The Auditory System and Human Sound-Localization Behavior
Answers Exercises Chapter 3: Linear Systems

Exercise 3.1:

For a Linear System, the output to a scaled input, say $x_1(t) = a \cdot x_0(t)$, should be the same scaled version of the output: $y_1(t) = a \cdot y_0(t)$. You can now infer what this means for the amplitude-duration and amplitude-peak velocity relationships of linear systems.

Exercise 3-2: Consider that the earth's surface plays a particular role, which is not incorporated in the formulation of this problem.

Exercise 3-3: First-Order High-Pass system

The impulse response of a first-order High-Pass system can be found from the Low-Pass result by using the idea (from Kirchhoff's voltage law!) that *High-Pass = All-Pass - Low-Pass*. Thus:

$$h(\tau) = \delta(\tau) - \frac{1}{RC} \cdot \exp\left(-\frac{\tau}{RC}\right) \tag{1}$$

The step response:

$$s(\tau) = \int_0^{\tau} h(t)dt = \exp\left(-\frac{\tau}{RC}\right) \tag{2}$$

Fourier analysis gives (using the above):

$$H(\omega) = 1 - \frac{1}{1+i\omega RC} = \frac{i\omega RC}{1+i\omega RC}$$

The amplitude and phase characteristics are:

$$G(\omega) = \frac{\omega T_c}{\sqrt{1+\omega^2 T_c^2}} \;\; \text{en} \;\; \Theta(\omega) = \arctan\left(\frac{1}{\omega T_c}\right) \tag{3}$$

Exercise 3-4: The ideal Integrator.

Since integration is a linear process, it should obey convolution:

$$y(t) = \int_0^t h(\tau)x(t-\tau)d\tau \equiv \int_0^t x(\tau)d\tau \quad \Rightarrow \quad h(\tau) = 1 \;\; \forall \tau \geq 0 \;\; \text{and} \;\; s(\tau) = \tau \;\; \forall\, \tau \geq 0 \tag{4}$$

The response of the integrator to a sinusoidal input is given as $y(t) = \int_0^t \sin(\omega\tau)d\tau$. The frequency characteristics for the integrator are thus:

$$G(\omega) = \frac{1}{\omega} \;\; \text{and} \;\; \Theta(\omega) = -\pi/2 \;\; \forall\, \omega \tag{5}$$

Exerxise 3-5: The ideal Differentiator.

Also differentiation is a linear process (it follows superposition principle). The derivative of the Dirac pulse is a Dirac pulse, immediately followed (at $t = 0$) by a negative Dirac pulse! This is a so-called *'doublet'* and can be made qualtitatively understandable by approximating the Dirac pulse by a rectangular pulse.
The frequency response of the differentiator is found by $y(t) = d\sin(\omega t)/dt$
Therefore:

$$s(\tau) = \delta(\tau) \quad G(\omega) = \omega \;\text{ en }\; \Theta(\omega) = \pi/2 \tag{6}$$

Exercise 3-6:
To create a **bandpass filter** (BP) we place a LP and a HP filter in series, but we should ensure that the respective frequency bands overlap (and therefore the cut-off frequencies are positioned correctly). The time constants thus:

$$T_{\text{low}} < T_{\text{high}} \Rightarrow (R_L C_L) < (R_H C_H)$$

The difference in cut-off frequencies then determines the bandwidth of the BP filter:

$$BW = \frac{1}{R_L C_L} - \frac{1}{R_H C_H}$$

The transfer characteristic of the filter then is:

$$H(s) = \frac{sT_H}{(1 + sT_L)(1 + sT_H)}$$

with two poles, in $s = -1/T_L$ and $s = -1/T_H$, respectively, with a d.c.-gain and an $\omega = \infty$ transfer of 0 (as it should be). For the amplitude characteristic:

$$\|H(s)\| = \|H(s)_L\| \cdot \|H(s)_H\| = \frac{\omega T_H}{\sqrt{1 + \omega^2 T_L^2}\sqrt{1 + \omega^2 T_H^2}}$$

for which you can verify that this is maximal at:

$$\omega_o = \frac{1}{\sqrt{T_L \cdot T_H}}$$

For a **band-stop** (BS) filter we demand the opposite: now the cut-off point of the LP filter should fall outside the HP cut-off (no overlap), but for any transfer at all, the two filters need to be placed parallel to each other! Now the transfer characteristic is:

$$H(s) = \frac{1}{(1 + sT_L)} + \frac{1}{(1 + sT_H)}$$

for which the amplitude and phase behavior can be readily assessed by applying some calculus.

