The Auditory System and Human Sound-Localization Behavior
Short Answers Exercises Chapter 6: Auditory Nerve

⋆ Exercise 6.1:

You have to calculate the FT of

$$h(t) = g(t) \cdot \cos(\omega_D t) \quad \text{for } t \geq 0$$

which is defined as

$$H(\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i\omega t} dt$$

Use Euler's relation

$$e^{ix} = \cos(x) + i \cdot \sin(x) \quad \text{from which:} \quad \cos(\omega_D t) = \frac{e^{i\omega_D t} + e^{-i\omega_D t}}{2}$$

to show that

$$H(\omega) = \frac{1}{2}\left(G(\omega + \omega_D) + G(\omega - \omega_D)\right)$$

Then find the FT of

$$G(\omega \pm \omega_D) = \int_0^{\infty} a \cdot t^{n-1} \cdot e^{-t\cdot(2\pi b + i(\omega \pm \omega_D))} dt$$

You can calculate the FT by applying partial integration (call $\omega' \equiv \omega \pm \omega_D$):

$$G(\omega') = \frac{-a}{2\pi b + i\omega'} \int_0^{\infty} t^{n-1} \cdot d(e^{-t(2\pi b + i\omega')})$$

This yields (use $\int_a^b f dg = fg|_a^b - \int_a^b g df$):

$$G(\omega') = \frac{a \cdot (n-1)}{2\pi b + i\omega'} \int_0^{\infty} t^{n-2} \cdot e^{-t(2\pi b + i\omega')} dt$$

You then repeatedely apply this procedure $(n-1)$ times, until the exponent of power function reaches zero, and

$$H(\omega) \approx \frac{a \cdot (n-1)!}{(2\pi b)^n} \left(\frac{1}{(1 + i \cdot \frac{f - f_D}{b})} \right)^n$$

where f is typically close to f_D. Thus, the gamma-tone filter can be modeled as a serial cascade of n first-order identical low-pass filters.

⋆ Exercise 6.2:

This is a tedious one, which can best be analyzed with the computer (done below). Still, an analytical treatment is possible (shown in part, here), which would proceed as follows:

The amplitude-modulated and rectified sinusoid is described by Eqns. 6.19-6.22:

$$r(t) = \text{MAX}[0, \sin(\omega_C t) + \frac{a_M}{2} \cdot (\sin((\omega_C + \omega_M)t) + \sin((\omega_C - \omega_M)t))$$

and describe the rectified signal as the product of two time functions:

$$r(t) = s(t) \cdot p(t)$$

in which

$$s(t) = [1 + a_M \sin(\omega_M t)] \cdot \sin(\omega_C t)]$$

and

$$p(t) = \begin{cases} 1 & \text{for } s(t) \geq 0 \\ 0 & \text{for } s(t) < 0 \end{cases}$$

The Fourier transform of $r(t)$ is then given by convolution in the frequency domain:

$$R(f) = P(f) \star S(f) \equiv \int_{-\infty}^{\infty} P(\lambda) \cdot S(f - \lambda) d\lambda$$

The Fourier series for the repetitive pulse with period $T_0 = 2\pi/\omega_C$ and duration $\Delta T = \pi/\omega_C$ (see also the example Eqn. 2.71 in Chapter 2) can be found by convolving a single pulse, height 1 and width $\pm T/4$ around the origin with a regularly spaced (period T) series of Dirac δ-pulses, yielding

$$P(f) = \frac{\sin(\pi f T/2)}{\pi f} \cdot \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T})$$

The modulated sine yields three sinusoids:

$$s(t) = \sin(\omega_C t) + \frac{1}{2} (\sin((\omega_C + \omega_M)t) + \sin((\omega_C - \omega_M)t))$$

and thus $S(f - \lambda)$ is readily found as a sum of six delta functions:

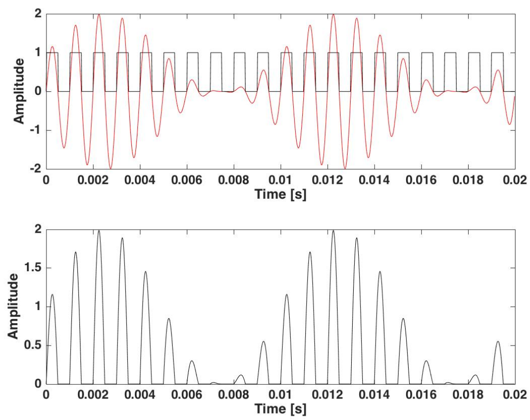
$$S(f - \lambda) = \frac{-i}{2} (\delta(f - f_C - \lambda) - \delta(f + f_C - \lambda) +$$

$$\frac{1}{2}\delta(f - (f_C + f_M) - \lambda) - \frac{1}{2}\delta(f + (f_C + f_M) - \lambda) +$$

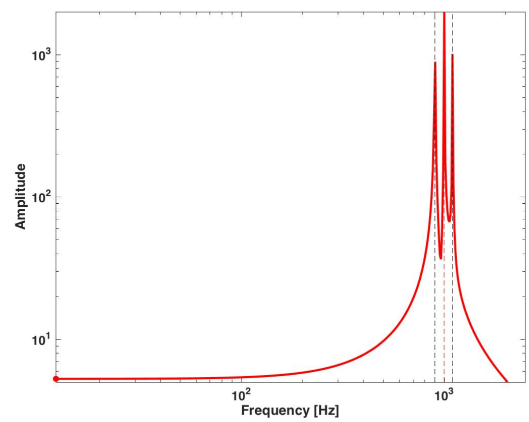
$$\frac{1}{2}\delta(f - (f_C - f_M) - \lambda) - \frac{1}{2}\delta(f + (f_C - f_M) - \lambda)\Big)$$

And yes, these two expressions should be convolved in the frequency domain to calculate the signal's spectrum.....

