The Auditory System and Human Sound-Localization Behavior
Short Answers to the Exercises of Chapter 4: Nonlinear Systems

Exercise 4.1:

(a) With the input given by $x(t) = \sin(\omega_1 t) + \sin(\omega_2 t)$, the output is found to be:

$$y(t) = a \cdot (\sin(\omega_1 t) + \sin(\omega_2 t)) + b \cdot (\sin(\omega_1 t) + \sin(\omega_2 t))^2 + c \cdot (\sin(\omega_1 t) + \sin(\omega_2 t))^3$$

Collecting terms yields the following 13 frequency components with their amplitudes (n.b.: all at phase 0 (positive) or π (negative)):

frequency	amplitude
0	b
ω_1	$(a + 5c/4)$
ω_2	$(b + 5c/4)$
$2\omega_1$	$-b/2$
$2\omega_2$	$-b/2$
$\omega_1 - \omega_2$	b
$\omega_1 + \omega_2$	b
$\omega_1 + 2\omega_2$	$-3c/4$
$\omega_1 - 2\omega_2$	$-3c/4$
$\omega_2 + 2\omega_1$	$-3c/4$
$\omega_2 - 2\omega_1$	$-3c/4$
$3\omega_1$	$-c/4$
$3\omega_2$	$-c/4$

(b) The Matlab script is found on the Book's web page as *Chapter4-Exc4-1.m*

Exercise 4-2:

(a) The function $y(t) = \log(x(t) + 1)$ is defined on $|x(t)| < 1$, for which the Taylor expansion thus yields

$$y(t) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

The associated Volterra kernels for this nonlinear function are thus determined by:

$$k_n(\tau_1, \tau_2, \cdots, \tau_n) = \frac{(-1)^{n+1}}{n} \Pi_{k=1}^{n} \delta(\tau_k)$$

(b) For the exponential function $y = exp(x(t))$, the Volterra series reads

$$k_0 = 1 \text{ and } k_n(\tau_1, \tau_2, \cdots, \tau_n) = \frac{1}{n!}\Pi_{k=1}^{n} g(\tau_k)$$

(c) The cosine function $y(t) = \cos(x(t))$ is expanded as

$$k_0 = 1 \text{ and } k_n(\tau_1, \tau_2, \cdots, \tau_n) = \frac{(-1)^n}{(2n)!}\Pi_{k=1}^{2n} \delta(\tau_k) \text{ for } n \geq 1$$

Exercise 4-3:

(a) The following holds: $\int_0^T f^2(x) \cdot dx < \infty$ ($f(x)$ is an absolute integrable function on the interval $[0, T]$). We substitute the orthogonality condition, to find:

$$I_N = \int_0^T f^2(x)dx - 2\sum_{n=1}^{N} a_n \int_0^T f(x)w_n(x)dx + \sum_{n=1}^{N} a_n^2$$

We rewrite:

$$I_N = \int_0^T f^2(x)dx + \sum_{n=1}^{N}\left[a_n - \int_0^T f(x)w_n(x)dx\right]^2 - \sum_{n=1}^{N}\left[\int_0^T f(x)w_n(x)dx\right]^2$$

We thus have to minimize this term, which happens when:

$$a_n = \int_0^T f(x)w_n(x)dx$$

(b) The miminum integrated error is:

$$I_{N(\text{min})} = \int_0^T f^2(x)dx - \sum_{n=1}^{N} a_n^2$$

And the following must hold

$$\sum_{n=1}^{N} a_n^2 \leq \int_0^T f^2(x)dx$$

This is the so-called Bessel inequality. When the orthogonal basis set is *complete*, it follows that

$$\lim_{N\to\infty} I_{N(\text{min})} = 0$$

Exercise 4-4:

The general second-order inhomogeneous Volterra functional reads:

$$G_2[x(t),P] = k_{0,2} + \int_0^\infty k_{12}(\tau)\cdot x(t-\tau)d\tau + \int_0^\infty\int_0^\infty k_{22}(\tau_1,\tau_2)\cdot x(t-\tau_1)\cdot x(t-\tau_2)d\tau_1 d\tau_2$$

and the first two Wiener functionals are

$$G_0 = h_0 \text{ and } G_1[x(t),P] = \int_0^\infty h_1(\tau)\cdot x(t-\tau)d\tau$$