An **All-Pass** system can be made by putting the HP and LP filters in parallel, but with their cut-off frequencies the same as for the BP filter: the time constant of the LP filter is shorter than for the HP filter. For the transfer characteristic we obtain the same relation as for the BS filter, but the time constants were chosen differently, and therefore the filter has different properties.

Exercise 3-7:

(a) For a series concatenation of two first-order LP filters with different time constants, we obtain the following impulse response:

$$h(t) = \int_0^\infty h_2(\tau) h_1(t-\tau) d\tau$$

and with $h_1(t-\tau) = 0$ for $\tau > t$ (causality) the integration boundaries become $[0, t]$.
Now substitute the expressions for the two first-order impulse-response functions:

$$h(t) = \frac{1}{T_1 - T_2} \cdot \left(\exp(-\frac{t}{T_1}) - \exp(-\frac{t}{T_2}) \right)$$

This function is 0 for $t = 0$ and approaches 0 for $t \to \infty$.

The step response of this system is given by integration of the impulse response:

$$s(\tau) = \int_0^\tau h(\sigma) d\sigma = \frac{T_1}{T_1 - T_2}[1 - \exp(-\tau/T_1)] - \frac{T_2}{T_1 - T_2}[1 - \exp(-\tau/T_2)]$$

a function which in its increasing behavior shows the influence of the two time constants.

(b) The transfer characteristic is described by:

$$H(\omega) = \frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)}$$

It's behavior is LP (approaches zero for $\omega = \infty$, and 1 for d.c.), and is characterized by two cut-off points because of the two time constants. Construct the Bode plot by summing the characteristics of the two LP filters.

Exercise 3-8:

The impulse response is:

$$h(t) = A \cdot U(t - \Delta T) \text{ for } t \geq 0$$

with $U(x)$ the unitary step. The step response:

$$s(t) = A \cdot r(t - \Delta T)$$

with

$$\begin{aligned} r(x) &= 0 \text{ for } x < 0 \\ &= x \text{ for } x \geq 0 \end{aligned} \tag{7}$$

The amplitude and the phase characteristics:

$$G(\omega) = \frac{A}{\omega} \text{ and } \Phi(\omega) = -\frac{\pi}{2} - \omega \cdot \Delta T$$

Exercise 3-9:

(a)

$$H(\omega) \equiv \frac{Y(\omega)}{X(\omega)} = \frac{H_1(\omega) \cdot H_2(\omega)}{1 - H_2(\omega) \cdot H_3(\omega) \cdot H_4(\omega)}$$

(b) This system will become unstable whenever

$$H_2(\omega) \cdot H_3(\omega) \cdot H_4(\omega) = 1$$

(c) If $H_3(\omega) = G$, and $H_2(\omega) = 1/(j\omega)$ (pure integrator), it follows that when

$$H_4(\omega) = j\omega G$$

the system is unstable $\forall\omega$.

(d) Now also $H_4(\omega) = A$, a constant gain. The total transfer characteristic of the system becomes

$$H(\omega) = \frac{H_1(\omega)/(j\omega)}{1 - AG/(\jmath\omega)} =$$

which can be rewritten as

$$H(\omega) = \frac{H_1(\omega) \cdot (AG + j\omega)}{A^2G^2 + \omega^2}$$

which is never unstable (provided $H_1(\omega)$ is stable).

