The Auditory System and Human Sound-Localization Behavior
Short Answers Exercises Chapter 5: Cochlea

Exercise 5.1:

(a) The traveling wave solution for dry water is proposed to be:

$$\psi_y(x,y,t) = A\cos(\omega t - kx)\left(e^{ky} - e^{-2kh}\cdot e^{-ky}\right)$$
$$\psi_x(x,y,t) = A\sin(\omega t - kx)\left(e^{ky} + e^{-2kh}\cdot e^{-ky}\right)$$

You verify the validity of these solutions by substitution into the required conditions:

$$\vec{\nabla}\vec{\psi} = 0 \quad \text{and} \quad \frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y} = 0$$

(b) In the deep-water wave approximation $h \gg \lambda$, from which the factor $\exp(-2kh) \downarrow 0$.

(c) In the shallow-water approximation, $h \ll \lambda$ and $y \ll \lambda$, so that $\exp(-2kh) \approx 1 - 2kh$, $\exp(ky) \approx 1 + ky$ and $\exp(-ky) \approx 1 - ky$.

Exercise 5.2:

(a) To find the dispersion reslation for gravitational waves, use the property of harmonic oscillations, described on page 122:

$$\omega^2 = g \cdot \left[\frac{\partial \psi_y}{\partial x}\right]_{y=0} / \left[\psi_x\right]_{y=0}$$

and substitute the wave functions ψ_x and ψ_y in the solution of the traveling waves.

(b) Again, for deep water the exponentials in the solution can be neglected, and you find

$$\omega^2 \approx gk$$

The phase velocity is:

$$v_\varphi = \frac{\omega}{k} \ \text{etc.}$$

and the group velocity is

$$v_g = \frac{d\omega}{dk} = \text{etc.}$$

When the phase velocity and group velocity are unequal, the wave shape of a pulse, consisting of multiple frequencies (wave lengths) will change. This happens in dispersive media (speed depends on the wave length).

(c) For shallow water, take $h \ll \lambda$, and thus approximate, to find:

$$\omega = k \cdot \sqrt{gh}$$

from which the phase velocity and group velocity result to be equal.

Exercise 5.3:

(a) The pressure from the surface tension is given by the product of the surface tension coefficient, T and the convec curvature. The curvature, K, is inversely proportional to the curvature radius, R_0, which in turn is determined by the shape of the wave function, $\psi_y(x)$:

$$K \equiv \frac{1}{R_0} = \frac{\frac{\partial^2 \psi_y}{\partial x^2}}{\left[1 + \left(\frac{\partial \psi_y}{\partial x}\right)^2\right]^{3/2}}$$

(this formula can be found in any calculus book, or Wikipedia). We assume here that the wave function has a sinusoidal shape, so we can take $\psi_y = A \sin kx$. You then readily find the curvature:

$$K \approx -k^2 \psi_y$$

We here assumed that the wave length is sufficiently large, so that we may approximate $1 + k^2 \approx 1$. As a result, the downward directed pressure from the surface tension is:

$$p(x) = Tk^2 \cdot \psi_y(x)$$

Following the same procedure as done for the gravity waves, you calculate the net force from the surface tension, by looking at a small volume with length Δx (small re. λ), height Δy, and length L (for this, you ignore the contribution of gravity). The force in the x-direction on this small volume element is given by the surface $L\Delta y$ times the pressure difference $p(x + \Delta x) - p(x)$:

$$\begin{aligned} F_x &= -L\Delta y \cdot [p(x + \Delta x) - p(x)] \\ &= -\Delta V \cdot Tk^2 \cdot \left[\frac{\partial \psi_y}{\partial x}\right]_{y=0} \\ &= -(\Delta M)\frac{Tk^2}{\rho} \cdot \left[\frac{\partial \psi_y}{\partial x}\right]_{y=0} \\ (Newton) &= (\Delta M) \cdot \frac{\partial^2 \psi_x}{\partial t^2} \end{aligned}$$

The dispersion relation at the surface ($y = 0$) then is:

