The Auditory System and Human Sound-Localization Behavior
Short Answers Exercises Chapter 8: Asessing Spatial Performance

Exercise 8.1:

(a) $x = x_1 + x_2$ where x_1 and x_2 are uncorrelated random variables. The probability that the summed random variable has some value x is given by the pdf $P(x)$, and is determined by the product (because of the independence) of two pdf's: $P_1(s)$ that x_1 has some value s, and that x_2 then has value $(x - s)$:

$$P(x) = P_1(s) \cdot P_2(x - s)$$

Since s was chosen arbitrarily, we have to sum over all possible values for s, to obtain:

$$P(x) = \int_{-\infty}^{\infty} P_1(s) \cdot P_2(x - s) ds \equiv P_1(x) \star P_2(x)$$

(b) The two pdf's are

$$P_1(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad \text{and} \quad P_2(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The convolution for the sum of $x = x_1 + x_2$ then yields

$$P(x) = \frac{1}{\sigma^2 2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{(x-\mu-s)^2}{2\sigma^2}\right) ds$$

We write

$$P(x) = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{s^2}{2\sigma^2}\right) \cdot \exp\left(\frac{(x-\mu)s}{\sigma^2}\right) ds$$

which is

$$P(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{s^2 - (x-\mu)s}{\sigma^2}\right) ds$$

The integral can be readily calculated to yield

$$P(x) = \frac{1}{2\sigma^3\sqrt{\pi^3}} \exp\left(-\frac{(x-\mu)^2}{4\sigma^2}\right)$$

which is a Gaussian with a standard deviation of $\sigma\sqrt{2}$

(c) Now, the two pdf's are

$$P_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right) \quad \text{and} \quad P_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma_2^2}\right)$$

The convolution for the sum of $x = x_1 + x_2$ then yields

$$P(x) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left(-\frac{s^2}{2\sigma_1^2}\right) \cdot \exp\left(-\frac{(x-\mu-s)^2}{2\sigma_2^2}\right) ds$$

which reduces to

$$P(x) = \gamma \cdot \exp\left(-(x-\mu)^2 \cdot \left\{\frac{2\sigma_1^2}{\sigma_2^2(\sigma_1^2+\sigma_2^2)} - \frac{1}{2\sigma_2^2}\right\}\right) = \gamma \cdot \exp\left(-(x-\mu)^2 \cdot \left\{\frac{3\sigma_1^2 - \sigma_2^2}{2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right\}\right)$$

with γ a constant scaling factor. This again describes a Gaussian with mean μ, but now with variance

$$\sigma^2 = \frac{2\sigma_2^2(\sigma_1^2+\sigma_2^2)}{3\sigma_1^2-\sigma_2^2}$$

Note that when $\sigma_1 = \sigma_2$ this expression reduces to the answer in (b):

$$\sigma = \sigma_1\sqrt{2}$$

Exercise 8.2:

The standard Gaussian is defined as

$$\phi_N(x) \equiv \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

whereas the general Gaussian for Δr is

$$\phi_G(x) \equiv \frac{1}{(\sigma\sqrt{2})\sqrt{2\pi}} \cdot e^{-\frac{(\Delta r-\mu)^2}{2\cdot(\sigma\sqrt{2})^2}}$$

By introducing the transformation

$$x = \frac{\Delta r - \mu}{\sigma\sqrt{2}}$$

the general Gaussian is mapped onto the standard Gaussian.

So, here the question is whether

$$\frac{1}{(\sigma\sqrt{2})\sqrt{2\pi}}\int_0^{\infty} \exp\left(-\frac{(s-\mu)^2}{2(\sigma\sqrt{2})^2}\right) ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\mu/\sigma\sqrt{2}} \exp\left(-\frac{x^2}{2}\right) dx$$

We rewrite the left integral, as follows: name $x \equiv \frac{s-\mu}{\sigma\sqrt{2}}$, then $ds = \sigma\sqrt{2}dx$ and for $s = 0$ we obtain $x = -\mu/\sigma\sqrt{2}$. So, the integral becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\mu/\sigma\sqrt{2}}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\mu/\sigma\sqrt{2}} \exp\left(-\frac{x^2}{2}\right) dx$$

because of symmetry with respect to $x = 0$. In short,

$$P_C(\Delta r > 0) = \Phi_G\left(\frac{d'}{\sqrt{2}}\right) \quad \text{with } d' \equiv \frac{\mu}{\sigma}$$

Exercise 8.3:

(a) From the figure below (adapted from Fig. 18.2) we see that the probability for a 'false alarm' (noise exceeds the criterion β, without a signal) is given by