Exercise 3-10: First recall the ideal differentiator ($y(t) = dx/dt$):

$$G(\omega) = \omega \quad \text{and} \quad \Phi(\omega) = +\pi/2$$

Note that the gain increases without bound with ω. This is bad news, because signals tend to have noise, which tends to contain high frequencies: differentiation increases the noise in the signal!

(a) The digital implementation of the differentiator with the difference algorithm:

$$\dot{y}(t) = \frac{y(t + \Delta T) - y(t)}{\Delta T}$$

where $DeltaT$ is a (short) delay. The transfer characteristic of this system is therefore:

$$H(j\omega) \equiv \frac{\dot{Y}}{Y} = \frac{e^{j\omega\Delta T} - 1}{\Delta T}$$

The amplitude- and phase-characteristics are:

$$G(\omega) = \frac{\sqrt{2 + 2\cos(\omega\Delta T)}}{\Delta T} \quad \Phi(\omega) = \arctan\frac{\sin(\omega\Delta T)}{\cos(\omega\Delta T) - 1}$$

Now that the gain does not indefinitely increase with frequency! In fact, the difference algorithm acts as a filter on the signal. To find this filter, we compare the relation between the real differentiator $(j\omega)$, and the difference algorithm, which yields:

$$H_{\text{filter}} = \frac{H_{\text{algor}}}{H_{\text{diff}}} = \frac{e^{j\omega\Delta T} - 1}{j\omega\Delta T}$$

which is a low-pass filter! Gain and phase:

$$G_{\text{filter}} = \frac{\sqrt{2 + 2\cos(\omega\Delta T)}}{\omega\Delta T} \quad \Phi_{\text{filter}}(\omega) = \arctan\left(\frac{\sin(\omega\Delta T)}{\cos(\omega\Delta T) - 1}\right) - \pi/2$$

(b) One could add a serial low-pass filter (time constant T) to the (pure) differentiator. The total transfer characteristic is then:

$$H(j\omega) = \frac{j\omega}{\jmath\omega T + 1} \Rightarrow G(\omega) = \frac{\omega}{\sqrt{\omega^2 T^2 + 1}} \quad \Phi(\omega) = \frac{\pi}{2} - \arctan\omega T$$

Exercise 3-11:

(a) The transfer function for a delay is:

$$H(s) = \exp(-s\Delta T)$$

which has an amplitude characteristic $|H(\omega)| = 1 \;\; \forall\omega$ and phase characteristic $\Phi(\omega) = -\omega\Delta T$.

(b) The transfer function of the total feedback system is:

$$H(s) = \frac{A\exp(-s\Delta T)}{1 + sT + A\exp(-s\Delta T)}$$

The loop gain is given by the product of all systems in the loop:

$$L(s) = \frac{A\exp(-s\Delta T)}{1 + sT}$$

for which the amplitude characteristic is:

$$|L(\omega)| = \frac{A}{\sqrt{\omega^2 + T^2}}$$

and the phase characteristic:

$$\Phi(\omega) = -\omega\Delta T - \arctan(\omega T)$$

Unstable behaviour of the system occurs when $\Phi(\omega_0) = -180^o$ and $|L(\omega_0)| > 1$. It is convenient to try to approximate this numerically.
(c) Lowering A or increasing T brings the gain below 1 and the instability disappears. Increasing A, or lowering T shifts the instability point to higher frequencies. If these frequencies fall beyond the operation range, this could also be an attractive solution to prevent instability.

Exercise 3-12:

(a) The microphone picks up a superposition of the direct, unfiltered, input and the filtered/delayed input. In the time domain this is written as

$$p^*(t) = p(t) + \int_0^\infty a(\tau) \cdot p(t - \Delta T - \tau) d\tau$$

(b) Fourier transformation yields the following transfer characteristic:

$$T(\omega) = \frac{P^*(\omega)}{P(\omega)} = 1 + A(\omega) \cdot \exp(-j\omega\Delta T)$$

