rényi probabilities

$$\begin{aligned}
 & \overline{e^{\beta \sum_{i<j} A_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}} \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \sum_{\mathbf{A}} \rho_{\text{ER}}(\mathbf{A}) \prod_{i<j} \left[e^{-i(\omega_i + \omega_j) A_{ij}} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \right] \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \sum_{\mathbf{A}} \prod_{i<j} \left[\left[\frac{\langle k \rangle}{N} \delta_{A_{ij},1} + \left(1 - \frac{\langle k \rangle}{N}\right) \delta_{A_{ij},0} \right] e^{-i(\omega_i + \omega_j) A_{ij}} \int dJ P(J) e^{\beta J A_{ij} \sigma_i \cdot \sigma_j} \right] \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \prod_{i<j} \left[\left(1 - \frac{\langle k \rangle}{N}\right) \cdot 1 + \frac{\langle k \rangle}{N} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \sigma_i \cdot \sigma_j} \right] \\
 &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k}} \prod_{i<j} \left[1 + \frac{\langle k \rangle}{N} \left(e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \sigma_i \cdot \sigma_j} - 1 \right) \right]
 \end{aligned}$$

Exploit graph sparseness

- $L_{ij} = \mathcal{O}(1)$, expand for large N

$$\prod_{i < j} \left[1 + \frac{1}{N} L_{ij} \right] = \prod_{i < j} e^{\log[1 + \frac{1}{N} L_{ij}]} = e^{\sum_{i < j} [\frac{1}{N} L_{ij} + \mathcal{O}(N^{-2})]} = e^{N^{-1} \sum_{i < j} L_{ij} + \mathcal{O}(1)}$$

$$\begin{aligned} & \overline{e^{\beta \sum_{i < j} A_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}} \\ &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{N} \sum_{i < j} \left[e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1} \right] + \mathcal{O}(1)} \\ &= \frac{1}{\mathcal{Z}} \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} \left[e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - 1} \right] + \mathcal{O}(1)} \\ & \beta = 0 : \quad \mathcal{Z} = \int_{-\pi}^{\pi} \frac{d\omega}{(2\pi)^N} e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} \left[e^{-i(\omega_i + \omega_j)} - 1 \right] + \mathcal{O}(1)} \end{aligned}$$

hence

$$\overline{e^{\beta \sum_{i < j} A_{ij} J_{ij} \sum_{\alpha=1}^n \sigma_i^\alpha \sigma_j^\alpha}} = \frac{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j} + \mathcal{O}(1)}}{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} + \mathcal{O}(1)}}$$

Disorder-averaged free energy density

use previous result

$$\begin{aligned}
 -\beta \bar{f} &= \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{nN} \log \sum_{\sigma_1 \dots \sigma_N} \frac{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \sigma_i \cdot \sigma_j} + \mathcal{O}(1)}}{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} + \mathcal{O}(1)}} \\
 &= \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{nN} \log \left\{ 2^{nN} \left\langle \frac{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \sigma_i \cdot \sigma_j} + \mathcal{O}(1)}}{\int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} + \mathcal{O}(1)}} \right\rangle_{\{\sigma_i\}} \right\} \\
 &= \log 2 + \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{nN} \log \left\langle \int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \sigma_i \cdot \sigma_j} + \mathcal{O}(1)} \right\rangle_{\{\sigma_i\}} \\
 &\quad - \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{nN} \log \int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} + \mathcal{O}(1)} \\
 &= \log 2 + \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} \frac{1}{n} [\Phi(\beta) - \Phi(0)]
 \end{aligned}$$

Left to do

$$\Phi(\beta) = \frac{1}{N} \log \left\langle \int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)} \int dJ P(J) e^{\beta J \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j} \right\rangle_{\{\boldsymbol{\sigma}_i\}}$$

notes:

- link with graph ensemble

$$\Phi(0) = \frac{1}{N} \log \int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k} + \frac{\langle k \rangle}{2N} \sum_{ij} e^{-i(\omega_i + \omega_j)}} = \log(2\pi) + \frac{1}{2} \langle k \rangle + \frac{1}{N} \log \mathcal{Z}$$