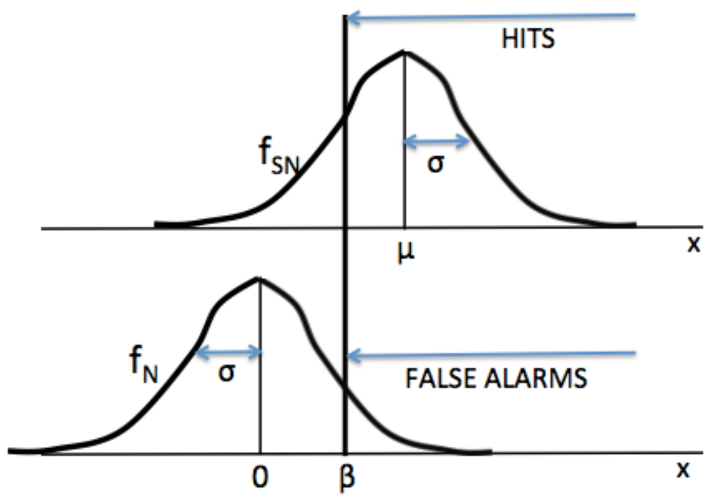
$$P_{FA} = \Phi_G(-\beta/\sigma)$$

Likewise, the probability for a 'hit' (signal exceeds the criterion) is given by

$$P_H = \Phi_G((\mu - \beta)/\sigma)$$

Taking the difference of the inverses eliminates the criterion:

$$\Phi_G^{-1}(P_H) - \Phi_G^{-1}(P_{FA}) = \frac{\mu - \beta}{\sigma} - \frac{-\beta}{\sigma} = \frac{\mu}{\sigma} \equiv d'$$



(b) The likelihood ratio, LR, is given by the probabilities of SN vs. N at the chosen criterion, β:

$$LR \equiv \frac{f_{SN}(\beta)}{f_N(\beta)} = \frac{\exp\left(-\frac{(\mu-\beta)^2}{2\sigma^2}\right)}{\exp\left(-\frac{\beta^2}{2\sigma^2}\right)}$$

and using the result from (a) this gives

$$LR = \exp\left(-\frac{1}{2}\left\{[\Phi^{-1}(P_H)]^2 - [\Phi^{-1}(P_{FA})]^2\right\}\right)$$

(c) When LR = 1, the probabilities at the criterion are the same, and hence, the cumulative tails of the two probability functions will be identical too. Therefore, in this situation, $P_H + P_{FA} = 1$

Exercise 8.4:

Pick a point P on any of the ROC curves, and draw the line perpendicular to the diagonal to Q. Then $D(\beta) = PQ$. The length of the line from the origin to P depends on β and is given by

$$R(\beta) = \sqrt{\Phi_H(\beta)^2 + \Phi_{FA}(\beta)^2}$$

The line OR makes an angle $\phi(\beta)$ with the horizontal axis. This angle is given by

$$\phi(\beta) = \arctan\left(\frac{\Phi_H(\beta)}{\Phi_{FA}(\beta)}\right)$$

The diagonal makes an angle of $\pi/4$, so that the angle between $R(\beta)$ and the main diagonal is $\Delta\phi(\beta) = \phi(\beta) - \pi/4$. Now we have the triangle OPQ within which the line PQ is determined by

$$D(\beta) = R(\beta) \cdot \sin(\Delta\phi(\beta))$$

Exercise 8.5:

(a) From $R(\varepsilon) = p + q \cdot T(\varepsilon) + r \cdot T(\alpha) + s \cdot T(\alpha)T(\varepsilon)$ we simply introduce the drift bias and drift gain as:

$$\Delta r \equiv r \cdot T(\alpha_{\mathsf{MAX}}) \quad \text{and} \quad \Delta s \equiv s \cdot T(\alpha_{\mathsf{MAX}})$$

which immediately leads to Eqn. (8.33).
For the azimuth components we apply a similar regression model:

$$R(\alpha) = k + m \cdot T(\varepsilon) + l \cdot T(\alpha) + n \cdot T(\varepsilon)T(\alpha)$$

for which we introduce the drift bias and drift gain as

$$\Delta m \equiv m \cdot T(\varepsilon_{\mathsf{MAX}}) \quad \text{and} \quad \Delta n \equiv n \cdot T(\varepsilon_{\mathsf{MAX}})$$

(b) When $(\Delta m, \Delta n) = (0,0)$ and $(\Delta r, \Delta s) = (0,0)$ we obtain the standard linear regression of

$$\begin{aligned} R(\alpha) &= k + l \cdot T(\alpha) \\ R(\varepsilon) &= p + q \cdot T(\varepsilon) \end{aligned}$$

