The Auditory System and Human Sound-Localization Behavior

Short Answers to the Exercises of Chapter 2

Problem 2.1

The thermal speed of gas molecules follows from the kinetic gas theory:

$$v_{Therm} = \sqrt{\frac{k_B T}{m}} = \sqrt{\frac{nRT}{M}}$$

with *m* the molecular mass, *n* the number of moles, and *M* the total mass of the gas within the volume. Substitute the ideal gas law in the expression for the adiabatic velocity by using the same trick to show that:

$$\frac{p_0}{\rho} = \frac{k_B T}{m}$$

Problem 2.2

a. Any function of the type $s(x,t) = f(x \pm v \cdot t)$ is a solution of the one-dimensional wave equation:

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 s}{\partial t^2}$$

You can demonstrate this by first introducing $\xi \equiv x \pm v \cdot t$. The perturbation is then a function of one variable, $s(x,t) = f(\xi)$. You may then show from the definition of $s(x,t) = f(x \pm v \cdot t)$ (and the chain rule) that

$$\frac{\partial^2 s}{\partial x^2} = \frac{d^2 f(\xi)}{d\xi^2} = \frac{1}{v^2} \cdot \frac{\partial^2 s}{\partial t^2}$$

b. Test of the superposition principle: if $s_1(x,t)$ and $s_2(x,t)$ are both a solution of the wave equation, then $s_{12}(x,t) \equiv a s_1(x,t) + b s_2(x,t)$ is also a solution can be demonstrated by substitution.

Problem 2.3

Substitute $s(x,t) = X(x) \cdot T(t)$ into the wave equation:

$$\frac{\partial^2 s}{\partial x^2} = \frac{\partial^2 [X(x) \cdot T(t)]}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 [X(x) \cdot T(t)]}{\partial t^2}$$

Since X only depends on x, and T only depends on t, this becomes:

$$\frac{1}{X} \cdot \frac{d^2 X}{dx^2} = \frac{1}{T \cdot v^2} \cdot \frac{d^2 T}{dt^2}$$

and note that this should hold for all *x and t*! This requirement can only be met if both the left- and right-hand sides are equal to an arbitrary constant, and will yield *harmonic* solutions provided this constant is *negative*. So, write the constant as a negative square: $-k^2$, which yields Eqn. 2.30.

Problem 2.4
Eqn. 2.32 gives the general spatial-temporal harmonic solution of the wave equation. The spatial component of the solution:

$$X(x) = \sum_{k=1}^{\infty} [A_k \cos kx + B_k \sin kx]$$

Demand that it is constrained by fixed boundary conditions at x=0 and x=L_0. Thus,

$$X(0) = X(L_0) = 0$$

This yields

$$kL_0 = n \cdot \pi \quad \text{for } n \in 1,2,3, \dots ..$$

You may now finish the solution.

Problem 2.5
(**a**) We concentrate on the spatial component of the solution with open boundary conditions. In that case the ends are free to move and undergo *no net force* in the transversal direction. That means that the spatial derivatives in x=0 and x=L_0 are zero:

$$\frac{\partial X}{\partial x}(0) = kB_k = 0$$

$$\frac{\partial X}{\partial x}(L_0) = -kA_k \sin kL_0 = 0$$

which fully determines the solution as

$$X(x) = \sum_{k=1}^{\infty} \left[A_k \cos \frac{n\pi x}{L_0} \right]$$

(**b**) For mixed boundary conditions (fixed at x=0 and open at x=L_0) the spatial solution reads:

$$X(x) = \sum_{k=0}^{\infty} \left[B_k \cos \frac{(2n+1)\pi x}{2L_0} \right]$$

(**c**) For periodic boundary conditions the ends are joined: they have the same amplitude at all times, and the same spatial derivative. You can now show that

$$kL_0 = 2n\pi \quad \text{for } \; n = 1,2,3, \dots .$$

$$X(x) = \sum_{k=1}^{\infty} \left[A_k \cos \frac{2n\pi x}{L_0} + B_k \sin \frac{2n\pi x}{L_0} \right]$$

