Finitely Connected Vector Spin Systems with Random Matrix Interactions

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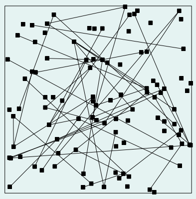


MOTIVATION

Physical systems

- *finite connectivity:*
  more realistic than full connectivity
- *vector spins:*
  more realistic than Ising or spherical spins
- *matrix interactions:*
  e.g. coupled oscillators, Josephson junctions

Theory

- *how far can we push our present techniques?*
- *more intuition on finitely connected continuous spins*

OVERVIEW

Definitions

vector spins $\boldsymbol{\sigma}_i \in S_{d-1}$ with random matrix interactions
on finitely connected random graphs

Replica calculation of free energy

saddle-point equations
replica-symmetric theory

d=2: XY spins

phase transitions
phase diagrams for different chirality distributions
theory versus simulations

d=3,4,...

phase transitions
diagrams for $d = 3$, classical Heisenberg spins
theory versus simulations

Summary

DEFINITIONS

N unit-length vector spins,
$\boldsymbol{\sigma}_i \in S_{d-1}, \quad \{\boldsymbol{\sigma}\} = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_N)$

$$H(\{\boldsymbol{\sigma}\}) = -J \sum_{i<j} c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{U}_{ij} \boldsymbol{\sigma}_j + \sum_i V(\boldsymbol{\sigma}_i)$$

quenched disorder:

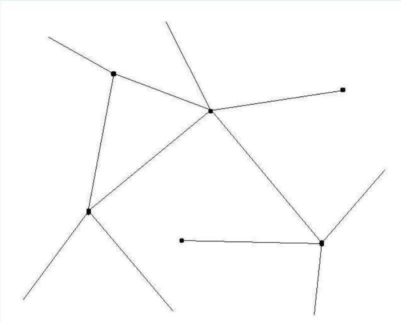
- *random lattice:* $c_{ij} \in \{0, 1\}$

  $$\mathrm{Prob}(c_{ij}) = \frac{c}{N} \delta_{c_{ij},1} + (1 - \frac{c}{N}) \delta_{c_{ij},0} \quad \text{for all} \;\; i < j$$

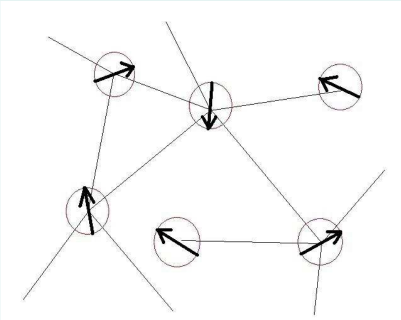
  $c = \mathcal{O}(N^0)$, *Erdös-Rényi graph*

- *interactions between spins:*

  random matrices $\boldsymbol{U}_{ij} \in \mathrm{SO}(3)$ *(rotations in* $\mathbb{R}^d$*)*
  independently drawn from $P(\boldsymbol{U})$, *with* $P(\boldsymbol{U}^\dagger) = P(\boldsymbol{U})$



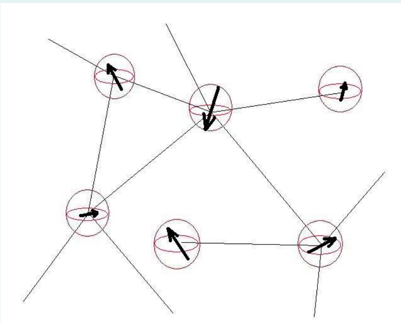
random graph: $\mathrm{Prob}(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1-\frac{c}{N})\delta_{c_{ij},0}$



$d=2$: $\boldsymbol{\sigma}_i = (\cos\phi_i, \sin\phi_i)$

$H = -J\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j - \omega_{ij}) + \sum_i V(\phi_i)$

random $\omega_{ij} \in [0, 2\pi]$



$d=3$: $\boldsymbol{\sigma}_i = (\sin\theta_i\cos\phi_i, \sin\theta_i\sin\phi_i, \cos\theta_i)$

$H = -J\sum_{i<j} c_{ij}\boldsymbol{\sigma}_i \cdot \boldsymbol{U}_{ij}\boldsymbol{\sigma}_j + \sum_i V(\boldsymbol{\sigma}_i)$

random $\boldsymbol{U}_{ij} \in \mathrm{SO}(3)$

Technicalities

Complicating model ingredients:

(i) spin variables continuous
(ii) spin variables vectorial
(iii) spin-interactions represented by random matrices

- *RS order parameter is a functional*
- *finding phase transitions:*
  - *involves functional moment expansions*
  - *nontrivial eigenvalue problems*
- *finding order parameters:*
  - *population dynamics nontrivial: iterating functionals*
  - *numerically demanding and potentially difficult to converge*
- *numerical simulations:*
  - *generating suitable random matrices*
  - *Langevin dynamics too slow*