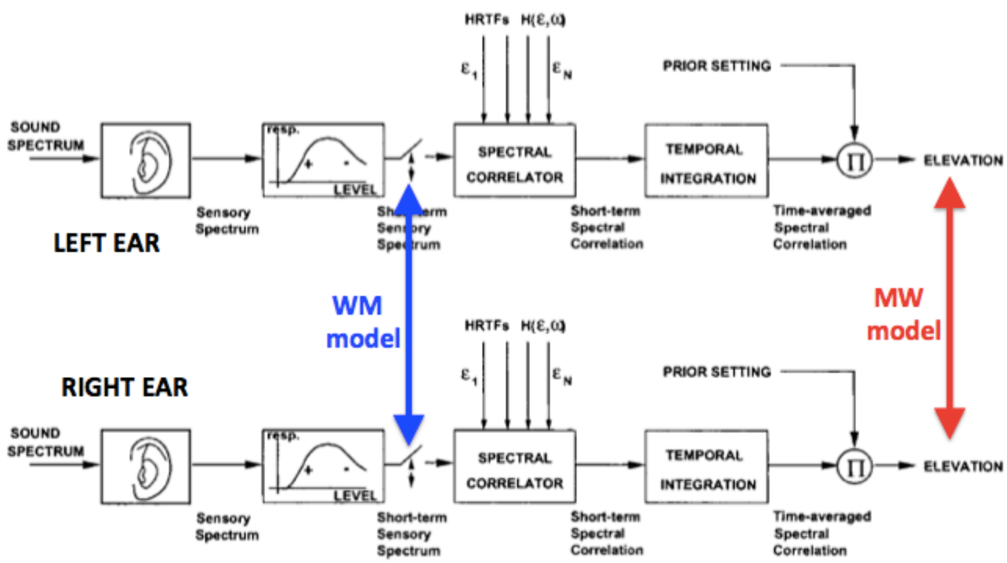
which yields a rectangular grid in Fig. 8.18A. When the biases $(k, p) = (0,0)$ this grid is centered on (0,0), and when the gains $(l, q) = (1,1)$ the grid is a square with edges along the 30 deg boundaries.

The effect of the drift biases $(\Delta m, \Delta r)$ is: when $\Delta m > 0$ the vertical center line of the grid runs oblique with a positive slope. When $\Delta r > 0$ the central horizontal line of the grid runs oblique with a positive slope.

The effect of the drift gains $(\Delta n, \Delta s)$: when $\Delta n > 0$ the length the vertical edge line of positive azimuths is longer than the edge line for negative azimuths, with a gradual decay. When $\Delta s > 0$ the horizontal top edge for upward elevations is longer than the horizontal bottom edge for downward elevations, with a gradual decay.

(c) If the drift gains, $(\Delta n, \Delta s)$ are both zero, the grids will be rectangular, where the sides are determined by the gains (l, q). If the drift biases are zero the lines of constant target elevation will run horizontal, and the lines of constant azimuth will all run vertical. The center lines will cross at the overall bias, at (k, p)

Exercise 8.6:



In the two different models of Fig. 8.19, here in extended format of Fig. 8.16, azimuth information enters the system either at the spectral input stage, where the spectral information from both ears is combined (WM model), or at the spatial stage where the elevation estimates of both ears interact (MW model).

Exercise 8.7:

The transfer (we take only the left ear) from left ear headphone to the eardrum is described by the following transformation in the frequency domain (Fig. 8.20):

$$Y_{2,L}(\omega; \alpha, \varepsilon) = M(\omega) \cdot H_L(\omega) \cdot X_{2,L}(\omega; \alpha, \varepsilon)$$

and from the free-field to the eardrum of the left ear:

$$Y_{1,L}(\omega; \alpha, \varepsilon) = M(\omega) \cdot HRTF_L(\omega; \alpha, \varepsilon) \cdot L(\omega) \cdot X_1(\omega)$$

A good simulation requires that $Y_{2,L} = Y_{1,L}$, so that

$$H_L(\omega) \cdot X_{2,L}(\omega; \alpha, \varepsilon) = HRTF_L(\omega; \alpha, \varepsilon) \cdot L(\omega) \cdot X_1(\omega)$$

and the transfer of free-field to headphone signals is determined by

$$T_L(\omega; \alpha, \varepsilon) \equiv \frac{X_{2,L}(\omega; \alpha, \varepsilon)}{X_1(\omega)} = \frac{HRTF_L(\omega; \alpha, \varepsilon) \cdot L(\omega)}{H_L(\omega)}$$