- all site-dependent variables appear in quantity of the form

$$\frac{1}{N} \sum_{ij} G(\omega_i, \boldsymbol{\sigma}_i; \omega_j, \boldsymbol{\sigma}_j)$$

Order parameters

• define
$$\mathcal{P}(\boldsymbol{\sigma}, \omega | \{\boldsymbol{\sigma}_i, \omega_i\}) = \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i)$$

$$\begin{aligned} \Phi(\beta) = & \frac{1}{N} \log \left\langle \int_{-\pi}^{\pi} d\omega e^{i\omega \cdot \mathbf{k}} \right. \\ & \times e^{\frac{1}{2} \langle k \rangle N \int d\omega d\omega' \sum_{\boldsymbol{\sigma} \boldsymbol{\sigma}'} \mathcal{P}(\boldsymbol{\sigma}, \omega | \dots) \mathcal{P}(\boldsymbol{\sigma}', \omega' | \dots) e^{-i(\omega + \omega')} \int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'} } \left. \right\rangle_{\{\boldsymbol{\sigma}_i\}} \end{aligned}$$

• introduce for *each* $(\boldsymbol{\sigma}, \omega)$

$$\begin{aligned} 1 &= \int d\mathcal{P}(\boldsymbol{\sigma}, \omega) \delta \left[\mathcal{P}(\boldsymbol{\sigma}, \omega) - \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i) \right] \\ &= \int \frac{d\mathcal{P}(\boldsymbol{\sigma}, \omega) d\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)}{2\pi} e^{i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \left[\mathcal{P}(\boldsymbol{\sigma}, \omega) - \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i) \right]} \end{aligned}$$

discretise ω ,

$$\hat{\mathcal{P}}(\dots) \rightarrow \hat{\mathcal{P}}(\dots) N \Delta\omega$$

$$1 = \int \frac{d\mathcal{P}(\boldsymbol{\sigma}, \omega) d\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)}{2\pi / N \Delta\omega} e^{iN\Delta\omega \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \left[\mathcal{P}(\boldsymbol{\sigma}, \omega) - \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i) \right]}$$

• use $\Delta\omega \sum_{\omega} \rightarrow \int d\omega$:

$$\begin{aligned}
 1 &= \lim_{\Delta\omega \rightarrow 0} \prod_{\omega, \boldsymbol{\sigma}} \int \frac{d\mathcal{P}(\boldsymbol{\sigma}, \omega) d\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)}{2\pi/N\Delta\omega} e^{iN\Delta\omega \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \left[\mathcal{P}(\boldsymbol{\sigma}, \omega) - \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i) \right]} \\
 &= \lim_{\Delta\omega \rightarrow 0} \int \left[\prod_{\omega, \boldsymbol{\sigma}} \frac{d\mathcal{P}(\boldsymbol{\sigma}, \omega) d\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)}{2\pi/N\Delta\omega} \right] e^{iN\Delta\omega \sum_{\omega, \boldsymbol{\sigma}} \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \left[\mathcal{P}(\boldsymbol{\sigma}, \omega) - \frac{1}{N} \sum_i \delta_{\boldsymbol{\sigma}, \boldsymbol{\sigma}_i} \delta(\omega - \omega_i) \right]} \\
 &= \int \{d\mathcal{P}d\hat{\mathcal{P}}\} e^{iN \sum_{\boldsymbol{\sigma}} \int d\omega \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \mathcal{P}(\boldsymbol{\sigma}, \omega) - i \sum_i \hat{\mathcal{P}}(\boldsymbol{\sigma}_i, \omega_i)}
 \end{aligned}$$

with short hand (path integral measure):

$$\{d\mathcal{P}d\hat{\mathcal{P}}\} = \lim_{\Delta\omega \rightarrow 0} \prod_{\omega, \boldsymbol{\sigma}} [\mathcal{P}(\boldsymbol{\sigma}, \omega) d\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) N\Delta\omega / 2\pi]$$

• result:

factorisation over sites!