REPLICA ANALYSIS

Disorder-averaged free energy per spin:

$$\overline{f} \;=\; \lim_{n\to 0}\frac{1}{n}\left\{-\lim_{N\to\infty}\frac{1}{\beta N}\log\int[\prod_i\prod_{\alpha=1}^{n}d\boldsymbol{\sigma}_i^\alpha]\;\overline{e^{-\beta\sum_{\alpha=1}^{n}H(\{\boldsymbol{\sigma}^\alpha\})}}\right\}$$

Disorder average:

$$\overline{e^{-\beta\sum_\alpha H(\{\boldsymbol{\sigma}^\alpha\})}} \;=\; \exp\left\{-\beta\sum_{i\alpha}V(\boldsymbol{\sigma}_i^\alpha)+\mathcal{O}(N^0)\right.$$
$$\left.+\frac{c}{2N}\sum_{ij}\left[\int d\boldsymbol{U}P(\boldsymbol{U})e^{\beta J\sum_\alpha \boldsymbol{\sigma}_i^\alpha\cdot\boldsymbol{U}\boldsymbol{\sigma}_j^\alpha}-1\right]\right\}$$

Order parameters:

$\{\boldsymbol{\sigma}\}=(\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n),\quad \boldsymbol{\sigma}^\alpha,\boldsymbol{\sigma}_i^\alpha\in S_{d-1}$ $\qquad P(\{\boldsymbol{\sigma}\})=\frac{1}{N}\sum_i\prod_\alpha\delta[\boldsymbol{\sigma}-\boldsymbol{\sigma}_i^\alpha]$

minor technicalities

- *functional δ-distributions to isolate order parameters* $P(\{\boldsymbol{\sigma}\})$
  integral representations: conjugate functions $\hat{P}(\{\boldsymbol{\sigma}\})$
- *continuum limit for domain* S_{d-1} *of* $\boldsymbol{\sigma}_\alpha$,
  gives path integral measure:

$$\prod_{\{\boldsymbol{\sigma}\}}[dP(\{\boldsymbol{\sigma}\})d\hat{P}(\{\boldsymbol{\sigma}\})/2\pi] = \{dPd\hat{P}\}$$

$$\begin{aligned}\overline{f} \;=\; -\lim_{n\to 0}\frac{1}{\beta n}\text{extr}_{\{P,\hat{P}\}}\Big\{\, i\!\int\!\{d\boldsymbol{\sigma}\}P(\{\boldsymbol{\sigma}\})\hat{P}(\{\boldsymbol{\sigma}\}) + \log\int\!\{d\boldsymbol{\sigma}\}\; e^{-\beta\sum_\alpha V(\boldsymbol{\sigma}_\alpha)-i\hat{P}(\{\boldsymbol{\sigma}\})} \\ +\frac{1}{2}c\int\!\{d\boldsymbol{\sigma}d\boldsymbol{\sigma}'\}P(\{\boldsymbol{\sigma}\})P(\{\boldsymbol{\sigma}'\})[\int d\boldsymbol{U}P(\boldsymbol{U})e^{\beta J\sum_\alpha \boldsymbol{\sigma}_\alpha\cdot\boldsymbol{U}\boldsymbol{\sigma}'_\alpha}-1]\Big\}\end{aligned}$$

saddle-point eqns

$$P(\{\boldsymbol{\sigma}\}) \;=\; \frac{e^{-\beta\sum_\alpha V(\boldsymbol{\sigma}_\alpha)-i\hat{P}(\{\boldsymbol{\sigma}\})}}{\int\{d\boldsymbol{\sigma}'\}\; e^{-\beta\sum_\alpha V(\boldsymbol{\sigma}'_\alpha)-i\hat{P}(\{\boldsymbol{\sigma}'\})}}$$

$$\hat{P}(\{\boldsymbol{\sigma}\}) \;=\; ic\int\!\{d\boldsymbol{\sigma}'\}\; P(\{\boldsymbol{\sigma}'\})[\int d\boldsymbol{U}P(\boldsymbol{U})e^{\beta J\sum_\alpha \boldsymbol{\sigma}_\alpha\cdot\boldsymbol{U}\boldsymbol{\sigma}'_\alpha}-1]$$

Replica symmetric theory

$$P(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_n) = \frac{1}{N}\sum_i \prod_{\alpha=1}^{n} \delta[\boldsymbol{\sigma}_\alpha - \boldsymbol{\sigma}_i^\alpha]$$

RS ansatz for
continuous variables:

- *let* $P[\phi|\boldsymbol{\mu}]$ *denote a complete parametrized family of functions on* S_{d-1}
  $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_2, \ldots)$

$$P_{\rm RS}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_n) = \int d\boldsymbol{\mu}\; w(\boldsymbol{\mu}) \prod_\alpha P[\boldsymbol{\sigma}_\alpha|\boldsymbol{\mu}], \qquad \int d\boldsymbol{\mu}\; w(\boldsymbol{\mu}) = 1$$