We have to determine $k_{02}, k_{12}(\tau)$ and $k_{22}(\tau_1,\tau_2)$. We demand orthogonality by setting the expectation values of the inner products to zero:

$$\overline{G_0 \cdot G_2} = 0 \text{ and } \overline{G_1 \cdot G_2} = 0$$

It then follows (using the autocorrelation properties of GWN, Ewn. 4.16):

$$G_2[x(t),P] = \int_0^\infty\int_0^\infty h_2(\tau_1,\tau_2)\cdot x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 - P\cdot\int_0^\infty h_2(\tau,\tau)d\tau$$

Exerxise 4-5:

The first-order Wiener functional is:

$$G_1[h_1;x(t),P] = \int h_1(\tau)\cdot x(t-\tau)d\tau$$

and the third-order Wiener functional reads:

$$\begin{aligned} G_3[h_3;x(t),P] = & \int\int\int h_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)\cdot x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \\ & -3P\cdot\int\int h_3(\tau_1,\tau_2,\tau_2)x(t-\tau_1)d\tau_1 d\tau_2 \end{aligned}$$

We calculate the expectation value of the inner product:

$$\overline{G_1[h_1;x(t),P]\cdot G_3[h_3;x(t),P]}$$

and note that the two terms of $G_1 \cdot G_3$ will contain a double product and a four-product of GWN samples. It follows that

$$\overline{G_1\cdot G_3} = 3P^2\int\int h_1(\tau_p)h_3(\tau_p,\tau_q,\tau_q)d\tau_p d\tau_q - 3P^2\int\int h_1(\tau_1)h_3(\tau_1,\tau_2,\tau_2)d\tau_1 d\tau_2 = 0$$

Exercise 4-6:

The output of the system in Fig 4.5 is described as follows:

$$\begin{aligned} y(t) = & \quad a \cdot w(t) + b \cdot w^2(t) \\ = & \int_0^\infty ag(\tau) \cdot x(t-\tau)d\tau + \int_0^\infty \int_0^\infty bg(\tau_1)g(\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 \end{aligned}$$

from which the Volterra kernels are given by:

$$\begin{aligned} k_0 = & \quad 0 \\ k_1(\tau) = & \quad ag(\tau) \\ k_2(\tau_1, \tau_2) = & \quad bg(\tau_1)g(\tau_2) \end{aligned}$$

The Lee and Schetzen method (Eqns. 4.30-4.31) applies crosscorrelation between input and output to determine the Wiener kernels. This is how this works (we omitted all odd-numbered products in $x(t)$, as these yield zero):

$$h_0 = E[y(t)] = Pb \int_0^\infty g^2(\tau_1) d\tau_1$$

$$h_1(\sigma) = \frac{1}{P} E[y(t)x(t-\sigma)] = ag(\sigma)$$

$$\begin{aligned} h_2(\sigma_1, \sigma_2) = & \quad \tfrac{1}{2P^2} E[(y(t) - h_0)x(t-\sigma_1)x(t-\sigma_2)] \\ = & \quad bg(\sigma_1)g(\sigma_2) \end{aligned}$$

Indeed, the first-order and second-order Volterra and Wiener kernels are identical.

Exercise 4-7:

The fourth Hermite polynomial with respect to a GWN signal with power P reads:

$$\mathsf{He}_4 = x^4 - 6Px^2 + 3P^2$$

From the analogy in Eqn. 4.18, the fourth-order Wiener functional can thus be constructed:

$$\begin{aligned} G_4[h_4; x(t), P] = & \quad \int\int\int\int h_4(\tau_1, \tau_2, \tau_3, \tau_4) \cdot \Pi_{n=1}^4 x(t-\tau_n) d\tau_n + \\ & -6P \int\int\int h_4(\tau_1, \tau_2, \tau_3, \tau_3) x(t-\tau_1)x(t-\tau_2) d\tau_1 d\tau_2 d\tau_3 + \\ & +3P^2 \int\int h_4(\tau_1, \tau_1, \tau_2, \tau_2) d\tau_1 d\tau_2 \end{aligned}$$