Problem 2.6
Also the inhomogeneous wave equation:

$$\rho(x) \cdot \frac{\partial^2 s}{\partial t^2} = \frac{\partial}{\partial x} \left(B(x) \cdot \left(\frac{\partial s}{\partial x} \right) \right)$$

can be solved by assuming spatial-temporal separability (see Problem 2.3). In this case the requirements become:

$$\frac{1}{T} \frac{\partial^2 T}{\partial t^2} = -k^2 = \frac{1}{X(x)\rho(x)} \cdot \frac{d}{dx} \left(B(x) \cdot \left(\frac{dX}{dx} \right) \right)$$

Now suppose harmonic solutions for the spatial and temporal functions and show that this only works for the temporal function, but not for the spatial function, unless B and ρ are constants.

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Problem 2.7
The wave equation for this inhomogeneous string is described by:

$$\frac{B_0}{\rho(x)}\frac{\partial^2 s}{\partial x^2}=\frac{\partial^2 s}{\partial t^2} \quad \text{from which} \quad \frac{B_0 x^2}{m}\frac{\partial^2 s}{\partial x^2}=\frac{\partial^2 s}{\partial t^2}$$

Separation of variables (try: s(x, t) ≡ A(x) cos(ωt + φ)) then yields the following differential equation for the spatial eigenmodes:

$$\frac{B_0 x^2}{m}\cdot\frac{d^2 A}{dx^2}=-\omega^2 A(x) \quad \Rightarrow \quad \text{'Ansatz': } A(x)=A_0\sqrt{\frac{x}{a}}\sin\left(k\cdot\ln\frac{x}{a}\right)$$

Substitute the Ansatz and show that it is a solution, provided the following dispersion relation, $\omega(k)$, holds:

$$\frac{B_0}{m}\left[-k^2-\tfrac{1}{4}\right]=-\omega^2$$

Problem 2.8
(a) The general form of the standing waves solution can be written as:

$$\mathrm{s(x,t)} = [\mathrm{A}\sin(\mathrm{kx}) + \mathrm{B}\cos(\mathrm{kx})]\cdot\cos(\omega \mathrm{t} - \varphi)$$

in which k = 2π/λ, and v = λ · f. However, there are two different domains to consider, because of the different mass densities of the rope. Therefore,

$$\text{for } -L\le x\le 0: \quad s_1(x,t)=[A_1\sin(k_1x)+B_1\cos(k_1x)]\cdot\cos(\omega_1 t-\varphi_1)$$
$$\text{for } 0\le x\le +L: \quad s_2(x,t)=[A_2\sin(k_2x)+B_2\cos(k_2x)]\cdot\cos(\omega_2 t-\varphi_2)$$

You can solve this problem by setting appropriate boundary conditions at x = ±L and at the transition in x = 0. In x = 0 the wave function, s(x, t), and the spatial derivative, ∂s/∂x have to be continuous (massless point, no net force).

(b) Substitution of the constraint from (a) gives:

$$-L\le x\le 0: \quad s_1(x,t)=A[\sin(\omega x/v_1)+\tan(\omega L/v_1)\cos(\omega x/v_1)]\cdot\cos(\omega t)$$
$$0\le x\le +L: \quad s_2(x,t)=A\frac{v_2}{v_1}[\sin(\omega x/v_2)-\tan(\omega L/v_2)\cos(\omega x/v_2)]\cdot\cos(\omega t)$$

(c) x=0 is a node when $s_1(0,t)=s_2(0,t)=0$ for all t. This requires that the constraint

$$v_1\tan\left(\frac{\omega L}{v_1}\right)=-v_2\tan\left(\frac{\omega L}{v_2}\right)=0$$

Problem 2.9

The kinetic energy flux (energy per m^2) in the gas cylinder (mass density ρ) is determined by:

$$\bar{K}_\lambda = \frac{1}{2}\rho \int_0^{\lambda=cT} \left(\frac{\partial s}{\partial t}\right)^2 dt$$

where $s(x,t) = s_{max}\cos(\omega t - kx)$.The total kinetic energy is found by multiplying the solution with the cylinder's cross section:

$$K_\lambda = \frac{A}{4}\rho\omega^2 s_{max}^2 \lambda$$

Problem 2.10

Intensity relates to pressure (p) and impedance (Z) through

$$I = \frac{p^2}{Z}$$

Define the incident intensity as I_i, the reflected intensity is I_r and the transmitted intensity as I_t, then: $I_t=I_i-I_r$