- *representation-independent formulation:*

$$P_{\rm RS}(\boldsymbol{\sigma}_1,\ldots,\boldsymbol{\sigma}_n) = \int \{dP\}\; W[\{P\}]\; \prod_\alpha P(\boldsymbol{\sigma}_\alpha)$$

RS order parameter:

$$\text{functional measure}\;\; W[\{P\}]$$

physical interpretation:

$$\int \{dP\}\; W[\{P\}] \prod_\alpha \left[\int d\boldsymbol{\sigma}\; P(\boldsymbol{\sigma}) f_\alpha(\boldsymbol{\sigma})\right] = \lim_{N\to\infty} \frac{1}{N}\sum_i \overline{\prod_\alpha \langle f_\alpha(\boldsymbol{\sigma}_i)\rangle}$$

insert RS ansatz:

$$\overline{f}_{\rm RS} = \frac{c}{2\beta}\int\{dP_1dP_2\}\; W[\{P_1\}]W[\{P_2\}]\int d\boldsymbol{U}\,P(\boldsymbol{U})\log\left[\int d\boldsymbol{\sigma}d\boldsymbol{\sigma}'P_1(\boldsymbol{\sigma})P_2(\boldsymbol{\sigma}')e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{U}\boldsymbol{\sigma}'}\right]$$
$$-\frac{1}{\beta}\sum_{\ell\geq 0}p_\ell\int\prod_{k=1}^{\ell}[\{dP_k\}W[\{P_k\}]d\boldsymbol{U}_kP(\boldsymbol{U}_k)]$$
$$\times\log\left[\int d\boldsymbol{\sigma}\;e^{-\beta V(\boldsymbol{\sigma})}\prod_{k=1}^{\ell}\int d\boldsymbol{\sigma}'P_k(\boldsymbol{\sigma}')e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{U}_k\boldsymbol{\sigma}'}\right]$$

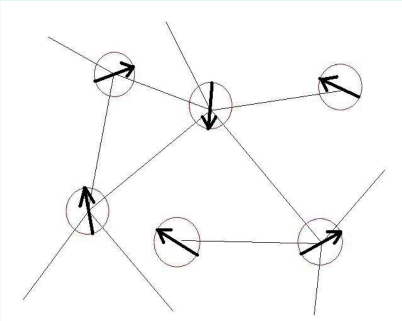
$p_k = e^{-c}c^k/k!$

$$W[\{P\}] = \sum_{\ell\geq 0}p_\ell\int\prod_{k\leq\ell}[\{dP_k\}W[\{P_k\}]d\boldsymbol{U}_kP(\boldsymbol{U}_k)]$$
$$\times\prod_{\boldsymbol{\sigma}\in S_{d-1}}\delta\left[P(\boldsymbol{\sigma})-\frac{e^{-\beta V(\boldsymbol{\sigma})}\prod_{k=1}^{\ell}\int d\boldsymbol{\sigma}'P_k(\boldsymbol{\sigma}')e^{\beta J\boldsymbol{\sigma}\cdot\boldsymbol{U}_k\boldsymbol{\sigma}'}}{\int d\boldsymbol{\sigma}''e^{-\beta V(\boldsymbol{\sigma}'')}\prod_{k=1}^{\ell}\int d\boldsymbol{\sigma}'P_k(\boldsymbol{\sigma}')e^{\beta J\boldsymbol{\sigma}''\cdot\boldsymbol{U}_k\boldsymbol{\sigma}'}}\right]$$

paramagnetic state:

$$\lim_{\beta\to 0}W[\{P\}] = \prod_{\boldsymbol{\sigma}\in S_{d-1}}\delta\left[P(\boldsymbol{\sigma})-\frac{1}{|S_{d-1}|}\right] \qquad \lim_{\beta\to 0}\beta\overline{f}_{\rm RS} = -\log|S_{d-1}|$$

d=2: XY SPINS WITH CHIRAL INTERACTIONS



$d = 2$: $\boldsymbol{\sigma}_i = (\cos\phi_i, \sin\phi_i)$

$$H = -J\sum_{i<j} c_{ij}\cos(\phi_i - \phi_j - \omega_{ij}) + \sum_i V(\phi_i)$$

random $\omega_{ij} \in [0, 2\pi]$

$V = 0$:

$$W[\{P\}] = \sum_{k\geq 0} p_k \int \prod_{\ell\leq k} [\{dP_\ell\} W[\{P_\ell\}] d\omega_\ell P(\omega_\ell)] \times \prod_{\phi\in[0,2\pi]} \delta\left[P(\phi) - \frac{\prod_{\ell=1}^k \int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi-\phi'-\omega_\ell)}}{\int d\phi'' \prod_{\ell=1}^k \int d\phi' P_\ell(\phi') e^{\beta J\cos(\phi''-\phi'-\omega_\ell)}}\right]$$

$$\text{paramagnetic state}: \quad \lim_{\beta\to 0} W[\{P\}] = \prod_{\phi\in[0,2\pi]} \delta\left[P(\phi) - \frac{1}{2\pi}\right]$$