$$\begin{aligned}
 \Phi(\beta) &= \frac{1}{N} \log \int \{d\mathcal{P}d\hat{\mathcal{P}}\} e^{iN \sum_{\boldsymbol{\sigma}} \int d\omega \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \mathcal{P}(\boldsymbol{\sigma}, \omega)} \\
 &\quad \times e^{\frac{1}{2} \langle k \rangle N \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \int d\omega d\omega' \mathcal{P}(\boldsymbol{\sigma}, \omega) \mathcal{P}(\boldsymbol{\sigma}', \omega') e^{-i(\omega + \omega')} \int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'}} \\
 &\quad \times \prod_i \left\langle \int_{-\pi}^{\pi} d\omega_i e^{i\omega_i k_i - i\hat{\mathcal{P}}(\boldsymbol{\sigma}_i, \omega_i)} \right\rangle_{\boldsymbol{\sigma}_i}
 \end{aligned}$$

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Saddle point equations

- simplify

$$\begin{aligned} \prod_i \left\langle \int_{-\pi}^{\pi} d\omega_i e^{i\omega_i k_i - i\hat{\mathcal{P}}(\boldsymbol{\sigma}_i, \omega_i)} \right\rangle_{\boldsymbol{\sigma}_i} &= e^{\sum_i \log \langle \int_{-\pi}^{\pi} d\omega e^{i\omega k_i - i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)} \rangle_{\boldsymbol{\sigma}}} \\ &= e^{N \sum_k p(k) \log \langle \int_{-\pi}^{\pi} d\omega e^{i\omega k - i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)} \rangle_{\boldsymbol{\sigma}}} + \mathcal{O}(\sqrt{N}) \end{aligned}$$

$$\lim_{N \rightarrow \infty} \Phi(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \int \{d\mathcal{P} d\hat{\mathcal{P}}\} e^{N\psi[\mathcal{P}, \hat{\mathcal{P}}] + \mathcal{O}(\sqrt{N})} = \text{extr}_{\{\mathcal{P}, \hat{\mathcal{P}}\}} \Psi_{\beta}[\mathcal{P}, \hat{\mathcal{P}}]$$

$$\begin{aligned} \Psi_{\beta}[\mathcal{P}, \hat{\mathcal{P}}] &= i \sum_{\boldsymbol{\sigma}} \int d\omega \hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) \mathcal{P}(\boldsymbol{\sigma}, \omega) + \sum_k p(k) \log \left\langle \int_{-\pi}^{\pi} d\omega e^{i\omega k - i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)} \right\rangle_{\boldsymbol{\sigma}} \\ &\quad + \frac{1}{2} \langle k \rangle \sum_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} \int d\omega d\omega' \mathcal{P}(\boldsymbol{\sigma}, \omega) \mathcal{P}(\boldsymbol{\sigma}', \omega') e^{-i(\omega + \omega')} \int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'} \end{aligned}$$

- extrema

$$\begin{aligned} \frac{\delta \Psi}{\delta \mathcal{P}} = 0 : \quad & i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega) = -\langle k \rangle e^{-i\omega} \sum_{\boldsymbol{\sigma}'} \int d\omega' \mathcal{P}(\boldsymbol{\sigma}', \omega') e^{-i\omega'} \int dJ P(J) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'} \\ \frac{\delta \Psi}{\delta \hat{\mathcal{P}}} = 0 : \quad & \mathcal{P}(\boldsymbol{\sigma}, \omega) = \sum_k p(k) \frac{e^{i\omega k - i\hat{\mathcal{P}}(\boldsymbol{\sigma}, \omega)}}{\sum_{\boldsymbol{\sigma}'} \int_{-\pi}^{\pi} d\omega' e^{i\omega' k - i\hat{\mathcal{P}}(\boldsymbol{\sigma}', \omega')}} \end{aligned}$$

- new quantities and short-hands

$$\mathcal{D}(\sigma) = \int_{-\pi}^{\pi} d\omega \mathcal{P}(\sigma, \omega) e^{-i\omega}, \quad \hat{\mathcal{P}}(\sigma, \phi) = i\langle k \rangle e^{-i\omega} \gamma(\sigma), \quad \int dJ P(J) \dots = \langle \dots \rangle_J$$

new eqns: $\gamma(\sigma) = \sum_{\sigma'} \mathcal{D}(\sigma') \langle e^{\beta J \sigma \cdot \sigma'} \rangle_J$

$$\mathcal{D}(\sigma) = \sum_k p(k) \frac{\int_{-\pi}^{\pi} d\omega e^{i\omega(k-1) + \langle k \rangle e^{-i\omega} \gamma(\sigma)}}{\sum_{\sigma'} \int_{-\pi}^{\pi} d\omega e^{i\omega k + \langle k \rangle e^{-i\omega} \gamma(\sigma')}} \quad \gamma(\sigma) = \sum_{\sigma'} \mathcal{D}(\sigma') \langle e^{\beta J \sigma \cdot \sigma'} \rangle_J$$