(c) In case of N reflectors in the environment, each with their own delay and filter, the time-domain and frequency domain expressions become:

$$p^*(t) = p(t) + \sum_{n=1}^{N} \int_0^\infty a_n(\tau) \cdot p(t - \Delta T_n - \tau) d\tau$$

with transfer characteristic:

$$T(\omega) = 1 + \sum_{n=1}^{N} A_n(\omega) \cdot \exp(-j\omega\Delta T_n)$$

(d) For N=1:

$$T(\omega) = 1 + A(\omega) \cdot \exp(-j\omega\Delta T)$$

with

$$G(\omega) = \sqrt{1 + 2 \cdot A(\omega)\cos(\omega\Delta T)} \quad \text{and} \quad \Phi(\omega) = -\arctan\left(\frac{A(\omega)\sin(\omega\Delta T)}{1 + A(\omega)\cos(\omega\Delta T)}\right)$$

The delay is measured directly from the amplitude spectrum by determining the distance between consecutive peaks. The distance follows by multiplying this delay by 340 m/s.

Exercise 3-13:

The autocorrelation function is defined by

$$\Phi_{xx}(\tau) = \int_{interval} x(t) \cdot x(t + \tau) dt$$

(a) This is a transient signal, and

$$\Phi_{xx}(\tau) = P^2 \cdot (2T - |\tau|) \text{ for } |\tau| \leq 2T \text{ and } \Phi_{xx}(\tau) = 0 \text{ elsewhere.}$$

(b) when $x(t) = A\cos(\omega t)$ the autocorrelation is calculated by integrating over a full period, $T = 2\pi/\omega$:

$$\Phi_{xx}(\tau) = \frac{A^2\omega}{2\pi}\int_0^T \cos(\omega t) \cdot \cos(\omega(t+\tau))dt$$

use $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and the orthogonality relations for harmonic functions, to find

$$\Phi_{xx}(\tau) = \frac{A^2}{2}\cos(\omega\tau)$$

(c) For the exponential, transient pulse ($t \geq 0$) we calculate:

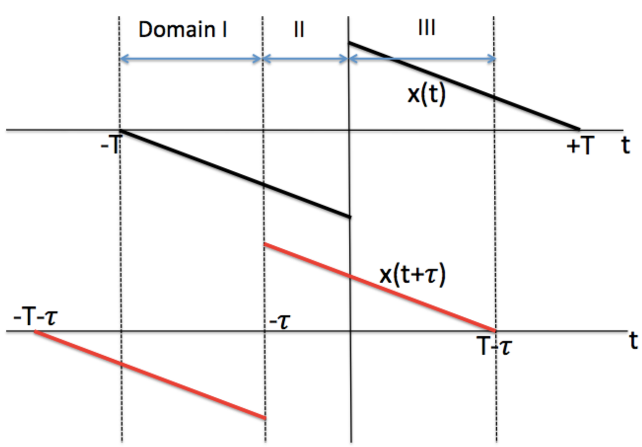
$$\Phi_{xx}(\tau) = \frac{\exp(-\alpha|\tau|)}{2\alpha}$$

(d) This is an odd, straight-line function, running between $-T$ and $+T$, with maximum $x(0) = A$. Like in (a), the integration is done over the intervals where the functions overlap, which is constrained to $|\tau| \leq 2T$. We obtain:

$$\Phi_{xx}(-2T \leq \tau \leq 2T) = A^2 \int_{-T}^{T-\tau} (sgn(t) - t/T) \cdot (sgn(t) - (t+\tau)/T)dt$$

In $\tau = 0$ the maximum is

$$\Phi_{xx}(0) = A^2 \int_{-T}^{T} (1 - t/T)^2)dt = \frac{8A^2T}{3}$$



The calculation requires integration over three different domains.....

We note that the autocorrelation function is always symmetric re. $\tau = 0$. We can suffice by calculating this function, e.g. for postivie values of τ, i.e. for the domain $0 \leq \tau \leq 2T$. In that case, we will have to distinguish three domains (see figure): I: $-T \leq t \leq -\tau$, II: $-\tau \leq t \leq 0$, III: $0 \leq t \leq T - \tau$ over which to integrate.