Phase transitions

Continuous bifurcations away from paramagnetic state located by Guzai (i.e. functional moment) expansion

- *transform:*

$$P(\phi) \to \frac{1}{2\pi} + \Delta(\phi), \qquad W[\{P\}] \to \tilde{W}[\{\Delta\}]$$

constraint:

$$\tilde{W}[\{\Delta\}] = 0 \quad \text{if} \quad \int_0^{2\pi} d\phi\ \Delta(\phi) \neq 0$$

- *expand saddle-point eqns in functional moments*

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \ldots \Delta(\phi_r)$$

- *assume: close to continuous bifurcation* $\exists \epsilon \ll 1$ *such that*

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\phi_1) \ldots \Delta(\phi_r) = \mathcal{O}(\epsilon^r)$$

Lowest order bifurcation ϵ^1

$$\Psi(\phi) = \frac{c}{2\pi I_0(\beta J)} \int_0^{2\pi} d\phi' \int d\omega \; P(\omega) e^{\beta J \cos(\phi - \phi' - \omega)} \Psi(\phi') \qquad \int_0^{2\pi} d\phi \; \Psi(\phi) = 0$$

$I_k(z)$:
modified Bessel functions

- *soln: Fourier modes* $\Psi(\phi) = e^{ik\phi}$

$$c = \min_{k>0} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(k\omega) \right\}^{-1}$$

- *bifurcating state:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\left\langle \begin{pmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{pmatrix} \right\rangle} \;=\; \frac{1}{2}\epsilon \; \delta_{k1} \begin{pmatrix} \cos(\lambda) \\ \sin(\lambda) \end{pmatrix} + \ldots$$

$$\mathrm{P} \to \mathrm{F}: \qquad c = \left\{ \frac{I_1(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(\omega) \right\}^{-1}$$

$$\mathrm{KT}: \qquad c = \min_{k>1} \left\{ \frac{I_k(\beta J)}{I_0(\beta J)} \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(k\omega) \right\}^{-1}$$

Lowest order bifurcation ϵ^2

$$\Psi(\phi_1,\phi_2) \;=\; c\int \frac{d\phi_1' d\phi_2'}{[2\pi I_0(\beta J)]^2}\left[\int d\omega\; P(\omega) e^{\beta J\cos(\phi_1-\phi_1'-\omega)+\beta J\cos(\phi_2-\phi_2'-\omega)}\right]\Psi(\phi_1',\phi_2')$$

$$\int d\phi_1\;\Psi(\phi_1,\phi_2) = \int d\phi_2\;\Psi(\phi_1,\phi_2) = 0$$

- *soln: Fourier modes* $\Psi(\phi_1,\phi_2) = e^{i(k_1\phi_1+k_2\phi_2)}$
- *bifurcating state:*
  no global ferromagn order, yet

$$\lim_{N\to\infty}\frac{1}{N}\sum_i\left[\overline{\langle\cos(\phi_i)\rangle^2}+\overline{\langle\sin(\phi_i)\rangle^2}\right] > 0$$

$$\mathrm{P}\to\mathrm{SG}: \qquad c = I_0^2(\beta J)/I_1^2(\beta J)$$

Phase diagrams

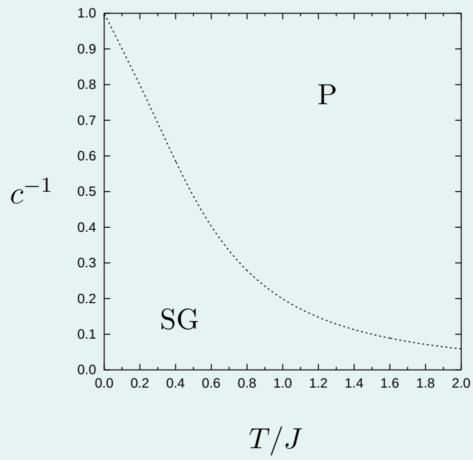
$$\text{P} \rightarrow \text{F}: \qquad c^{-1} = [I_1(\beta J)/I_0(\beta J)] \int_{-\pi}^{\pi} d\omega \; P(\omega) \cos(\omega)$$

$$\text{P} \rightarrow \text{SG}: \qquad c^{-1} = [I_1(\beta J)/I_0(\beta J)]^2$$

F → SG : cannot yet calculate ...
Parisi − Toulouse hypothesis ?