- do ω -integrals:

$$\begin{aligned} \int_{-\pi}^{\pi} d\omega e^{i\omega k + \langle k \rangle e^{-i\omega} \gamma} &= \sum_{\ell \geq 0} \frac{\langle k \rangle^\ell \gamma^\ell}{\ell!} \int_{-\pi}^{\pi} d\omega e^{i\omega k - i\omega \ell} \\ &= 2\pi \sum_{\ell \geq 0} \frac{\langle k \rangle^\ell \gamma^\ell}{\ell!} \delta_{k\ell} = \begin{cases} 2\pi \langle k \rangle^k \gamma^k / k! & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases} \end{aligned}$$

$$\mathcal{D}(\sigma) = \sum_{k \geq 0} p(k) \frac{k}{\langle k \rangle} \frac{\gamma^{k-1}(\sigma)}{\sum_{\sigma'} \gamma^k(\sigma')}, \quad \gamma(\sigma) = \sum_{\sigma'} \mathcal{D}(\sigma') \langle e^{\beta J \sigma \cdot \sigma'} \rangle_J$$

use saddle point eqns

$$\begin{aligned}\lim_{N \rightarrow \infty} \Phi(\beta) &= \text{extr}_{\{\mathcal{P}, \hat{\mathcal{P}}\}} \Psi_{\beta}[\mathcal{P}, \hat{\mathcal{P}}] \\ &= \log(2\pi) - \frac{1}{2} \langle k \rangle + \sum_k p(k) \log \langle \gamma^k(\sigma) \rangle_{\sigma} + \sum_k p(k) \log [\langle k \rangle^k / k!]\end{aligned}$$

$$\beta = 0: \quad \gamma(\sigma) = 1, \quad \mathcal{D}(\sigma) = 2^{-n}$$

$$\lim_{N \rightarrow \infty} \Phi(0) = \log(2\pi) - \frac{1}{2} \langle k \rangle + \sum_k p(k) \log [\langle k \rangle^k / k!]$$

• free energy

$$\begin{aligned}-\beta \bar{f} &= \log 2 + \lim_{N \rightarrow \infty} \lim_{n \rightarrow 0} n^{-1} [\Phi(\beta) - \Phi(0)] \\ &= \lim_{n \rightarrow 0} \frac{1}{n} \sum_k p(k) \log \langle \gamma^k(\sigma) \rangle_{\sigma}\end{aligned}$$

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Replica symmetric solution

$\mathcal{D}(\sigma)$ invariant under all permutations of $\{1, \dots, n\}$

$$\mathcal{D}(\sigma) = \int dh \mathcal{D}(h) \frac{e^{\beta h \sum_{\alpha} \sigma_{\alpha}}}{[2 \cosh(\beta h)]^n}$$

• work out $\gamma(\sigma)$

$$\begin{aligned} \gamma(\sigma) &= \int \frac{dh \mathcal{D}(h)}{[2 \cosh(\beta h)]^n} \left\langle \sum_{\sigma'} e^{\beta h \sum_{\alpha} \sigma'_{\alpha} + \beta J \sigma \cdot \sigma'} \right\rangle_J \\ &= \int \frac{dh \mathcal{D}(h)}{\cosh^n(\beta h)} \left\langle \prod_{\alpha=1}^n \cosh[\beta(h + J\sigma_{\alpha})] \right\rangle_J \end{aligned}$$

• any $F(\sigma = \pm 1)$:

$$F(\sigma) = a e^{b\sigma} : \quad a = \sqrt{F(1)F(-1)}, \quad b = \log \sqrt{F(1)/F(-1)}$$

$$\cosh[\beta(h + J\sigma)] = \sqrt{\cosh[\beta(h + J)] \cosh[\beta(h - J)]} e^{\frac{1}{2} \sigma \log[\cosh[\beta(h + J)] / \cosh[\beta(h - J)]]}$$