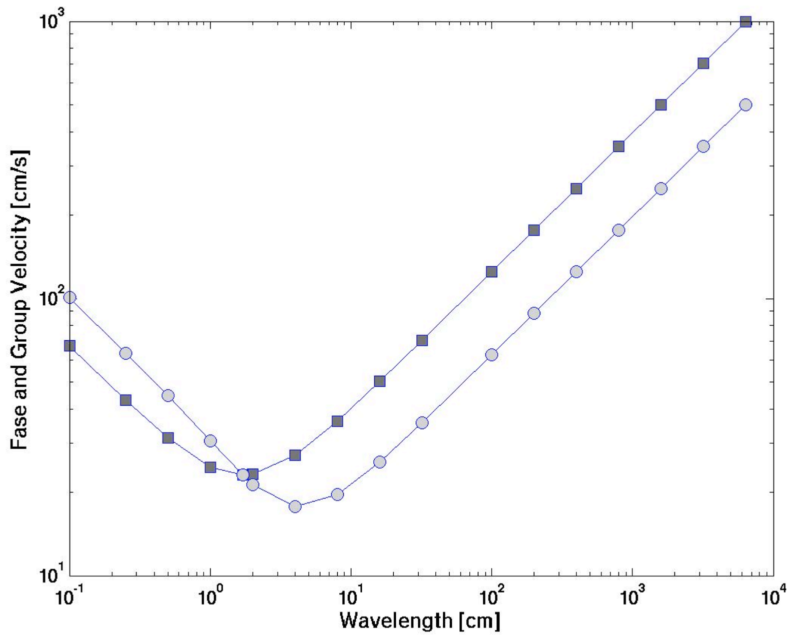
$$\omega^2 = \frac{Tk^2}{\rho} \cdot \left[\frac{\partial \psi_y}{\partial x}\right]_{y=0} \Big/ [\psi_x]_{y=0}$$

in which you insert the solutions ψ_y and ψ_x for the surface tension waves:

$$\omega^2(k) = \frac{Tk^3}{\rho} \cdot \tanh(kh)$$

The total combined effects of gravity and surface tension gives the sum of their contributions:

$$\omega^2(k) = \left(gk + \frac{Tk^3}{\rho}\right) \cdot \tanh(kh)$$



Square symbols: phase velocity; circular symbols: group velocity; both are shown as function of wave length (on log-log scale). The phase velocity is minimal at $\lambda = 1.7$ cm, the group velocity at $\lambda \approx 4$ cm.

(b) In the deep-water approximation you neglect the right-hand factor:

$$\omega = k\sqrt{\frac{g}{k} + \frac{Tk}{\rho}}$$

so that the phase velocity is given as

$$v_\varphi = \frac{\omega}{k} = \text{etc.}$$

The group velocity you get from

$$v_g = \frac{d\omega}{dk} = \text{etc.}$$

Phase- and group velocities are then found to be equal when:

$$k = \sqrt{\frac{g\rho}{T}} \quad \text{and} \quad \lambda = 2\pi\sqrt{\frac{T}{\rho g}}$$

From the literature it is found that the surface tension of water is $T = 72$ dyne/cm = 0.072 N/m. This yields:

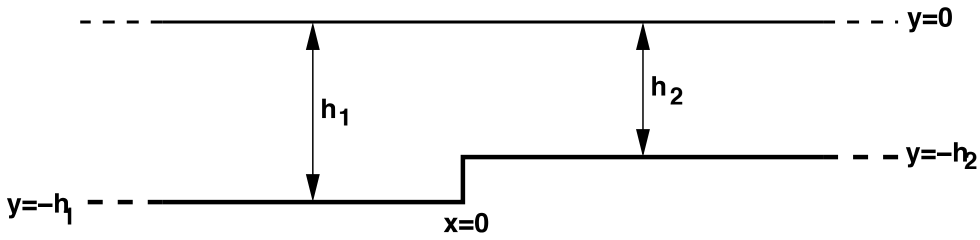
$$\lambda = 2\pi\sqrt{\frac{0.072}{1000 \cdot 9.81}} = 1.7 \text{ cm}$$

(c) The table below is generated with Matlab function *Chapter5-Table-Exc5-3.m* (see Matlab resources)

λ (cm)	f (Hz)	v_φ (cm/s)	v_g (cm/s)	$\frac{v_g}{v_\varphi}$
0.1	673.8	67.4	100.8	1.5
0.25	172.0	43.0	63.6	1.5
0.5	62.7	31.4	44.5	1.4
1.0	24.7	24.7	30.7	1.2
$\Rightarrow$ **1.7**	13.6	**23.1**	23.1	**1.0**$\Leftarrow$
2.0	11.6	23.2	21.4	0.9
4.0	6.8	27.2	17.7	0.7
8.0	4.5	36.1	19.6	0.5
16	3.1	50.3	25.7	0.5
32	2.2	70.8	35.6	0.5
100	1.2	125.0	62.5	0.5
200	0.9	176.7	88.4	0.5
400	0.6	249.9	125.0	0.5
800	0.4	353.4	176.7	0.5
1600	0.3	499.8	249.9	0.5
3200	0.2	706.8	353.4	0.5
6400	0.2	999.6	499.8	0.5