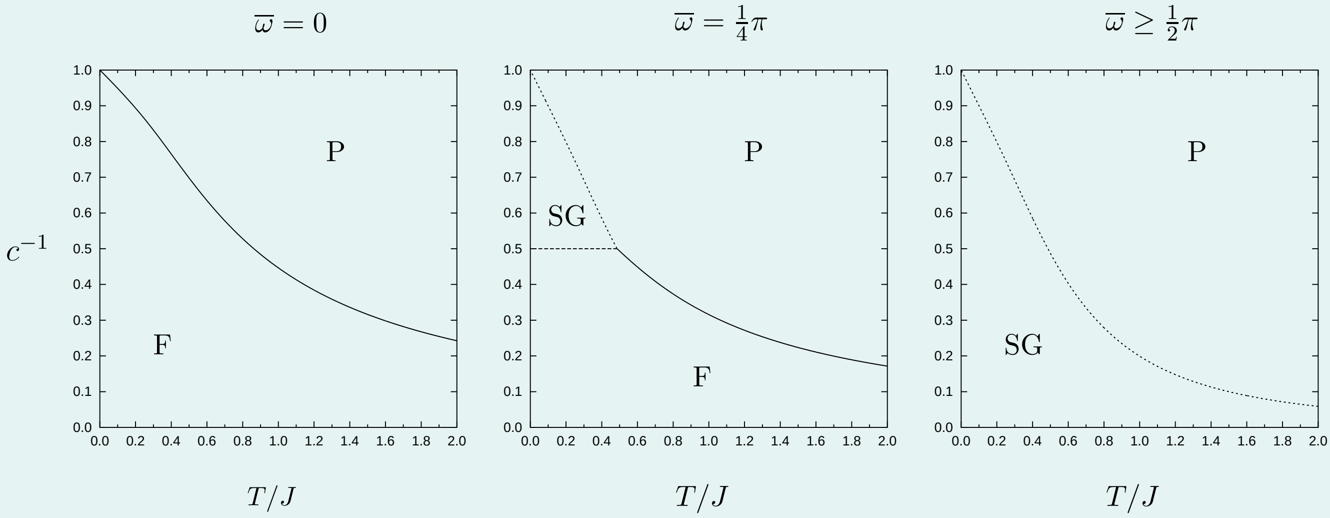
Example:

$P(\omega) = 1/2\pi$



Example:

$$P(\omega) = \tfrac{1}{2}\delta(\omega - \overline{\omega}) + \tfrac{1}{2}\delta(\omega + \overline{\omega})$$



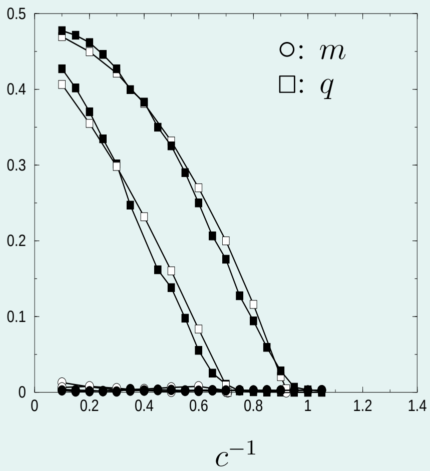
Theory versus simulations

Numerical solution of order parameter equations:
population dynamics with truncated parametrizations $w(\boldsymbol{\mu})$ *of functional* $W[\{P\}]$

$$
\begin{aligned}
m^2 &= \left[\frac{1}{N}\sum_i \overline{\langle\cos(\phi_i)\rangle}\right]^2 + \left[\frac{1}{N}\sum_i \overline{\langle\sin(\phi_i)\rangle}\right]^2 \\
q &= \frac{1}{2N}\sum_i \left[\overline{\langle\cos(\phi_i)\rangle^2} + \overline{\langle\sin(\phi_i)\rangle^2}\right]
\end{aligned}
$$

Example:

$P(\omega) = 1/2\pi$



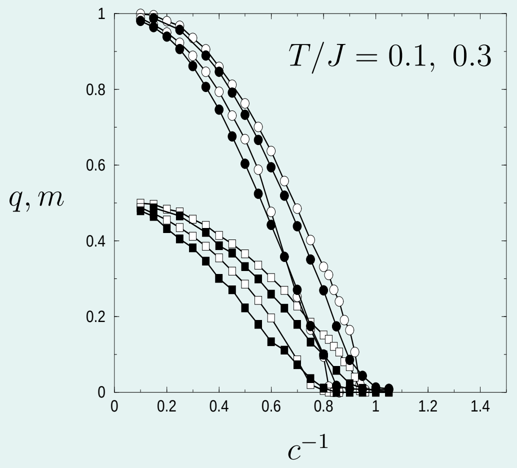
upper: $T/J = 0.1$
lower: $T/J = 0.3$

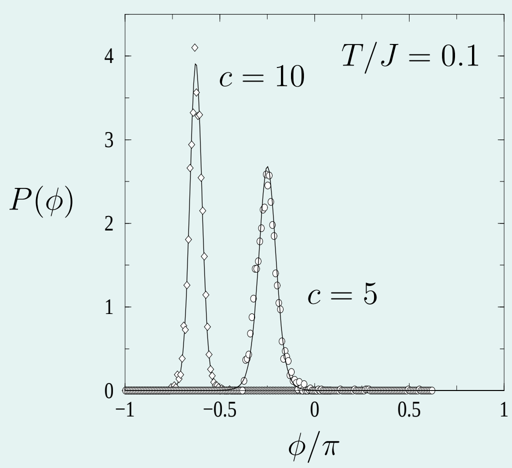
Example:

$P(\omega) = \delta(\omega)$

○: m

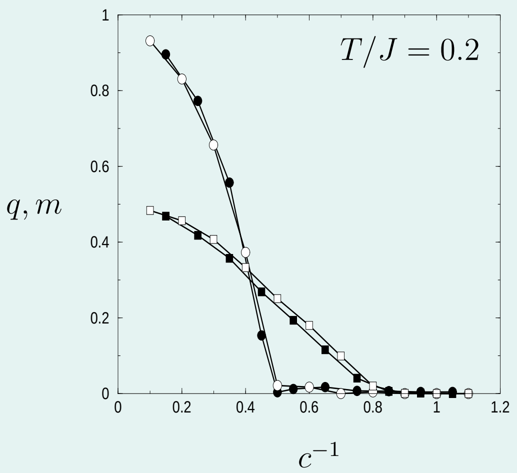
□: q





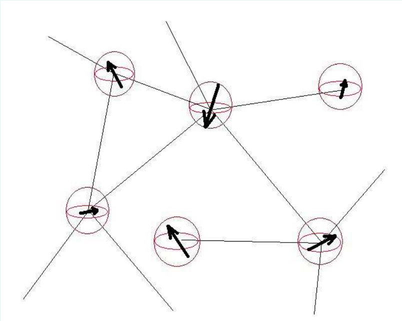
Example:

$P(\omega) = \frac{1}{2}\delta(\omega - \frac{\pi}{4}) + \frac{1}{2}\delta(\omega + \frac{\pi}{4})$



confirms
Parisi-Toulouse
for F→SG transition

d=3,4,...: HEISENBERG SPINS AND BEYOND



$\boldsymbol{\sigma}_i \in S_{d-1}$

$H = -J \sum_{i<j} c_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{U}_{ij} \boldsymbol{\sigma}_j + \sum_i V(\boldsymbol{\sigma}_i)$

random $\boldsymbol{U}_{ij} \in \mathrm{SO}(3)$

$V = 0$:

$$W[\{P\}] = \sum_{\ell \geq 0} p_\ell \int \prod_{k \leq \ell} [\{dP_k\} W[\{P_k\}] d\boldsymbol{U}_k P(\boldsymbol{U}_k)] \times \prod_{\boldsymbol{\sigma} \in S_{d-1}} \delta \left[P(\boldsymbol{\sigma}) - \frac{\prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{U}_k \boldsymbol{\sigma}'}}{\int d\boldsymbol{\sigma}'' \prod_{k=1}^{\ell} \int d\boldsymbol{\sigma}' P_k(\boldsymbol{\sigma}') e^{\beta J \boldsymbol{\sigma}'' \cdot \boldsymbol{U}_k \boldsymbol{\sigma}'}} \right]$$

$$\text{paramagnetic state :} \qquad \lim_{\beta \to 0} W[\{P\}] = \prod_{\boldsymbol{\sigma} \in S_{d-1}} \delta \left[P(\boldsymbol{\sigma}) - \frac{1}{|S_{d-1}|} \right]$$

Phase transitions

Continuous bifurcations away from paramagnetic state located by Guzai (i.e. functional moment) expansion

$$P(\boldsymbol{\sigma}) \to |S_{d-1}|^{-1} + \Delta(\boldsymbol{\sigma}), \qquad W[\{P\}] \to \tilde{W}[\{\Delta\}]$$

functional moments

$$\int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\boldsymbol{\sigma}_1) \ldots \Delta(\boldsymbol{\sigma}_r)$$

assume

$$\exists \epsilon \ll 1 \ \text{ such that } \ \int \{d\Delta\} \tilde{W}[\{\Delta\}] \Delta(\boldsymbol{\sigma}_1) \ldots \Delta(\boldsymbol{\sigma}_r) = \mathcal{O}(\epsilon^r)$$

generalized modified Bessel function:

$I_{0,d}(z) = |S_{d-1}|^{-1} \int_{S_{d-1}} d\boldsymbol{\sigma} \ e^{z\sigma_1}$

Lowest order ϵ^1

$$\Psi(\boldsymbol{\sigma}) = \frac{c}{I_{0,d}(\beta J)} \int_{S_{d-1}} \frac{d\boldsymbol{\sigma'}}{|S_{d-1}|} \Psi(\boldsymbol{\sigma}') \int d\boldsymbol{U} P(\boldsymbol{U}) e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{U} \boldsymbol{\sigma}'} \qquad \int_{S_{d-1}} d\boldsymbol{\sigma} \ \Psi(\boldsymbol{\sigma}) = 0$$

commuting operators:

$$KL\Psi = c^{-1} I_{0,d}(\beta J)\Psi$$

$$(K\Psi)(\boldsymbol{\sigma}) = \int_{S_{d-1}} \frac{d\boldsymbol{\tau}}{|S_{d-1}|} e^{\beta J \boldsymbol{\sigma} \cdot \boldsymbol{\tau}} \Psi(\boldsymbol{\tau}) \qquad (L\Psi)(\boldsymbol{\sigma}) = \int d\boldsymbol{U} P(\boldsymbol{U}) \Psi(\boldsymbol{U}^\dagger \boldsymbol{\sigma})$$