• any A, B :

$$\frac{1}{2} \log[\cosh(A+B)/\cosh(A-B)] = \operatorname{atanh}[\tanh(A) \tanh(B)]$$

$$\gamma(\sigma) = \int dh \mathcal{D}(h) \left\langle \left[\frac{\sqrt{\cosh[\beta(h+J)] \cosh[\beta(h-J)]}}{\cosh(\beta h)} \right]^n e^{\text{atanh}[\tanh(\beta h) \tanh(\beta J)] \sum_{\alpha} \sigma_{\alpha}} \right\rangle_J$$

- let $C_{n,k} = \sum_{\sigma} \gamma^k(\sigma)$,
remaining eqn:

$$\begin{aligned} \int dh \mathcal{D}(h) \frac{e^{\beta h \sum_{\alpha} \sigma_{\alpha}}}{[2 \cosh(\beta h)]^n} &= \sum_{k>0} p(k) \frac{k}{\langle k \rangle C_{n,k}} \gamma^{k-1}(\sigma) \\ &= \sum_{k>0} p(k) \frac{k}{\langle k \rangle C_{n,k}} \int \left[\prod_{\ell < k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \left[\prod_{\ell < k} \frac{\sqrt{\cosh[\beta(h_{\ell}+J_{\ell})] \cosh[\beta(h_{\ell}-J_{\ell})]}}{\cosh(\beta h_{\ell})} \right]^n \right. \\ &\quad \left. \times e^{\sum_{\alpha} \sigma_{\alpha} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})]} \right\rangle_{J_1 \dots J_{k-1}} \\ &= \int dh \frac{e^{\beta h \sum_{\alpha} \sigma_{\alpha}}}{[2 \cosh(\beta h)]^n} \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \frac{[2 \cosh(\beta h)]^n}{C_{n,k}} \int \left[\prod_{\ell < k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \times \\ &\quad \left\langle \left[\prod_{\ell < k} \frac{\sqrt{\cosh[\beta(h_{\ell}+J_{\ell})] \cosh[\beta(h_{\ell}-J_{\ell})]}}{\cosh(\beta h_{\ell})} \right]^n \delta \left[h - \frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})] \right] \right\rangle_{J_1 \dots J_{k-1}} \end{aligned}$$

after $n \rightarrow 0$

$$\mathcal{D}(h) = \sum_{k>0} \frac{p(k)k}{\langle k \rangle C_{0,k}} \int \left[\prod_{\ell < k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \delta \left[h - \frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})] \right] \right\rangle_{J_1 \dots J_{k-1}}$$

- integrate over h ,
let $d = \int dh \mathcal{D}(h)$

$$d = \sum_{k>0} \frac{p(k)k d^{k-1}}{\langle k \rangle C_{0,k}}$$

$$\begin{aligned} C_{0,k} &= \lim_{n \rightarrow 0} \sum_{\sigma} \gamma^k(\sigma) \\ &= \lim_{n \rightarrow 0} \int \left[\prod_{\ell \leq k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \left[\prod_{\ell \leq k} \frac{\sqrt{\cosh[\beta(h_{\ell} + J_{\ell})] \cosh[\beta(h_{\ell} - J_{\ell})]}}{\cosh(\beta h_{\ell})} \right]^n \right. \\ &\quad \left. \times \sum_{\sigma} e^{\sum_{\alpha} \sigma_{\alpha} \sum_{\ell \leq k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})]} \right\rangle_{J_1 \dots J_k} \\ &= \lim_{n \rightarrow 0} \int \left[\prod_{\ell \leq k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \left[2 \cosh \left(\sum_{\ell \leq k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})] \right) \right]^n \right\rangle_{J_1 \dots J_k} \\ &= d^k \end{aligned}$$

$$\text{hence: } d^2 = \langle k \rangle^{-1} \sum_k p(k)k = 1 \Rightarrow \int dh \mathcal{D}(h) = 1, C_{0,k} = 1$$

Interpretation?