Exercise 4-8:

From Fig. 4.11, we obtain the following relationships:

$$y(t) = \int_0^\infty \int_0^\infty ak(\sigma)h(\tau)x(t-\sigma-\tau)d\sigma d\tau + \\ + \int_0^\infty \int_0^\infty \int_0^\infty bk(\sigma)h(\tau_1)h(\tau_2)x(t-\sigma-\tau_1)\cdot x(t-\sigma-\tau_2)d\sigma d\tau_1 d\tau_2$$

For the Wiener kernels we take $x(t)$=GWN, power P and mean zero.
Taking the expectation value of the output yields the zero-order Wiener kernel (we leave out the odd-numbered products in $x(t)$):

$$h_0 = E[y(t)] = P \cdot b \cdot \left[\int_0^\infty h(\tau)d\tau\right]^2 \cdot \left[\int_0^\infty k(\sigma)d\sigma\right]$$

The first-order Wiener kernel is found by cross-correlation:

$$h_1(\lambda) = aP \cdot \int_0^\infty k(\sigma)h(\lambda-\sigma)d\sigma$$

Finally, we find the second-order Wiener kernel from the third-order cross-correlation:

$$\phi_{yxx}(\lambda_1,\lambda_2) = E[y(t)x(t-\lambda_1)x(t-\lambda_2)]$$

Corrected for the average, h_0, the second-order Wiener kernel is given by:

$$h_2(\lambda_1,\lambda_2) = \frac{\phi_{(y-h_0)xx}}{2P^2} = \frac{b}{2}\int_0^\infty k(\sigma)h(\sigma-\lambda_1)h(\sigma-\lambda_2)d\sigma$$

Exercise 4-9:
In Fig. 4.13 we calculate:

$$u(t) = \int_0^\infty g(\tau)x(t-\tau)d\tau \quad \text{and} \quad y(t) = [u(t)]^2$$

so that

$$y(t) = \int_0^\infty \int_0^\infty g(\tau_1)g(\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

from which it follows that $k_0 = 0$, $k_1(\tau) = 0$ and $k_n(\tau_1, \tau_2, \cdots, \tau_n) = 0$ for $n \geq 3$, and

$$k_2(\tau_1,\tau_2) = A^2 \cdot \exp(-k(\tau_1+\tau_2)) \cdot \sin(m\tau_1) \cdot \sin(m\tau_2)$$

with $A = 6.67$, $k = 0.08$ and $m = 0.3$.

Exercise 4-10:
For this exercise, the reader is referred to the Matlab section for *Chapter4-Exc4-10.m*

Exercise 4-11:

We use the 'hint' to determine the n-th order expectation value of the output $y(t)$ (or: the n-th order autocorrelation function of $y(t)$), which is given by taking the following time-average:

$$\overline{y(t-\sigma_1)\cdots y(t-\sigma_n)} = \int_0^\infty \cdots \int_0^\infty h(\tau_1)\cdots h(\tau_n) \times \overline{x(t-\tau_1-\sigma_1)\cdots x(t-\tau_n-\sigma_n)} d\tau_1 \cdots d\tau_n$$

We assume that $x(t)$ is a Gaussian process, with average zero. For odd values of n, i.e., when it can be written as $n = 2m+1$ for $m = 0, 1, 2, 3, \cdots$, we see that

$$\overline{y(t-\sigma_1)y(t-\sigma_2)\cdots y(t-\sigma_{2m+1})} = 0$$

For even values of n, when we write $n = 2m$, the following holds (using Eqn. 4.16):

$$\overline{y(t-\sigma_1)\cdots y(t-\sigma_{2m})} = \Sigma\Pi\overline{y(t-\tau_i-\sigma_i)y(t-\tau_j-\sigma_j)}$$

We conclude that also $y(t)$ has to be a Gaussian process!