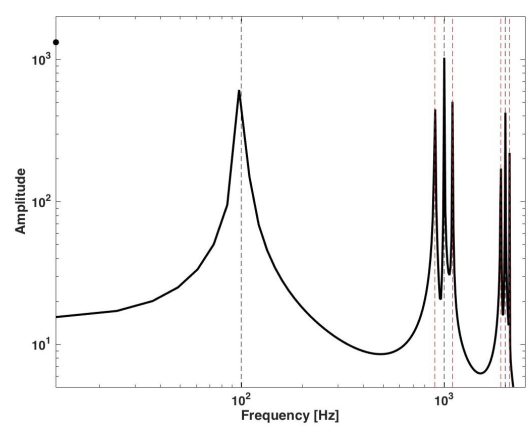
The figures below show the calculated signals and spectra with Matlab script *Chapter6-Exc6-2.m*. You can use this function (see Matlab materials) to inspect the spectra for different modulation frequencies and modulation depths.



The original signal (top, red) with the pulsed rectifier (black), and the rectified AN fiber response (bottom).



The spectrum of the original modulated sinusoid contains three spectral components.



The spectrum of the rectified modulated sinusoid contains many more spectral components, among which the modulation frequency at 100 Hz and the three original frequencies, a dc (dot at $f = 0$) and the harmonic complex at 1900, 2000, 2100 Hz.

Exercise 6.3:

The rate-level function is modeled as

$$R(p, \Theta) = SR + R_{\mathsf{MAX}} \cdot \frac{(p-\Theta)^\alpha}{S + (p-\Theta)^\alpha}$$

We define

$$R_N(p, \Theta) \equiv \frac{R(p, \Theta) - SR}{R_{\mathsf{MAX}}}$$

and you may see immediately that

$$R_N(p, \Theta) = \frac{1}{1 + S \cdot (p-\Theta)^{-\alpha}}$$

Exercise 6.4:

(a) Write

$$a \cdot e^{i\omega_1 t + \phi_1} \equiv a \cdot e^{ix} = a\cos(x) + ia\sin(x) \quad \text{and} \quad b \cdot e^{i\omega_2 t + \phi_2} \equiv b \cdot e^{iy} = b\cos(y) + ib\sin(y)$$

So,

$$s(t) = a \cdot e^{ix} + b \cdot e^{iy} = (a\cos(x) + b\cos(y)) + i \cdot (a\sin(x) + b\sin(y)) = \mathsf{Re}[s(t)] + i \cdot \mathsf{Im}(s(t)]$$

From this we readily find for the amplitude of the envelope:

$$|s|^2 = A^2 = \mathsf{Re}[s]^2 + \mathsf{Im}[s]^2 = (a\cos(x) + b\cos(y))^2 + (a\sin(x) + b\sin(y))^2 = a^2 + b^2 + 2ab \cdot \cos(x-y)$$

(which is indeed Eqn. 6.25 for $n = 2$).

(b) In a similar way you can extend this result to three components by

$$s(t) = a \cdot e^{ix} + b \cdot e^{iy} + c \cdot e^{iz} = (a\cos(x) + b\cos(y) + c\cos(z)) + i \cdot (a\sin(x) + b\sin(y) + c\sin(z))$$

and find

$$|s|^2 = A^2 = a^2 + b^2 + c^2 + 2ab \cdot \cos(x-y) + 2ac \cdot \cos(x-z) + 2bc \cdot \cos(y-z)$$

where

$$x \equiv \omega_1 t + \phi_1 \qquad y \equiv \omega_2 t + \phi_2 \qquad z \equiv \omega_3 t + \phi_3$$

(which indeed corresponds to the summary of Eqn. 6.25 for $n = 3$).

Exercise 6.5:

The dynamic time constant is described by

$$\tau(t) = \tau_N \cdot \left\{ R_0 + (1 - R_0) \cdot \left(\frac{\frac{\tau_W}{\tau_N} - R_0}{1 - R_0} \right)^{\beta |V_{LP}(t)|} \right\}$$

where

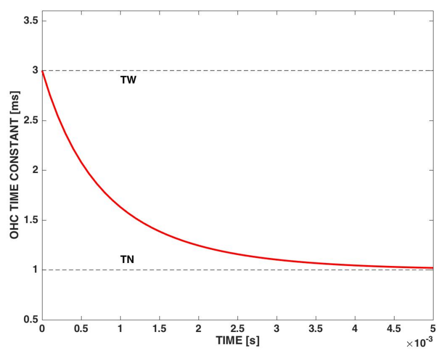
$$\tau_W > \tau_N \qquad R_0 = 0.05 \quad \beta = 1/0.37 \quad V_{LP}(t) = V_0 e^{-800t}$$

the exponent is scaled such that it runs between 0 and 1 ($V_0 = 1/\beta$).
Clearly, when $|V_{LP}(t)| \downarrow 0$

$$\tau(t) = \tau_N \cdot (R_0 + (1 - R_0)) \downarrow \tau_N$$

and when $|V_{LP}(t)| \uparrow 1$

$$\tau(t) = \tau_N \cdot \left\{ R_0 + (1 - R_0) \cdot \left(\frac{\frac{\tau_W}{\tau_N} - R_0}{1 - R_0} \right) \right\} \uparrow \tau_W$$



Calculation of $\tau(t)$ by Chapter6-Exc6-5.m