Note that for $\lambda = 1.7$ cm the phase velocity of the waves reaches a *minimum* (and note that the group velocity is minimal at a different wave length!)
This specific wave length of 1.7 cm separates two different 'regimes' for water waves (see figure).

Exercise 5.4:



(a) We deal with reflection and transmission in $x = 0$ when the impedances for $x < 0$ and $x > 0$ differ. For the impedance the following holds: $Z = \rho \cdot v_\varphi$. The density of the water is the same throughout, so we only have to look at the velocities in the two compartments. These are found from the dispersion relation:

$$\omega^2 = gk \cdot \tanh(kh) \quad \Rightarrow \quad v_\varphi = \sqrt{\frac{g\lambda}{2\pi}\tanh(kh)}$$

In the deep-water case ($\lambda \ll h$, so thats $hk \to \infty$) and we can take $\tanh(kh) \approx 1$. In other words: $\omega^2 \approx gk$. For the velocity we thus find:

$$v_\varphi = \frac{\omega}{k} = \text{etc.}$$

Note that the velocity depends on wavelength (dispersive), but **not** on depth! Thus, the velocities are the same in both compartments. As a consequence, the impedances are equal, and the reflection coefficient will be zero, and the transmission $T = 1$. The wave passes the step in depth unperturbed.

(b) For the shallow case $\lambda \gg h$, and therefore you may approximate $\tanh(kh) \approx kh$. Now, the dispersion relation is $\omega^2 \approx ghk^2$, and the velocity:

$$v_\varphi = \frac{\omega}{k} = \text{etc.}$$

The velocity is independent of wave length (non-dispersive), bue it does depend on the depth. As a consequence, the impedances in the two media will differ and there will be reflection at $x = 0$. The reflection coefficient is given by:

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{\sqrt{h_1} - \sqrt{h_2}}{\sqrt{h_1} + \sqrt{h_2}}$$

and this positive as $h_1 > h_2$. For the transmitted wave we obtain

$$T = \frac{2Z_1}{Z_1 + Z_2} = \frac{2\sqrt{h_1}}{\sqrt{h_1} + \sqrt{h_2}}$$

Exercise 5.5:

(a) Suppose that the LT of $x(t)$ is given by $X(s)$, then the delayed signal becomes:

$$Y(s) = X(s) \cdot e^{-s\Delta T}$$

The transfer function for a delay is therefore $H \equiv Y/X$:

$$H(s) = \exp(-s\Delta T)$$

In the frequency domain we have: substitute $s = j\omega$:

$$H(\omega) = \exp(-j\omega\Delta T)$$

which has an amplitude characteristic $|H(\omega)| = 1 \quad \forall\omega$ and phase characteristic $\Phi(\omega) = -\omega\Delta T$.

(b) For the transfer function of the total feedback system we obtain:

$$H(s) = \frac{A\exp(-s\Delta T)}{1 + sT + A\exp(-s\Delta T)}$$

The loop gain is given by the product of all systems in the loop:

$$L(s) = \frac{A\exp(-s\Delta T)}{1 + sT}$$

for which the amplitude characteristic is:

$$|L(\omega)| = \frac{A}{\sqrt{\omega^2 + T^2}}$$

and the phase characteristic:

$$\Phi(\omega) = -\omega\Delta T - \arctan(\omega T)$$

Unstable behaviour of the system occurs when $\Phi(\omega_0) = -180^o$ and $|L(\omega_0)| > 1$. It is convenient to try to approximate this numerically. Figure 5.26 illustrates the situation by showing the Bode plots for the two subsystems.

(c) Lowering A or increasing T brings the gain below 1 and the instability disappears.

Note that if $\Delta T \ll T$, the phase curve of the delay moves rightward, and the gain of the system remains below 1 (0 dB), preventing instability. However, when the delay approaches the time constant T the situation becomes problematic! In the CNS such situations could occur.