Lowest order ϵ^2

$$\Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \frac{c}{I_{0,d}^2(\beta J)} \int_{S_{d-1}} \frac{d\boldsymbol{\tau}^1 d\boldsymbol{\tau}^2}{|S_{d-1}|^2} \Psi(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2) \int d\boldsymbol{U} P(\boldsymbol{U}) e^{\beta J (\boldsymbol{\sigma}^1 \cdot \boldsymbol{U} \boldsymbol{\tau}^1 + \boldsymbol{\sigma}^2 \cdot \boldsymbol{U} \boldsymbol{\tau}^2)}$$

$$\int_{S_{d-1}} d\boldsymbol{\sigma}^1 \ \Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = \int_{S_{d-1}} d\boldsymbol{\sigma}^2 \ \Psi(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = 0$$

commuting operators:

$$KL\Psi = c^{-1} I_{0,d}^2(\beta J)\Psi$$

$$\begin{aligned} (K\Psi)(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) &= \int_{S_{d-1}} \frac{d\boldsymbol{\tau}^1 d\boldsymbol{\tau}^2}{|S_{d-1}|^2} e^{\beta J (\boldsymbol{\sigma}^1 \cdot \boldsymbol{\tau}^1 + \boldsymbol{\sigma}^2 \cdot \boldsymbol{\tau}^2)} \Psi(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2) \\ (L\Psi)(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) &= \int d\boldsymbol{U} P(\boldsymbol{U}) \Psi(\boldsymbol{U}^\dagger \boldsymbol{\sigma}^1, \boldsymbol{U}^\dagger \boldsymbol{\sigma}^2) \end{aligned}$$

Continuous phase transitions for $d = 3$: classical Heisenberg spins

Euler angle representation
of random rotations in $\mathbb{R}^3$*:*

$$P(\boldsymbol{U}) = \epsilon\delta[\boldsymbol{U} - \mathbb{1}] + \frac{1-\epsilon}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{2\pi} d\gamma \;\sin(\beta)\delta\left[\boldsymbol{U} - R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\right]$$

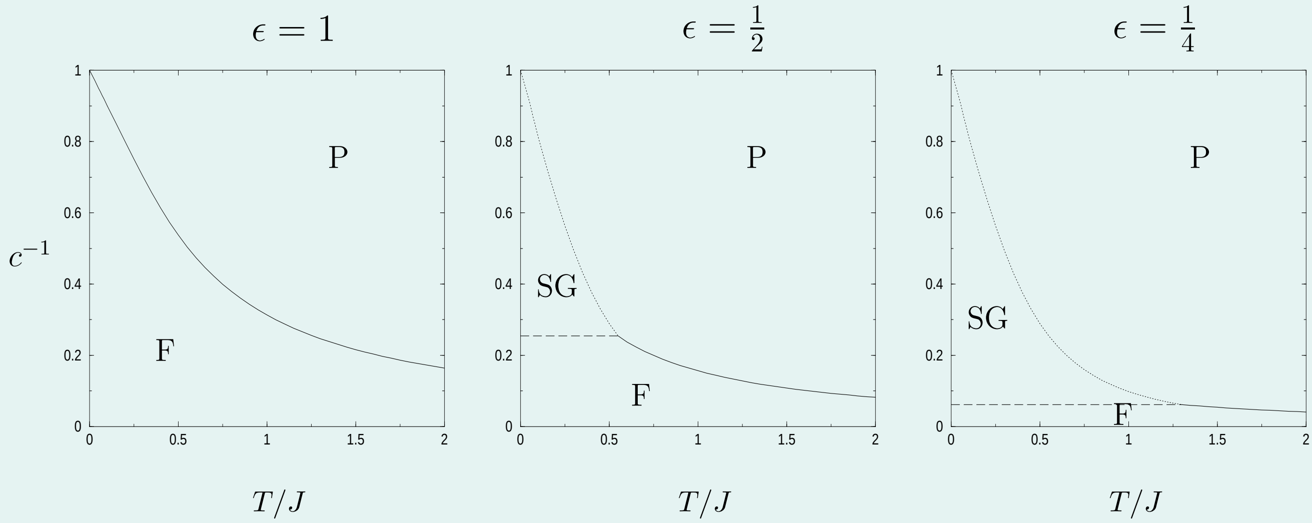
$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \quad R_z(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{P}\!\to\!\mathrm{F}: \quad \int_0^1 dy\; I_0(\beta J\sqrt{1-x^2}\sqrt{1-y^2})\sinh(\beta Jxy)\rho(y) = \frac{\sinh(\beta J)}{\epsilon\beta Jc}\rho(x)$$