- return to $\mathcal{P}(\sigma, \omega)$:

$$\mathcal{P}(\sigma) = \int_{-\pi}^{\pi} d\omega \mathcal{P}(\sigma, \omega) = \lim_{n \rightarrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \lim_{n \rightarrow 0} \overline{\left\langle \prod_{\alpha \leq n} \delta_{\sigma_\alpha, \sigma_i^\alpha} \right\rangle}$$

hence

$$m = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle} = \lim_{n \rightarrow 0} \sum_{\sigma} \mathcal{P}(\sigma) \frac{1}{n} \sum_{\alpha=1}^n \sigma_\alpha$$

$$q = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i \rangle^2} = \lim_{n \rightarrow 0} \sum_{\sigma} \mathcal{P}(\sigma) \frac{1}{n(n-1)} \sum_{\alpha \neq \beta=1}^n \sigma_\alpha \sigma_\beta$$

- $\mathcal{P}(\sigma)$ in terms of $\mathcal{D}(h)$:

$$\begin{aligned} \mathcal{P}(\sigma) &= \sum_k p(k) \frac{\int_{-\pi}^{\pi} d\omega e^{i\omega k + \langle k \rangle e^{-i\omega} \gamma(\sigma)}}{\sum_{\sigma'} \int_{-\pi}^{\pi} d\omega e^{i\omega k + \langle k \rangle e^{-i\omega} \gamma(\sigma')}} = \sum_k \frac{p(k)}{C_{n,k}} \gamma^k(\sigma) \\ &= \sum_k p(k) \int \left[\prod_{\ell \leq k} dh_\ell \mathcal{D}(h_\ell) \right] \left\langle e^{\sum_{\alpha} \sigma_{\alpha} \sum_{\ell \leq k} \text{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)]} \left[1 + \mathcal{O}(n) \right] \right\rangle_{J_1 \dots J_k} \end{aligned}$$

result:

$$m = \int dh \, \mathcal{D}(h) \tanh(\beta h), \quad q = \int dh \, \mathcal{D}(h) \tanh^2(\beta h)$$

$$\mathcal{D}(h) = \sum_k p(k) \int \left[\prod_{\ell \leq k} dh_\ell \, \mathcal{D}(h_\ell) \right] \left\langle \delta \left[h - \frac{1}{\beta} \sum_{\ell \leq k} \text{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)] \right] \right\rangle_{J_1 \dots J_k}$$

$$\mathcal{D}(h) = \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \left[\prod_{\ell < k} dh_\ell \, \mathcal{D}(h_\ell) \right] \left\langle \delta \left[h - \frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)] \right] \right\rangle_{J_1 \dots J_{k-1}}$$

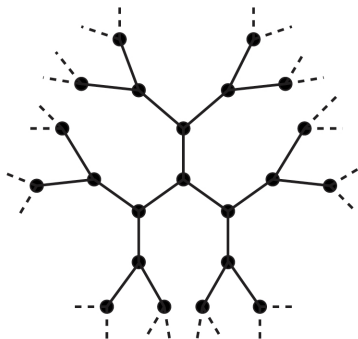
effective field distr:

$\mathcal{D}(h)$: for *true* graph

$\mathcal{D}(h)$: for *cavity* graph

typical form of order parameter eqns
for **locally tree-like** graphs

alternative derivation via
belief propagation or cavity methods



- 1 Topological heterogeneity
- 2 Replica analysis
- 3 Saddle point equations
- 4 RS solution
- 5 Phase transitions**

Continuous phase transitions

$$\mathcal{D}(h) = \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \left[\prod_{\ell < k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \delta \left[h - \frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})] \right] \right\rangle_{J_1 \dots J_{k-1}}$$

- at $T = \beta^{-1} = \infty$: $\mathcal{D}(h) = \delta(h)$

gives: $D(h) = \delta(h)$, $m = q = 0$

paramagnetic state, is a soln at any β

- bifurcations away from $\delta(h)$:
expand eqns in width of $\mathcal{D}(h)$
bifurcating state always has
 $\int dh P(h) h^2 > 0$

$$\mathcal{D}(h) = \epsilon^{-1} W(h/\epsilon), \quad 0 < \epsilon \leq 1$$

1st moment bifurcation : $\delta(h) \rightarrow P(h)$ with $\int dh P(h) h \neq 0$