Exercise 3-14:

Application of the cross-correlation function to the low-pass filter.

(a) The cross correlation function is defined as

$$\Phi_{yx}(\tau) \equiv \int_{-\infty}^{\infty} y(t)x(t+\tau)dt$$

with $y(t)$ the filter's output function (a *transient* signal), and $x(t)$ =GWN. Substitute the convolution integral for $y(t)$ to find:

$$\Phi_{yx}(\tau) = \frac{P}{RC}\exp(-|\tau|/(RC))$$

where we used the auto correlation property of GWN: $\Phi(\tau)_{xx} = P\delta(\tau)$

(b) For the autocorrelation function of the response use the definition:

$$\Phi_{yy}(\tau) \equiv \int_{-\infty}^{\infty} y(t)y(t+\tau)dt$$

leading to

$$\Phi_{yy}(\tau) = \frac{P}{2RC}\exp(-|\tau|/(RC))$$

(c) Calculating the output when $x(t) = P$ for $-T \leq t \leq T$. Note that the pulse can be written as a *superposition* of two steps, and then apply the superposition principle. The steps are delivered at $t = -T$ and $t = +T$, repsectively:

$$x_P(t) = P \cdot U(t+T) - P \cdot U(t-T)$$

We recall that the step response of the LP filter to a unity step at $t = 0$ is determined by:

$$y_{\text{step at } t=0}(t) = 1 - \exp(-t/RC) \;\; \text{for} \;\; t \geq 0$$

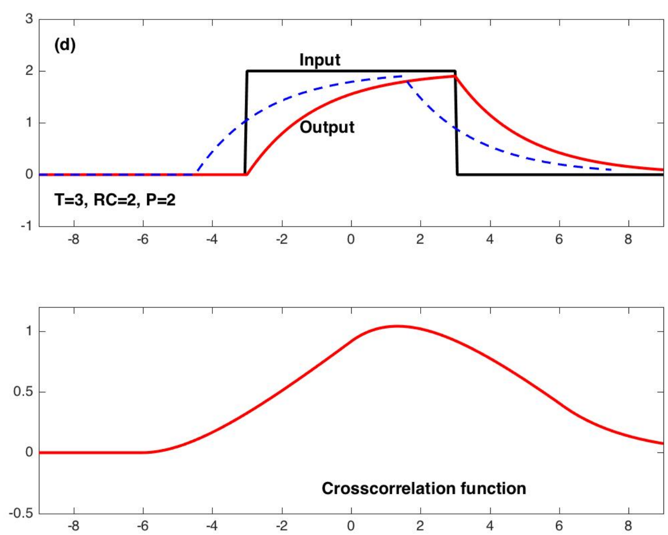
The system is time-invariant. Therefore, the response to the two steps at $t = -T$ and $t = +T$ is:

$$y_P(t) = \begin{cases} 0 \;\; \text{for} \;\; t \leq -T \\ P \cdot (1 - \exp(-(t+T)/RC)) \;\; \text{for} \;\; -T \leq t \leq T \\ P \cdot \exp(-t/RC) \cdot [\exp(T/RC) - \exp(-T/RC)] \;\; \text{for} \;\; t \geq T \end{cases}$$

(d) The explicit calculation of the cross-correlation function becomes a bit tedious, because of the different domains (like in Exc. 3-13d):

$$\Phi_{y_P x}(\tau) = \int_{-\infty}^{\infty} y_P(t) \cdot x_P(t+\tau) dt$$

Note that for $\tau \leq -2T$ the crosscorrelation function is zero, but on the positive side τ extends (at least in principle) to infinity. Also the maximum is found at $\tau > 0$ (the red curve in the figure below, when shifted leftward by $T/2$ (blue dashed curve), gives a maximum overlap ('correlation'!) with $x(t)$). The cross-correlation function is therefore asymmetric re. $\tau = 0$, peaking at $t = T/2$. The figure below (generated by Matlab script Exc3-14.m) shows its full behavior.



The input (b) and output (r) functions (top) with their associated crosscorrelation (botom).