Exercise 5.6:
(we saw this exercise before, in a slightly different form, in 4.1)

(a) With the input given as $p(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$, the output is found to be:

$$q(t) = a \cdot (\cos(\omega_1 t) + \cos(\omega_2 t)) + b \cdot (\cos(\omega_1 t) + \cos(\omega_2 t))^2 + c \cdot (\cos(\omega_1 t) + \cos(\omega_2 t))^3$$

Note that $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$ and $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$, and that

$$(\cos x + \cos y)^3 = \cos^3 x + 3\cos^2 x \cos y + 3\cos x \cos^2 y + \cos^3 y$$

Collecting terms yields the following 13 frequency components with their amplitudes (n.b.: all at phase 0 (positive)):

frequency	amplitude
0	b
ω_1	$(a + 5c/4)$
ω_2	$(b + 5c/4)$
$2\omega_1$	$b/2$
$2\omega_2$	$b/2$
$\omega_1 - \omega_2$	b
$\omega_1 + \omega_2$	b
$\omega_1 + 2\omega_2$	$3c/4$
$\omega_1 - 2\omega_2$	$3c/4$
$\omega_2 + 2\omega_1$	$3c/4$
$\omega_2 - 2\omega_1$	$3c/4$
$3\omega_1$	$c/4$
$3\omega_2$	$c/4$

The 'F'-percept is due to the third-order nonlinearity.

(b) When $p(t) = A\cos(\omega_1 t)$, all terms containing ω_2 disappear, and the amplitudes of the remaining signals are:

$$q(t) = A \cdot b/2 + A \cdot (a + 3c/4) \cdot \cos(\omega_1 t) + A \cdot (b/2) \cdot \cos(2\omega_1 t) + A \cdot (c/4) \cdot \cos(3\omega_1 t)$$

The spectrum now contains 4 frequency components.

Exercise 5.7:
We have to assess the (instability) of the equilibria of this system for different parameter values. We take the radius $r(t) \geq 0\ \forall t$, and consider for the parameters a, μ the four different possibilities

$(>0, <0)$. Note that whenever $\dot{r} < 0$ the trajectories will always converge to the stable equilibrium point for $t \to \infty$.

$$\frac{dr}{dt} = r \cdot (a - \mu r^2)$$
$$\frac{d\theta}{dt} = \omega_0$$

with $\omega_0 > 0$ trajectories in the (x, y) plane follow a counter-clockwise motion.

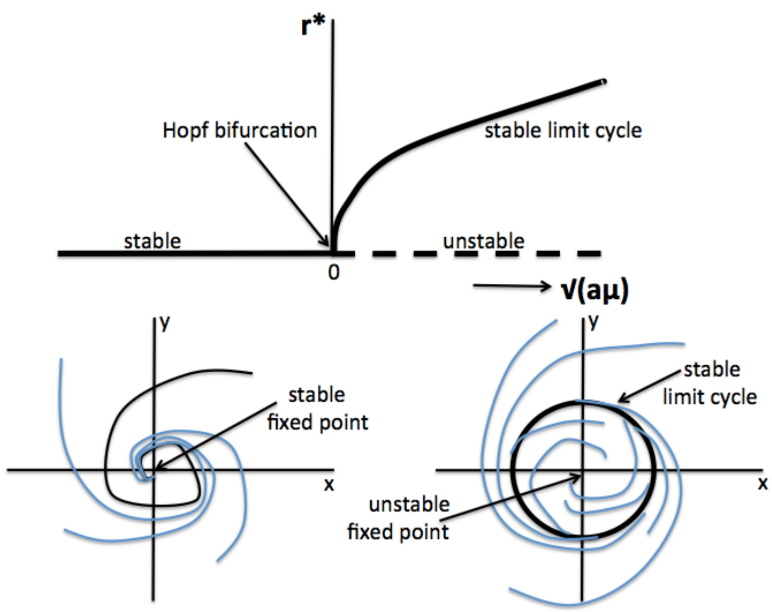
Rewrite the first equation as

$$\frac{dr}{dt} = \frac{r}{\mu} \cdot (a \cdot \mu - r^2)$$

The equilibria are given by $\dot{r} = 0$:

$$r_1^* = 0$$
$$r_2^* = \sqrt{a\mu}$$

You may now verify the trajectories from the Figure below, by looking at the different conditions for the parameters.



A supercritical Hopf bifurcation at $\sqrt{a\mu} = 0$