$$\mathrm{P}\!\to\!\mathrm{SG}: \quad \int_{-1}^1 \frac{dydsdt}{4\pi}\; I_0(\beta J\sqrt{1-s^2}\sqrt{1-x^2})\; e^{\beta J[sx+t]}\; \frac{\theta[(1-s^2)(1-t^2)-(y-st)^2]}{\sqrt{(1-s^2)(1-t^2)-(y-st)^2}}\;\psi(y) = \frac{\sinh^2(\beta J)}{c(\beta J)^2}\psi(x)$$

subject to constraint $\quad \int_{-1}^1 dy\;\psi(y) = 0$

Phase diagrams for $d = 3$



$$P(\boldsymbol{U}) = \epsilon\ \delta[\boldsymbol{U} - \mathbb{1}] + \frac{1-\epsilon}{8\pi^2} \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta \int_0^{2\pi} d\gamma\ \ \sin(\beta)\ \delta\left[\boldsymbol{U} - R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\right]$$

Theory versus simulations

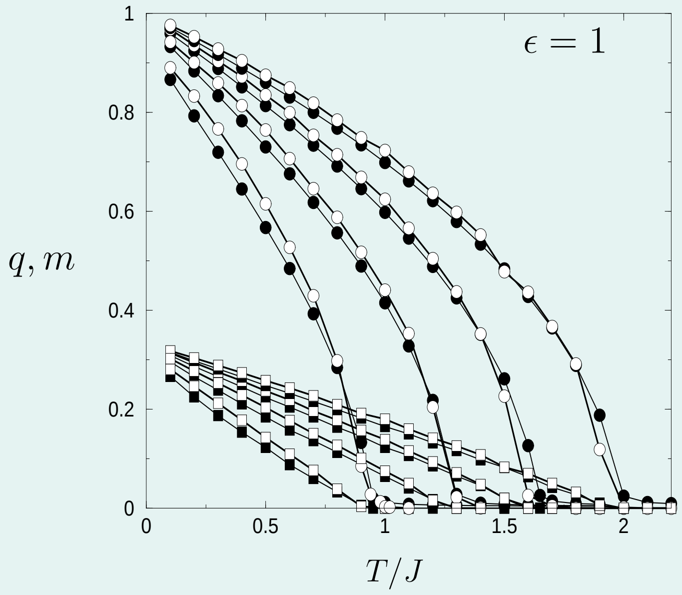
population dynamics with truncated parametrizations $w(\boldsymbol{\mu})$ of functional $W[\{P\}]$

$$m = \sqrt{m_x^2 + m_y^2 + m_z^2} \qquad q = \frac{1}{3}(q_x + q_y + q_z)$$

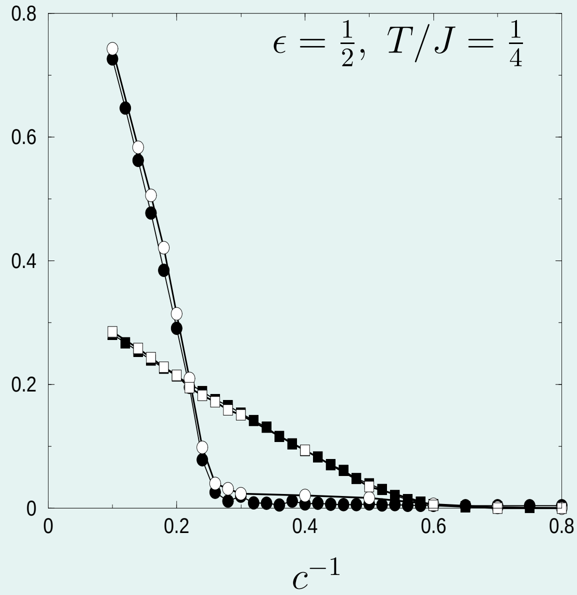
$$m_x = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \cos(\phi_i)\sin(\theta_i)\rangle} \qquad q_x = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \cos(\phi_i)\sin(\theta_i)\rangle^2}$$

$$m_y = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \sin(\phi_i)\sin(\theta_i)\rangle} \qquad q_y = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \sin(\phi_i)\sin(\theta_i)\rangle^2}$$

$$m_z = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \cos(\theta_i)\rangle} \qquad q_z = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \cos(\theta_i)\rangle^2}$$



$c = 3, 4, 5, 6$ *(left to right)*



confirmation of Parisi-Toulouse

SUMMARY

- *finitely connected vector spin models with random matrix interactions are solvable with presently available methods*
- *RS order parameter is a functional* $W[\{P\}]$
- *continuous transition lines within RS can be calculated exactly using Guzai (functional moment) expansions of the functional* $W[\{P\}]$
- *Parisi-Toulouse hypothesis (F→SG transition) appears correct*
- *RSB effects are only modest, limited to very low temperatures*
- *obvious possible extensions:*
  - *RSB*
  - *non-Poissonian graphs*
- *less obvious extensions: dynamics*