2nd moment bifurcation : $\delta(h) \rightarrow P(h)$ with $\int dh P(h) h = 0$

- 1st moment bifurcates first:

$$\begin{aligned}
 \int dh \mathcal{D}(h) h &= \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \left[\prod_{\ell < k} dh_{\ell} \mathcal{D}(h_{\ell}) \right] \left\langle \frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_{\ell}) \tanh(\beta J_{\ell})] \right\rangle_{J_1 \dots J_{k-1}} \\
 \epsilon \int dx W(x) x &= \frac{1}{\beta} \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \int dx W(x) \left\langle \text{atanh}[\tanh(\beta \epsilon x) \tanh(\beta J)] \right\rangle_J \\
 &= \epsilon \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \int dx W(x) x \langle \tanh(\beta J) \rangle_J + \mathcal{O}(\epsilon^2)
 \end{aligned}$$

bifurcation of ferromagn state
with $m, q \neq 0$ at:

$$P \rightarrow F: \quad (\langle k^2 \rangle / \langle k \rangle - 1) \int dJ P(J) \tanh(\beta J) = 1$$

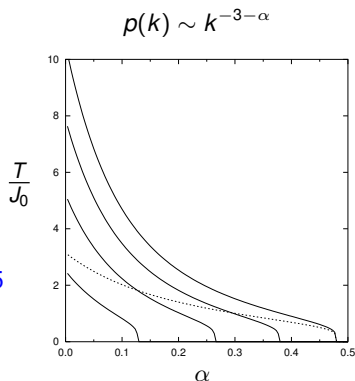
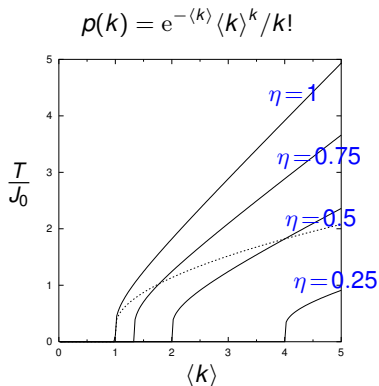
- 2nd moment bifurcates first:

$$\begin{aligned}
 \int dh \mathcal{D}(h) h^2 &= \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \int \left[\prod_{\ell < k} dh_\ell \mathcal{D}(h_\ell) \right] \left\langle \left(\frac{1}{\beta} \sum_{\ell < k} \text{atanh}[\tanh(\beta h_\ell) \tanh(\beta J_\ell)] \right)^2 \right\rangle_{J_1 \dots J_{k-1}} \\
 \epsilon^2 \int dx W(x) x^2 &= \frac{1}{\beta^2} \sum_{k>0} p(k) \frac{k}{\langle k \rangle} \left\{ (k-1) \int dx W(x) \left\langle \text{atanh}^2[\tanh(\beta \epsilon x) \tanh(\beta J)] \right\rangle_J \right. \\
 &\quad \left. + (k-1)(k-2) \left[\int dx W(x) \left\langle \text{atanh}[\tanh(\beta \epsilon x) \tanh(\beta J)] \right\rangle_J \right]^2 \right\} \\
 &= \epsilon^2 \sum_{k>0} p(k) \frac{k(k-1)}{\langle k \rangle} \int dx W(x) x^2 \langle \tanh^2(\beta J) \rangle_J + \mathcal{O}(\epsilon^3)
 \end{aligned}$$

bifurcation of spin-glass state
with $m = 0$, $q \neq 0$ at:

$$P \rightarrow SG: \quad (\langle k^2 \rangle / \langle k \rangle - 1) \int dJ P(J) \tanh^2(\beta J) = 1$$

Phase diagrams



$$P(J) = \frac{1}{2}(1+\eta)\delta(J-J_0) + \frac{1}{2}(1-\eta)\delta(J+J_0), \quad J_0 \geq 0,$$

$\eta = 1$: ferromagnet,

$\eta = 0$: antiferromagnet

solid : $P \rightarrow F, \quad \eta (\langle k^2 \rangle - \langle k \rangle) \tanh(\beta J_0) = 1$

dashed : $P \rightarrow SG, \quad (\langle k^2 \rangle - \langle k \rangle) \tanh^2(\beta J_0) = 1$