The relative reflected intensity equals R^2, and thus

$$I_t = (1-R^2)\cdot I_i \quad \text{and} \quad I_r = R^2 \cdot I_i$$

it follows immediately that

$$I_r = \left(\frac{1-Z_2/Z_1}{1+Z_2/Z_1}\right)^2 \quad \text{and} \quad I_t = \frac{4Z_2/Z_1}{(1+Z_2/Z_1)^2}$$

Problem 2.11

To show:

$$\frac{2}{T}\int_0^T \sin(n\omega t)\cdot\sin(m\omega t)\,dt = \delta_{nm}$$

you use $\sin p \sin q = \frac{1}{2}[\cos(p-q) - \cos(p+q)]$ and distinguish the two conditions:

$$n \neq m$$
$$n = m$$

In the same way you show that sine and cosine are always mutually orthogonal, regardless their frequency:

$$\frac{2}{T}\int_0^T \sin(n\omega t)\cdot\cos(m\omega t)\,dt = 0$$

by using the identity $\sin p \cos q = \frac{1}{2}[\sin(p+q) + \sin(p-q)]$

Problem 2.12

Using the orthogonality relations, we can now derive the Fourier series. So, the series read:

$$f(t) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \quad \text{with} \quad \omega_0 = 2\pi/T$$

First, we take the time average of f(t) over the full period, T:

$$\frac{1}{T}\int_0^T f(t)dt$$

which shows that the constant term is the time-average of the function. Then multiply *f(t)* with *cos(mωt)* and take the time average:

to immediately obtain the even Fourier coefficients:

$$a_n = \frac{2}{T}\int_0^T f(t)\cdot\cos(n\omega_0 t)\,dt$$

Likewise, by multiplying f(t) with sin*(mωt)* and taking the time average yields b_n.

Problem 2.13

$$f(t) = t^2 - t \text{ on the interval } 0 \le t \le 1$$

a) Odd expansion: on the interval -1≤t≤0 the function should be defined such that f(t)=-f(-t), i.e.:
$f(t) = -(t^2 + t)$ on the interval $-1 \le t \le 0$ and the period of the function is T=2. Because the function is odd, all a_n=0, and one can find the Fourier series by calculating the b_n coefficients:

$$b_n = \begin{cases} 0 & n = \text{even} \\ -\dfrac{8}{(n\pi)^3} & n = \text{odd} \end{cases}$$

and the Fourier series finally reads:

$$f(t) = -\frac{8}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin[(2n+1)\pi t]$$

b) Even expansion: on the interval -1≤t≤0 the function should be defined such that f(t)=f(-t), i.e.: $f(t) = (t^2 + t)$ on the interval $-1 \le t \le 0$ and the period of the function is T=1. Because the function is even, all b_n=0, and one can find the Fourier series by calculating the a_n coefficients: $a_n = \frac{1}{(n\pi)^2}$ and $a_0 = -\frac{1}{3}$ (check!)
The even Fourier series is:

$$f(t) = -\frac{1}{6} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(n)^2}\cos(2\pi n t)$$

c) The odd expansion converges faster than the even expansion

Problem 2.14

b) You only calculate the even coefficients, as all b_n=0. You find:

$$a_0 = 1/c$$

$$a_n = \frac{2c}{(n\pi)^2}(1 - \cos(n\pi/c))$$

c) For the limit that c→∞ you obtain a flat Fourier spectrum:

$$a_n \approx \frac{1}{c} = a_0 = \text{constant}$$

Problem 2.15
Note that the initial condition at t=0 is an odd function, but beware: the period of this function is not $2D = 5\pi$ but $2D = 2\pi$!
You only need to calculate the b_n. The Fourier series can thus be written as:

$$s(x,t) = \sin(x)\cos(5t) + \sin(2x)\cos(10t)$$